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The algebraic dimension of linear metric spaces and Baire properties of their hyperspaces

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Abstract. Answering a question of Halbeisen we prove (by two different methods) that the algebraic dimension of each infinite-dimensional complete linear metric space X equals the size of X. A topological method gives a bit more: the algebraic dimension of a linear metric space X equals $|X|$ provided the hyperspace $K(X)$ of compact subsets of X is a Baire space. Studying the interplay between Baire properties of a linear metric space X and its hyperspace, we construct a hereditarily Baire linear metric space X with meager hyperspace $K(X)$. Also under $(\mathfrak{d} = \mathfrak{c})$ we construct a metrizable separable non-complete linear metric space with hereditarily Baire hyperspace. We do not know if such a space can be constructed in ZFC.

Dimensión algebraica de espacios métricos lineales y propiedades de Baire de sus hiperespacios

Resumen. En contestación a una pregunta de Halbeisen se demuestra (mediante dos técnicas distintas) que la dimensión algebraica de cada espacio métrico lineal completo de dimensión infinita X iguala el tamaño de X. Si se utiliza un método topológico aún puede obtenerse más: la dimensión algebraica de un espacio métrico lineal X es igual a $|X|$ si el hiperespacio $K(X)$ de subconjuntos de X compactos es un espacio de Baire. Si se estudia la relación entre las propiedades de Baire de un espacio métrico lineal X y su hiperespacio, se construye un espacio métrico lineal hereditariamente Baire con un hiperespacio $K(X)$ magro. También en $(\mathfrak{d} = \mathfrak{c})$ puede construirse un espacio métrico lineal, separable y no-completo con un hiperespacio hereditariamente Baire. No sabemos si dicho espacio puede ser construido en ZFC.

It is well-known that the *algebraic dimension* (= the size of a Hamel basis) of each infinite-dimensional Banach space X equals the size of X (see [8] or [7] for an elementary proof). In [5] Lorenz Halbeisen asked if the same is true for all complete linear metric spaces. The precise question was if the algebraic dimension of any separable complete linear space equals continuum.

In this paper we answer this question affirmatively using two alternative approaches: analytical and topological. The latter approach leads to interesting open problems on interplay between Baire properties of a linear metric space and its hyperspace.

1. Analytic approach

We start with analytical approach. The basic tool is the following theorem that can have an independent value.

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Theorem 1 *For an infinite-dimensional separable Banach space X and an infinite-dimensional separable complete linear metric space Y there is an injective compact linear operator $T : X \to Y$ with dense range.*

PROOF. In case of Banach Y this theorem is well known (see, e.g., [1, 2.1]) and follows from the existence of a Markushevich basis in every separable Banach space [9]. The general case can be reduced to the "Banach" case as follows. Given a complete linear metric space Y apply [11, p.9] to find an invariant metric d on Y such that $d(\lambda y, 0) \le d(y, 0)$ for all $y \in Y$ and all scalars λ with $|\lambda| \le 1$. The invariant metric d induces a "norm" $\|y\| = d(y, 0)$ on Y with the property $\|\lambda y\| \le \|y\|$ if $|\lambda| \le 1$.

Being infinite-dimensional and separable, the space Y contains a linearly independent sequence $(y_i)_{i\in\omega}$ such that $\|y_i\| \le 2^{-i}$ for all $i \in \omega$ and $\mathrm{lin}(y_i)_{i\in\omega}$ is dense in Y. Let ℓ_1^f denote the linear hull of the standard basis in the Banach space ℓ_1 and let $A : \ell_1^f \to Y$ be the linear operator assigning to each (eventually null) sequence $\vec{t} = (t_i)_{i\in\omega} \in \ell_1^f$ the vector $A(\vec{t}) = \sum_{i\in\omega} t_i y_i$. It follows from the convergence of the series $\sum_{i=1}^{\infty} \|y_i\|$ that the operator $A : \ell_1^f \to Y$ is uniformly continuous and hence extends to a continuous linear operator $A : \ell_1 \to Y$. This operator is factorized through the quotient space $Z = \ell_1/\mathrm{Ker}A$, which is infinite-dimensional because the kernel of A misses the subspace ℓ_1^f. Let $\tilde{A} : Z \to Y$ be an injective continuous operator induced by A. As we mentioned, for every separable Banach space X there is a compact linear injective operator $C : X \to Z$ with dense range. Then the composition $T = \tilde{A} \circ C : X \to Y$ is a compact injective operator from X into Y with dense range.

Corollary 1 *Each infinite-dimensional complete linear metric space Y contains a linearly independent copy of any compact metric space Q.*

PROOF. Of course, we can suppose Q to be infinite. Let $C(Q)$ be the Banach space of continuous functions on Q and $C^*(Q)$ be the dual Banach space, endowed with the weak* topology. Identifying each point $x \in Q$ with the Dirac measure δ_x, we embed Q into $C^*(Q)$ as a linearly independent set. By Theorem 1, there is a linear injective compact operator $T : \ell_2 \to C(Q)$ with dense range. Then $T^* : C^*(Q) \to \ell_2^*$ is injective, compact and weakly* continuous. Consequently, $T^*(Q)$ is linearly independent and endowed with weak* topology is homeomorphic to Q. Since $T^*(Q)$ is norm compact, $T^*(Q)$ endowed with norm topology is homeomorphic to Q too.

Given any infinite-dimensional complete linear metric space Y, we may apply Theorem 1 to find an injective compact linear operator $C : \ell_2^* \to Y$. Then the composition $C \circ T^*|_Q$ maps Q onto a linearly independent compact subset of Y, homeomoprhic to Q.

In particular, we can take as Q the Hilbert cube $[0, 1]^\omega$. So, in a standard way Corollary 1 implies another corollary that answers the question of Halbeisen.

Corollary 2 *The algebraic dimension of each infinite-dimensional complete linear metric space X equals $|X|$.*

2. A topological approach

In this section we give an alternative topological proof of Corollary 2 based on Mycielski-Kuratowski Theorem, see [6, 19.1]. We shall show that a completely-metrizable subset X of a linear metric space L contains a linearly independent Cantor set iff X has uncountable algebraic dimension.

At first we recall some definitions. By the *algebraic dimension* of a subset X of a linear space L we understand the algebraic dimension of the linear hull $\mathrm{lin}X$ of X in L. A *Cantor set* is any topological copy of the Cantor cube $\{0, 1\}^\omega$. By [6, 6.2], each uncountable Polish space contains a Cantor set, and by the classical Brouwer Theorem [6, 7.4] a topological space is a Cantor set if and only if it is compact, metrizable, zero-dimensional and has no isolated point.

Given a metric space (X, d) by $K(X)$ we denote the hyperspace of compact subsets of X, endowed with the Hausdorff metric $d_H(A, B) = \inf\{\varepsilon > 0 : A \subset O(B, \varepsilon),\ B \subset O(A, \varepsilon)\}$. It is known that this metric is complete if and only if the metric of X is complete.

A subset $K \subset X$ is called *perfect* if it has no isolated points. It is well known (and easy to see) that for a perfect metrizable space X the family of perfect compacta is comeager in $K(X)$. We recall that a subset A of a topological space X is *comeager* if its complement $X \setminus A$ is a meager subset of X.

For a set A let $(A)^n = \{(a_i) \in A^n : a_i \neq a_j \text{ for } i \neq j\}$.

Our basic tool is the classical theorem of Mycielski and Kuratowski, see [6, 19.1].

Theorem 2 *Let X be a metrizable space and for each $n \in \mathbb{N}$ let R_n be a comeager set in X^n. Then the set $\{K \in K(X) : \forall n\ (K)^n \subset R_n\}$ is comeager in $K(X)$. So, if X has no isolated points and the hyperspace $K(X)$ is Baire (which happens if X is complete), then there is a Cantor set $C \subset X$ with $(C)^n \subset R_n$ for all $n \in \mathbb{N}$.*

This theorem will be used to prove

Proposition 1 *Let X be a (metric) subspace of a linear metric space L such that each non-empty open subset of X has infinite algebraic dimension. If the hyperspace $K(X)$ of X is Baire, then X contains a linearly independent Cantor set.*

PROOF. For each $n, m \in \mathbb{N}$ consider the open dense subset

$$R_{n,m} = \left\{(x_i) \in X^n : \forall (\alpha_i) \in \mathbb{R}^n \ \ \tfrac{1}{m} \leq \max_{i \leq n} |\alpha_i| \leq m \ \Rightarrow\ \alpha_1 x_1 + \cdots + \alpha_n x_n \neq 0\right\}$$

in X^n (the density of $R_{n,m}$ follows from the fact that non-empty open subsets of X have infinite algebraic dimension). Using the Mycielski-Kuratowski Theorem, find a Cantor set $C \subset X$ such that $(C)^n \subset \bigcap_{m \in \mathbb{N}} R_{n,m}$ for all $n \in \mathbb{N}$. It is clear that C is a linearly independent subset in X.

Corollary 3 *Each infinite-dimensional linear metric space X with Baire hyperspace $K(X)$ contains a linearly independent Cantor set and hence has algebraic dimension equal to $|X|$.*

It light of Corollary 3 it is interesting to remark that a Baire normed space need not have continual algebraic dimension. A counterexample exists under the set-theoretic assumption $\mathrm{non}(\mathcal{M}) < \mathfrak{c}$ where $\mathrm{non}(\mathcal{M})$ stands for the smallest size of a non-meager subset of the real line, see [13].

Proposition 2 *Each separable complete linear metric space X contains a dense linear subspace $L \subset X$ which is Baire and has algebraic dimension* $\dim(L) \leq \mathrm{non}(\mathcal{M})$.

PROOF. By the Lavrentiev's Theorem [6, 3.9], each countable dense subset $D \subset X$ can be enlarged to a zero-dimensional G_δ-subspace $G \subset D$. By Aleksandrov-Urysohn Theorem [6, 7.7], this subspace is homeomorphic to the space of irrationals and hence contains a dense non-meager subspace $B \subset G$ of size $|B| = \mathrm{non}(\mathcal{M})$. The linear hull L of B has algeraic dimension $\dim(L) \leq |B| \leq \mathrm{non}(\mathcal{M})$ and is non-meager because it contains a dense non-meager subspace B. Being topologicall homogeneous, the space L is Baire.

Since the hyperspace of any complete metric space is complete (and hence Baire), Corollary 3 generalizes Corollary 2 and gives an alternative answer the Halbeisen question.

We shall derive from Proposition 1 another corollary that will be applied in the next section.

Corollary 4 *If a Polish subspace X of a linear metric space L has uncountable algebraic dimension, then X contains a linearly independent Cantor set.*

PROOF. Assume that X has uncountable algebraic dimension and consider the set U of all points $x \in X$ having a neighborhood with countable algebraic dimension. It is clear that U is an open set and the complement $Y = X \setminus U$ is a closed subset whose any non-empty open subset has uncountable algebraic dimension. Since X is Polish, the closed subspace Y of X is Polish as well and so it its hyperspace $K(Y)$. In this case it is legal to apply Proposition 1 to find a linearly independent Cantor set $C \subset Y$.

In light of Corollary 3 the following problem arises naturally.

Problem 1 *Characterize linear metric spaces whose hyperspaces are Baire spaces.*

3. On spaces with (hereditarily) Baire hyperspaces

In this section, inspired by Problem 1 we study the Baire properties of hyperspaces of linear metric spaces. We recall that a space X is *hereditarily Baire* if all its closed subspaces are Baire. The following characterization of metric hereditarily Baire spaces is well-known and follows from [3].

Lemma 1 *For a subspace X of a complete metric space $\tilde{X}$ the following conditions are equivalent:*

1. *X is hereditarily Baire;*
2. *X contains no closed countable subset without isolated points;*
3. *For any Cantor set $C \subset \tilde{X}$ the intersection $C \cap X$ fails to be countable and dense in C.*

Since each metrizable space X is homeomorphic to a closed subset of the hyperspace $K(X)$, X is hereditarily Baire if so is its hyperspace. Similarly, X is Baire if $K(X)$ is Baire, see [10]. Thus we have the following diagram describing the interplay between the Baire properties of a metric space X and its hyperspace:

X is :	complete	$\Rightarrow$	hereditarily Baire	$\Rightarrow$	Baire
	$\Updownarrow$		$\Uparrow$		$\Uparrow$
$K(X)$ is:	complete	$\Rightarrow$	hereditarily Baire	$\Rightarrow$	Baire

None of these implications can be reversed (at least in ZFC). The simplest example of a hereditarily Baire space X with meager hyperspace is a Bernstein subset B on the real line $\mathbb{R}$, that is a subset such that both B and $\mathbb{R} \setminus B$ contains no uncountable compact subset. It can be shown that B is a hereditarily Baire space but its hyperspace $K(X)$ is meager. Bernstein sets can be easily constructed by transfinite induction. We modify this construction to get a linear version of a Bernstein set.

Theorem 3 *In each infinite-dimensional complete metric linear space L there is a hereditarily Baire subspace $X \subset L$ whose hyperspace $K(X)$ is meager. That is so because each compact subset of X has at most countable algebraic dimension.*

PROOF. Replacing L with a suitable subspace, we can assume that L is separable complete metric linear space. Let $\mathcal{K}$ be the family of compact subsets K of X having uncountable algebraic dimension. By Corollary 4, each compact set $K \in \mathcal{K}$ contains a linearly independent Cantor set.

Since L, being metrizable and separable, contains at most continuum many compact sets, the family $\mathcal{K}$ has size continuum $\mathfrak{c}$ and hence can be enumerated as $\mathcal{K} = \{K_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction of length $\mathfrak{c}$ in each compact set K_α choose two points x_α, y_α so that x_α does not belong to the linear hull of the set $\{x_\beta, y_\beta : \beta < \alpha\}$ and y_α does not belong to the linear hull of the set $\{x_\alpha, x_\beta, y_\beta : \beta < \alpha\}$. Such a choice is always possible since the set K_α contains a compact linearly independent subset of size continuum.

After completing the inductive construction, consider the linear hull X of the set $\{x_\alpha : \alpha < \mathfrak{c}\}$. We claim that X is a hereditarily Baire subspace of L. Assuming the converse, find, by Lemma 1, a countable closed subset $Q \subset X$ having no isolated points. We claim that the closure $\overline{Q}$ of Q in L has uncountable algebraic dimension. Assuming the converse and using Baire theorem, we would find a non-empty open subset $U \subset \overline{Q}$ lying in a finite-dimensional subspace $F \subset L$. Then $Q \cap F \subset X$, being linearly independent, is finite, which is not true because this space contains a subspace $Q \cap U$ having no isolated point.

Thus $\overline{Q}$ has uncountable linear dimension and by Corollary 4, the set $\overline{Q}$ contains an uncountable linearly independent compactum K. Replacing K by a smaller uncountable compact set, we may assume that $Q \cap K = \emptyset$ and thus $X \cap K \subset X \cap (\overline{Q} \setminus Q) = \emptyset$, which contradicts the fact that X meets each compact subset of uncountable linear dimension. The latter property of X implies also that X is dense in L and thus is infinite-dimensional.

To show that the hyperspace $K(X)$ is meager, let us note that each compact subsets K of X has countable algebraic dimension and thus lies in a countable union of finite-dimensional linear subspaces. Then the Baire Theorem implies that some non-empty open set of K lies in a finite-dimensional space. Consequently, $K(X) = \bigcup_{n,m} \mathcal{K}_{n,m}$ where

$$\mathcal{K}_{n,m} = \{K \in K(X) : \exists x \in K \text{ such that } K \cap B(x, \tfrac{1}{m}) \text{ has algebraic dimension} \leq n\}.$$

It remains to check that each set $\mathcal{K}_{n,m}$ is closed and nowhere dense. To show that $\mathcal{K}_{n,m}$ is closed, take any sequence $(K_i)_{i=1}^{\infty} \subset \mathcal{K}_{n,m}$ convergent to a compact set $K_\infty \in \mathcal{K}(X)$ in the hyperspace $K(X)$. For each K_i find a point $x_i \in K_i$ with $K_i \cap B(x_i, \frac{1}{m})$ having algebraic dimension $\leq n$. It follows from the convergence $K_i \to K_\infty$ that the sequence (x_i) has a subsequence convergent to some point $x_\infty \in K_\infty$. Replacing the sequence (K_i) by a suitable subsequence we may assume that $x_i \to x_\infty$. We claim that $K_\infty \cap B(x_\infty, \frac{1}{m})$ has algebraic dimension $\leq n$ and thus $K_\infty \in \mathcal{K}_{n,m}$. Assuming that it is not so, find $(n+1)$ linearly independent points $z_0, \dots, z_n \in K_\infty \cap B(x_\infty, \frac{1}{m})$. These points have neighborhoods $U_0, \dots, U_n$ such that any points $z'_0, \dots, z'_n$ with $z'_i \in U_i$, $i \leq n$, are linearly independent.

Let $\delta = \frac{1}{m} - \max_{i \leq n} d(x_\infty, z_i)$ and find a number m_1 such that $d(x_m, x_\infty) < \frac{\delta}{2}$ for all $m \geq m_1$. The convergence of the sequence (K_m) to K_∞ yields a number $m \geq m_1$ such that for every $i \leq n$ there is a point $z'_i \in K_m \cap U_i \cap B(z_i, \frac{\delta}{2})$. Then the points $z'_0, \dots, z'_n$ are linearly independent and lie in $K_m \cap B(x_m, \frac{1}{m})$ which is not possible as the latter set has algebraic dimension $\leq n$. So, each set $\mathcal{K}_{n,m}$ is closed in $K(X)$.

Next, we show that it is nowhere dense. Given any $\varepsilon > 0$ and a compact set $K \in K(X)$ we should find a compact set $K_\varepsilon \notin \mathcal{K}_{n,m}$ with $d_H(K, K_\varepsilon) < \varepsilon$. For this take any finite cover $\mathcal{U}$ of K by open subsets of X such that any open ball $B(x, \frac{1}{m})$ centered at a point $x \in K$ contains some set $U \in \mathcal{U}$. We may assume that $U \cap K \neq \emptyset$ for all $U \in \mathcal{U}$. Using the infinite-dimensionality of X in each set U pick a finite linearly independent subset $F_U \subset U$ of size $|F_U| > n$. Consider the compact set $K_\varepsilon = K \cup \bigcup\{F_U : U \in \mathcal{U}\}$ and note that $d_H(K_\varepsilon, K) < \varepsilon$ and $K_\varepsilon \notin \mathcal{K}_{n,m}$. This completes the proof of the nowhere density of $\mathcal{K}_{n,m}$ in $K(X)$. Being the countable union of closed nowhere dense sets, the hyperspace $K(X) = \bigcup_{n,m} \mathcal{K}_{n,m}$ is meager.

Next, we construct a (consistent) example of a non-complete separable linear metric space X whose hyperspace $K(X)$ is hereditarily Baire. Up to our knowledge it is still an open question if there is a ZFC-example of a separable metrizable space X which is not Polish but has hereditarily Baire hyperspace. On the other hand, there is a metrizable space X such that the hyperspace $K(X)$ is hereditarily Baire but the topology of X is not generated by a complete metric, see [2]. This space has an additional feature that each closed separable subspace of X is completely-metrizable, see [12]. However such a pathological space cannot happen among linear metric spaces.

Our construction is carried out under a relatively mild set-theoretic assumption $\mathfrak{d} = \mathfrak{c}$, where $\mathfrak{c}$ stands for the cardinality of continuum and $\mathfrak{d}$ is the smallest number of compact sets that cover the countable product $\mathbb{N}^\omega$. The equality $\mathfrak{d} = \mathfrak{c}$ holds under Martin Axiom but fails in some models of ZFC, see [13], [4]. We shall use the following property of the cardinal $\mathfrak{d}$.

Lemma 2 *Let $\mathcal{U}$ be a family of open subsets of a Cantor set C with $|\mathcal{U}| < \mathfrak{d}$. If the intersection $\bigcap\mathcal{U}$ is dense in C, then it is uncountable.*

PROOF. Assuming that $\cap\mathcal{U}$ is countable and dense in C, consider the complement $C \setminus \cap\mathcal{U}$ and note that it is homeomorphic to $\mathbb{N}^\omega$, being a Polish zero-dimensional nowhere locally compact space, see [6, 7.7]. Observe also that $C \setminus \cap\mathcal{U} = \bigcup_{U \in \mathcal{U}} C \setminus U$ is the union of $|\mathcal{U}| < \mathfrak{d}$ many compacta, which contradicts the definition of $\mathfrak{d}$.

Now we are able to state the promised

Theorem 4 *If $\mathfrak{d} = \mathfrak{c}$, then each infinite-dimensional complete linear metric space X contains a non-closed linear subspace Y whose hyperspace $K(Y)$ is hereditarily Baire.*

PROOF. Without loss of generality we can assume that the space X is separable. In this case the hyperspace $K(X)$ has size continuum and hence can be listed as $K(X) = \{K_\alpha : \alpha < \mathfrak{c}\}$. Without loss of generality we can assume that the linear hull $\mathrm{lin}K_0$ is dense in X and the compact set K_1 contains a point $y_1 \notin \mathrm{lin}K_0$.

The family $\mathfrak{Z} \subset K(K(X))$ of Cantor sets $\mathcal{Z} \subset K(X)$ also has size continuum and can be listed as $\mathfrak{Z} = \{\mathcal{Z}_\alpha : \alpha < \mathfrak{c}\}$ so that for every Cantor set $\mathcal{Z} \in \mathfrak{Z}$ the set $\{\alpha < \mathfrak{c} : \mathcal{Z} \neq \mathcal{Z}_\alpha\}$ is unbounded in $[0, \mathfrak{c})$.

By induction we shall construct transfinite sequences $(C_\alpha)_{\alpha<\mathfrak{c}}$ of compact subsets of X, $(L_\alpha)_{\alpha<\mathfrak{c}}$ of linear subspaces of X, and $(Y_\alpha)_{\alpha<\mathfrak{c}}$ of subsets of X so that for every non-zero ordinal $\alpha < \mathfrak{c}$ the following conditions are satisfied:

(1) $\bigcup_{\beta<\alpha} Y_\beta \subset Y_\alpha$;

(2) $|Y_\alpha| \leq |\alpha|$;

(3) $L_\alpha = L_{<\alpha} + \mathrm{lin}C_\alpha$, where $L_{<\alpha} = \bigcup_{\beta<\alpha} L_\beta$;

(4) $L_\alpha \cap Y_\alpha = \emptyset$;

(5) If $K_\alpha \not\subset L_{<\alpha}$, then $K_\alpha \cap Y_\alpha \neq \emptyset$;

(6) If $\mathcal{Z}_\alpha \cap K(L_{<\alpha})$ is countable and dense in $\mathcal{Z}_\alpha$, then $C_\alpha \in \mathcal{Z}_\alpha \setminus K(L_{<\alpha})$.

To start the induction we put $Y_0 = \emptyset$, $Y_1 = \{y_1\}$, $C_1 = K_0$, $L_0 = \{0\}$, and $L_1 = \mathrm{lin}C_1$. Assume that for some ordinal α the sets Y_β, L_β, C_β have been constructed for all $\beta < \alpha$.

We shall construct sets Y_α, C_α and L_α. Let $Y_{<\alpha} = \bigcup_{\beta<\alpha} Y_\beta$ and $L_{<\alpha} = \bigcup_{\beta<\alpha} L_{<\beta}$. If $K_\alpha \not\subset L_{<\alpha}$, then pick any point $y_\alpha \in K_\alpha \setminus L_{<\alpha}$ and put $Y_\alpha = \{y_\alpha\} \cup Y_{<\alpha}$. If $\mathcal{Z}_\alpha \cap K(L_{<\alpha})$ is not countable and dense in $\mathcal{Z}_\alpha$, then we put $C_\alpha = \emptyset$, $L_\alpha = L_{<\alpha}$ and finish the inductive step.

If $\mathcal{Z}_\alpha \cap K(L_{<\alpha})$ is countable and dense in $\mathcal{Z}_\alpha$, then some extra work is required. First we prove that the set $\mathcal{C} = \{C \in \mathcal{Z}_\alpha : Y_\alpha \cap (L_{<\alpha} + \mathrm{lin}C) = \emptyset\}$ is uncountable. Note that $L_{<\alpha}$, being the linear hull of the union $C_{<\alpha} = \bigcup_{\beta<\alpha} C_\beta$ of α many compacta, can be written as the union $L_{<\alpha} = \cup\mathcal{L}_\alpha$ of some family $\mathcal{L}_\alpha$ of compacta with $|\mathcal{L}_\alpha| < \mathfrak{c}$. Then the set $\mathcal{L}_\alpha \times [0, \alpha] \times \mathbb{N}$ has size $< \mathfrak{c} = \mathfrak{d}$.

For every quadruple $(K, \beta, n, m) \in \mathcal{L}_\alpha \times [0, \alpha] \times \mathbb{N} \times \mathbb{N}$ consider the set $\mathcal{C}_{K,\beta,n,m}$ of all compact sets $C \in \mathcal{Z}_\alpha \subset K(X)$ such that for any n-tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ with $\frac{1}{m} \leq \max\{|\lambda_1|, \dots, |\lambda_n|\} \leq m$ and any points $c_1, \dots, c_n \in C$ with $d(c_i, c_j) \geq \frac{1}{m}$ for $i \neq j$ we get $y_\beta \notin K + \lambda_1 c_1 + \dots \lambda_n c_n$. It is easy to check that the set $\mathcal{C}_{K,\beta,n,m}$ is open in $\mathcal{Z}_\alpha$. It is also clear that

$$\mathcal{C} = \bigcap_{K,\beta,n,m} \mathcal{C}_{K,\beta,n,m}$$

is the intersection of $< \mathfrak{c} = \mathfrak{d}$ many open sets and is dense in $\mathcal{Z}_\alpha$ because it contains the dense subset $\mathcal{Z}_\alpha \cap K(L_{<\alpha})$. By Lemma 2, this intersection is uncountable and hence we can find a set $C \in \mathcal{C} \setminus K(L_<\alpha)$. Put $C_\alpha = C$ and $L_\alpha = L_{<\alpha} + \mathrm{lin}C$. It follows from the inclusion $C_\alpha \in \mathcal{C}$ that $L_\alpha \cap Y_\alpha = \emptyset$. This completes the inductive construction.

Finally consider the linear space $L = \bigcup_{\alpha<\mathfrak{c}} L_\alpha$. It is dense in X because it contains the dense subspace L_1. Next, $L \neq X$ because $y_1 \notin L$. So, L is a dense non-complete subspace of X. We claim that the hyperspace $K(L)$ is hereditarily Baire. First notice that $K(L) = \bigcup_{\beta<\mathfrak{c}} K(L_\beta)$. Indeed, for any compact subset $K \subset L$ we can find an ordinal $\alpha < \mathfrak{c}$ with $K = K_\alpha$. If $K_\alpha \subset L_{<\alpha}$, then $K \in L_\alpha \subset \bigcup_{\beta<\mathfrak{c}} L_\beta$ and we are done. Otherwise, $\emptyset \neq K \cap Y_\alpha \subset K \cap X \setminus L = K \setminus L$, which contradicts the inclusion $K \subset L$.

To prove that $K(L)$ is hereditarily Baire it suffices to check that for any Cantor set $\mathcal{Z}$ in $K(X)$ the intersection $\mathcal{Z} \cap K(L)$ is not countable and dense in $\mathcal{Z}$. Assuming the converse, find an ordinal α with $\mathcal{Z} \cap K(L) \subset \mathcal{Z} \cap K(L_{<\alpha})$. Then $\mathcal{Z} \cap K(L_{<\alpha}) = \mathcal{Z} \cap K(L_\beta)$ for all $\beta \geq \alpha$. Replacing the ordinal α by a larger ordinal, if necessary, we can assume that $\mathcal{Z} = \mathcal{Z}_\alpha$ (this is possible due to the choice of the enumeration $(\mathcal{Z}_\alpha)$). Then the conditions (3) and (6) of the inductive construction imply that $\mathcal{Z}_\alpha \cap K(L_\alpha) \neq \mathcal{Z}_\alpha \cap K(L_{<\alpha})$, which is a contradiction.

Question 1 *Is there a ZFC-example of a non-complete linear metric space X with (hereditarily) Baire hyperspace $K(X)$?*

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