RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **100** (1-2), 2006, pp. 9–30
Análisis Matemático / Mathematical Analysis
Artículo panorámico / Survey

Denseness of norm attaining mappings

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Dedicated to the memory of Vladimir Gurarii. Hope that the enthusiasm that he devoted to Functional Analysis and to be helpful for other researchers will live forever.

Abstract. The Bishop-Phelps Theorem states that the set of (bounded and linear) functionals on a Banach space that attain their norms is dense in the dual. In the complex case, Lomonosov proved that there may be a closed, convex and bounded subset C of a Banach space such that the set of functionals whose maximum modulus is attained on C is not dense in the dual. This paper contains a survey of versions for operators, multilinear forms and polynomials of the Bishop-Phelps Theorem. Lindenstrauss provided examples of Banach spaces X and Y such that the set of norm attaining operators from X to Y is not dense. He also gave isometric conditions on X for which the set of norm attaining operators from X to Y are dense in the space of all operators between these Banach spaces. If the above conclusion holds for every Y, X is said to have the property A. Also, there are known sufficient conditions on the range space Y in order to have the same denseness conditions for every X. In such a case, Y has the property B. Bourgain proved that every space satisfying the Radon-Nikodým property has the property A. For classical Banach spaces, it is known that $C[0,1]$, ℓ_p $(1 < p < \infty)$ and any infinite-dimensional $L_1(\mu)$ do not satisfy the property B (results due to Schachermayer, Gowers and Acosta, respectively). When both X and Y are either $C(K)$ or $L_1(\mu)$, there are positive results due to Johnson and Wolfe, and Iwanik, respectively. Finet and Payá proved that there is also positive result for $X = L_1$ and $Y = L_\infty$.

For multilinear mappings, Aron, Finet and Werner initiated the research and gave sufficient conditions on a Banach space X in order to satisfy the denseness of the set of norm attaining N-linear mappings in the set of all the N-linear mappings (Radon-Nikodým property, for instance). Choi showed that the space $L_1[0,1]$ does not satisfy the denseness of the set of norm attaining bilinear forms. Alaminos, Choi, Kim and Payá proved that for any scattered compact space K, the set of norm attaining N-linear forms on $C(K)$ is dense in the space of all N-linear forms, and for the bilinear case no restriction on the compact is needed. Acosta, García and Maestre proved that the set of N-linear forms whose Arens extensions to the bidual attains the norm is dense in the space of all the N-linear forms on a product of N Banach spaces. For polynomials and for holomorphic mappings, there are some results along the same line, but more open problems than for the multilinear case.

Densidad de funciones que alcanzan la norma

Resumen. El Teorema de Bishop-Phelps afirma que para cualquier espacio de Banach X, el conjunto de los funcionales (lineales y continuos) que alcanzan la norma es denso en el dual. En el caso complejo, Lomonosov dio ejemplos de conjuntos convexos, cerrados y acotados C, tales que el conjunto de los funcionales cuyo máximo se alcanza en C no es denso en el dual. Este trabajo contiene diversas versiones del Teorema de Bishop-Phelps para operadores, formas multilineales y polinomios. Lindenstrauss dio

Presentado por Vicente Montesinos Santalucía.
Recibido: 07/03/2006. *Aceptado*: 24/05/2006
Palabras clave / Keywords: Norm attaining functionals; Bishop-Phelps Theorem; norm attaining operators; multilinear mappings; polynomials.
Mathematics Subject Classifications: Primary 46B20, 47B99; Secondary 47H60.

ejemplos de dos espacios de Banach X e Y tales que el conjunto de operadores de X en Y que alcanzan la norma no es denso en el correspondiente espacio de operadores. Dio también condiciones isométricas suficientes para que un espacio X verifique la densidad del conjunto de operadores de X en Y que alcanzan la norma para cualquier espacio de Banach Y. Si X verifica esta última condición para todo espacio Y se dice que X verifica la propiedad A. Existen condiciones suficientes para que un espacio de Banach Y tenga la propiedad B (esto es, cualquier operador de X en Y se pueda aproximar por operadores que alcanzan la norma, para cualquier X). Bourgain demostró que la propiedad de Radon-Nikodým es una condición suficiente para la propiedad A. Para espacios de Banach clásicos, destacamos los siguientes resultados: $C[0,1]$, ℓ_p $(1 < p < \infty)$ y cualquier espacio $L_1(\mu)$ que sea de dimensión infinita, no verifican la propiedad B (resultados de Schachermayer, Gowers y Acosta, respect.). Cuando los dos espacios X e Y son ambos, o bien de tipo $C(K)$ o $L_1(\mu)$, se sabe que hay resultados positivos (gracias a Johnson y Wolfe, y a Iwanik, respectivamente). Finet y Payá probaron que para $X = L_1$ y $Y = L_\infty$ también se verifica un resultado positivo.

En el caso multilineal, Aron, Finet y Werner plantearon el problema análogo para formas multilineales y dieron condiciones suficientes (por ejemplo la propiedad de Radon-Nikodým) para que un espacio de Banach X verifique la densidad de las formas N-lineales que alcanzan su norma en el correspondiente espacio de todas las formas N-lineales. Choi probó que el espacio $L_1[0,1]$ no verifica una versión del Teorema de Bishop-Phelps para formas bilineales. Alaminos, Choi, Kim y Payá demostraron que para cualquier espacio compacto disperso K, el conjunto de las formas N-lineales que alcanzan su norma en $C(K)$ es denso en el espacio de las formas N-lineales en $C(K)$. Para formas bilineales, el mismo resultado es válido para cualquier compacto K. Acosta, García y Maestre han probado que, para cualesquiera N espacios de Banach, el conjunto de las formas N-lineales que verifican que (todas) sus extensiones de Arens al producto de los biduales alcanzan la norma es denso en el espacio de todas las formas N-lineales. Para polinomios y funciones holomorfas, hay algunos resultados en las misma línea, pero más problemas abiertos que en el caso multilineal.

1. Introduction

The Bishop-Phelps Theorem was the origin of a series of "perturbed optimization" results for different class of functions. The first extensions were done for operators, then for multilinear mappings or polynomials and, very recently, results for holomorphic mappings were obtained. There are no known positive results (valid for every Banach space) for other class of functions different from the space of bounded and linear functionals. We will devote the following sections to state some known results on this topic for different class of functions:
§2. Functionals
§3. Operators
§4. Multilinear and polynomials.
Finally we will pose some open questions in §5.

2. Functionals

The first version of the Bishop-Phelps Theorem appeared in 1961 and it states that the set of (bounded and linear) norm attaining functionals is dense in the dual of every Banach space [19]. This solved a question posed by Klee [47]. That result does not hold in general just for normed spaces. In fact, the space of real polynomials on $[0,1]$ with the sup norm is an example of a space not satisfying the conclusion of the Bishop-Phelps Theorem [54]. Essentially with the same proof of the first version, Bishop and Phelps gave a more general positive result [20] that we state for Banach spaces.

Theorem 1 . *If C is a bounded closed and convex subset of a (real) Banach space X, then the subset of functionals whose sup at C is attained is norm dense in X^*. In fact, the subset of elements in C which are supports points of C (where the sup of some functional is attained) is dense in the boundary of C.*

Afterwards, Bollobás [21] proved a quantitative version of that result which has been very useful related to numerical ranges, for instance. In the following, if X is a normed space, S_X will be the unit sphere of X and B_X its closed unit ball.

Theorem 2 *If X is a Banach space, $x \in S_X, x^* \in S_{X^*}$ are such that $|x^*(x) - 1| \leq \frac{\varepsilon^2}{2}$ for some $0 < \varepsilon < 1/2$, then there are elements $y \in S_X$ and $y^* \in S_{Y^*}$ such that $y^*(y) = 1$, $\|y - x\| \leq \varepsilon$ and $\|y^* - x^*\| \leq \varepsilon + \varepsilon^2$.*

There is a more general variational principle than Bishop-Phelps Theorem due to Brønsted and Rockafellar [23]. Its original proof is based in the arguments used by Bishop and Phelps. In order to see such result and a proof of it that uses Ekeland's variational principle one can see, for instance, [31, Proposition I.3.6].

When dealing with a balanced set C, of course, the modulus of a functional x^* attains the supremum on C if, and only if, $\text{Re}\, x^*$ does it. In the non balanced case, there are two very different extensions of the notion of support functional on the unit ball, by using the modulus of the functional or its real part. In the second case, since this really depends upon the underlying real Banach space, Bishop-Phelps Theorem can be applied in order to obtain a positive result of denseness of the support functionals of the set. With respect to the first extension, Phelps proved that for real Banach spaces there was a version of the Bishop-Phelps result for functionals whose modulus attains its supremum on a bounded, closed and convex set [55, Proposition 7]. In the complex case, whether or not there was a possible version of the Bishop-Phelps Theorem under this setting was an open question for many years. That question was posed by Klee [47] and remarked also by Phelps [55]. Recently it was solved by Lomonosov in the negative [49].

Theorem 3 *([49, Theorem 1]). There is a complex Banach space X such that X^* is $\mathcal{H}^\infty$, the space of bounded and holomorphic functions on the open unit disk, endowed with its usual sup norm and a bounded, closed and convex subset $C \subset X$ such that the set of functionals attaining its maximum modulus at C is a 1- dimensional linear space. In fact, the closed convex hull of the set $\{\delta_z : z \in D\} \subset X^{**}$ given by*

$$\delta_z(h) = h(z) \quad (h \in \mathcal{H}^\infty)\,,$$

where D is the open complex unit disk, is a subset C contained in X satisfying the above property.

Because of results due to Bourgain and Stegall that we will state in the next section (Theorem 10), in a complex Banach space X with the Radon-Nikodým property, every bounded, closed and convex subset of X satisfies the denseness of the subset of their modulus-support functionals in X^*. For more recent results on this complex version of the Bishop-Phelps Theorem see [50]. However, even for some classical (complex) Banach spaces without the Radon-Nikodým property, like c_0 and $L_1([0,1])$, there is no known result along this line until now.

3. Operators

In [19] Bishop and Phelps also raised the question whether or not there was a possible extension of their result for operators. Lindenstrauss was the first author who initiated a systematic study, gave the first counterexample and obtained several positive results.

For two Banach spaces X and Y, $L(X,Y)$ will denote the Banach space of all (bounded and linear) operators from X to Y and $NA(X,Y)$ will be the subset of the norm attaining operators.

Theorem 4 *([48, Proposition 4]) If X is a strictly convex Banach space and there is a non-compact operator from c_0 to X, then the subset of norm attaining operators from c_0 to X is not dense in $L(c_0, X)$. As a consequence, if X is strictly convex space isomorphic to c_0, then the subset $NA(c_0, X)$ is not dense in $L(c_0, X)$.*

The key idea which is behind the proof is the following: If an injective operator $T \in L(Y, Z)$ attains its norm at an element y_0 of the unit ball of Y and Ty_0 is an extreme point of the ball $\|Ty_0\|B_Y$, then y_0 is an extreme point of B_Y. So, if B_Y lacks of extreme points and Z is rotund, such injective operators cannot attain the norm.

Lindenstrauss also introduced two properties on a Banach space, called A and B, as follows. X has property A if $NA(X, Y)$ is dense in $L(X, Y)$ for every Banach space Y. The Banach space Y has property B if $NA(X, Y)$ is dense in $L(X, Y)$ for every Banach space X. He gave intrinsical isometric sufficient conditions to have either property A or property B.

Theorem 5 *([48, Proposition 1]) Assume that X is a Banach space such that there is a subset $A \subset S_X$ satisfying $B_X = \overline{|\mathrm{co}|}(A)$ ($\overline{|\mathrm{co}|}$ denotes the closure of the circled convex hull) and A is a subset of uniformly strongly exposed points of the unit ball, then X has property A.*

The elements of $A \subset S_X$ are *uniformly strongly exposed points* if for every $a \in A$, there is $s(a) \in S_{X^*}$ such that

$$\forall \varepsilon > 0 \; \exists \delta > 0 : a \in A, x \in B_X \;\; \mathrm{Re}\, s(a)(x) > 1 - \delta \;\Rightarrow\; \|x - a\| < \varepsilon \; .$$

For instance, the usual vector basis of ℓ_1 is a subset satisfying the above condition.

We recall now a certain isometric assumption that Lindenstrauss used as a sufficient condition for property B. A Banach space Y has **property** β if there is a subset $\{(y_i, y_i^*) : i \in I\} \subset S_Y \times S_{Y^*}$ such that $y_i^*(y_i) = 1$ for every $i \in I$, the set of functionals $\{y_i^* : i \in I\}$ is a 1-norming set in Y and there is a constant $0 \leq \rho < 1$ such that $|y_j^*(y_i)| \leq \rho$ for every $i, j \in I, i \neq j$. The space c_0 and ℓ_∞ clearly satisfy such condition.

Theorem 6 *([48, Proposition 3]). If Y has property β, then it has property B of Lindenstrauss.*

In the proof of Theorem 5, from an operator $T \in L(X, Y)$, it is constructed a nuclear operator N with small norm such that $T + N$ attains its norm. Theorem 6 is proved by using the Bishop-Phelps-Bollobás Theorem applied to a functional expressed as $T^* y_\alpha^*$, where $\|T^* y_\alpha^*\|$ is close enough to $\|T\|$, and the perturbation is a rank one operator in this case.

The real and finite-dimensional normed spaces satisfying the property β are precisely those spaces whose unit ball is polyhedral. There are also positive results by assuming a property weaker than β, that we introduce now.

A Banach space Y has **property quasi-**β if there is a subset $A \subset S_{Y^*}$, a mapping $\sigma : A \longrightarrow S_Y$ and a real valued function ρ on A satisfying the following conditions:

(i) $y^*(\sigma(y^*)) = 1$ for $y^* \in A$.

(ii) $|z^*(\sigma(y^*))| \leq \rho(y^*) < 1$ for $y^*, z^* \in A, y^* \neq z^*$.

(iii) For every extreme point e^* in the unit ball of Y^*, there is a subset A_{e^*} of A and a scalar t with $|t| = 1$ such that te^* lies in the w^*-closure of A_{e^*} and $\sup\{\rho(y^*) : y^* \in A_{e^*}\} < 1$.

Theorem 7 *([7, Theorem 2]). Property quasi-β implies property B.*

As a consequence of the last result, new examples of finite-dimensional spaces with the property B of Lindenstrauss appeared (see [7, Example 5]) . In the infinite-dimensional case, every canonical predual of a Lorentz sequence space is a space that has property quasi-β, but not β. Also this new property satisfies some stability requirements shared also by the property B. For instance, both are stable under c_0-sums (see ([7, Propositions 3 and 4]).

However, in the finite-dimensional case, until now, it is far from being known a characterization of those spaces with the property B. In fact, it is not known whether the euclidean 2-dimensional space has it or not.

Also Lindenstrauss proved a general positive result of denseness by considering operators whose second adjoints attain their norms. In general, if T attains the norm, T^* also does it (at any support functional of Tx_0, where $x_0 \in S_X$ is such that $\|Tx_0\| = \|T\|$) and easy examples show that such inclusion between sets of operators can be proper (if X is not reflexive).

Theorem 8 *([48, Theorem 1]). For every Banach spaces X and Y, the set*

$$\{T \in L(X,Y) : T^{**} \in NA(X^{**}, Y^{**})\}$$

is norm dense in $L(X,Y)$.

Corollary 1 . *If X is a reflexive Banach space, then X has property A.*

Enflo, Kover and Smithies gave a constructive proof of the above result for the case of a Hilbert space [34, Theorem 1]. Theorem 8 was improved in [61, Proposition 4] by Zizler who proved that in fact

$$\{T \in L(X,Y) : T^* \in NA(Y^*, X^*)\}$$

is dense in $L(X,Y)$. The perturbation used by Lindenstrauss to prove Theorem 8 is a nuclear operator. Afterwards, Poliquin and Zizler obtained a general optimization principle from which it can be deduced that the result due to Zizler can be shown by summing an appropriate rank-one operator to a fixed operator.

Theorem 9 *([56, Theorem 1]). Let φ be a w^*-lower semicontinuous convex and Lipschitz function defined on a w^*-compact convex set C contained in the dual of a Banach space X. Given $\varepsilon > 0$, there is an $x \in X$, with $\|x\| \leq \varepsilon$, such that $\varphi + x$ attains its supremum on C at an extreme point of C.*

Also Lindenstrauss showed some other results giving necessary conditions in order to have property A. Under some extra assumptions on the space, property A on a Banach space X implies that the unit ball of X has a good extremal structure (see for instance [48, Theorem 2]). However, until now no intrinsical characterization of property A is known.

In 1977 Bourgain gave a more general sufficient condition than the one in Corollary 1 to have property A.

Theorem 10 *([22, Theorem 5]) Let $C \subset X$ be a nonempty, bounded, closed and absolutely convex Radon-Nikodým set. Then, for every Banach space Y, the subset of operators $T \in L(X,Y)$ such that $\sup_{x \in C} \|Tx\|$ is attained contains a G_δ-dense set in $L(X,Y)$.*

Recall that a subset C of a Banach space X is said to have the Radon-Nikodým property if every (nonempty) subset of C is dentable, that is, every subset of C has slices with arbitrarily small diameter. As a consequence of the results due to Davis and Phelps [30] and Huff [40], if a Banach space X has the Radon-Nikodým property (defined in terms of measure theory), then every nonempty bounded subset of X satisfies that property.

Afterwards, Stegall [59] gave a direct proof of a general perturbation principle, stated as follows:

Theorem 11 . *Let $C \subset B_X$ be a (nonempty) closed, bounded and convex Radon-Nikodým set and $f : C \longrightarrow \mathbb{R}$ be a bounded above and upper-semicontinuos function. Then the subset*

$$\{x^* \in X^* : f + \text{Re}\, x^* \text{ attains its maximum on } C\}$$

contains a G_δ-dense set.

The special case $f = 0$ of the previous result is due to Bourgain [22, Theorem 8]. Bourgain also proved a partial converse result of Theorem 10 (see [22, Proposition 1]). Afterwards, Huff [41] completed the following characterization:

Theorem 12 . *Let X be a Banach space. Then X has the Radon-Nikodým property if, and only if, for every Banach spaces Z and Y isomorphic to X, $NA(Z,Y)$ is dense in $L(Z,Y)$.*

In the special case of Theorem 11 that X is isometric to a dual space, then there is also a version where the functionals used are w^*-continuous. This version is useful if one wants to perturb bounded functions which are w^*-continuous and obtain a new one attaining its supremum and which is still w^*-continuous. For instance, that principle can be used in the case that Y is an Asplund space, $T \in L(X,Y)$, by taking $f = \|T^*(y^*)\|$. By applying Theorem 13, it is easy to obtain a rank-operator $S \in L(X,Y)$ with arbitrarily small norm and such that $(T+S)^*$ attains its norm. In that special case, also the result due to Zizler and sated above can be applied. Of course, the next result may have many other applications.

Theorem 13 *([9]). Let $C \subset B_{X^*}$ be a (nonempty) bounded, convex and closed Radon-Nikodým set and $f : C \longrightarrow \mathbb{R}$ be a bounded above and upper-semicontinuos function, then the subset*

$$\{x \in X : f + \text{Re}\, x \text{ attains its maximum on } C\}$$

contains a G_δ-dense set.

For classical spaces, Lindenstrauss proved that $L_1(\mu)$ has property A if and only if the measure μ is purely atomic. In the case that K is a metrizable compact space, he also showed that $C(K)$ has property A if and only if K is finite [48, Propostion 2].

In 1976, Uhl proved the following nice result for operators from L_1 to some other Banach space:

Theorem 14 *([60, Theorems 1 and 3]). If μ is a finite positive measure and Y has the Radon-Nikodým property, then $NA(L_1(\mu),Y)$ is dense in $L(L_1(\mu),Y)$. If Y is a strictly convex Banach space and $NA(L_1,Y)$ is dense in $L(L_1,Y)$, then Y has the Radon-Nikodým property.*

For classical spaces not covered by the previous theorems also specific independent results were proved. The techniques of those results strongly depend on the concrete Banach spaces. Here we state some of them.

Theorem 15 *([44, Theorem 1]). For every compact and Hausdorff topological spaces K and S, the set $NA(C(K), C(S))$ is dense in $L(C(K), C(S))$ (here just the real valued spaces of functions are considered).*

Iwanik obtained the corresponding positive result for $L_1(\mu)$ spaces. This author also gave an argument to extend the next result to any Lebesgue spaces [42].

Theorem 16 *([42, Theorem 2]). For every finite positive Borel measures μ, ν on the unit interval of $\mathbb{R}$, $NA(L_1(\mu), L_1(\nu))$ is dense in $L(L_1(\mu), L_1(\nu))$.*

Schachermayer proved the following results for real Banach spaces solving two questions posed in [44]. In fact, he gave the first example of a classical space without property B.

Theorem 17 *([57, Theorems B and A]).*

i) If K is a Hausdorff and compact topological space and X is any (real) Banach space, then every weakly compact operator from $C(K)$ (real valued functions) to X can be approximated by norm attaining weakly compact operators. Hence $NA(C(K), \ell_2)$ is dense in $L(C(K), \ell_2)$ and $NA(C(K), L_1[0, 1])$ is dense in $L(C(K), L_1[0, 1])$.

ii) $NA(L_1[0, 1], C[0, 1])$ is not dense in $L(L_1[0, 1], C[0, 1])$.

A more easy counterexample on denseness of norm attaining operators on the same kind of classical Banach spaces is due to Johnson and Wolfe:

Theorem 18 *([45, Corollary 2]). There is a compact set $K \subset (B_{L_\infty[0,1]}, \sigma(L_\infty[0, 1], L_1[0, 1]))$ such that the subset $NA(L_1[0, 1], C(K))$ is not dense in $L(L_1[0, 1], C(K))$.*

Alaminos, Choi, Kim and Payá obtained the following extension of Theorem 17.(i).

Theorem 19 *([13, Theorem 2]). For every Hausdorff and locally compact topological space L, the subset of the norm attaining weakly compact operators between $C_0(L)$ (either real or complex valued functions) to a Banach space Y is dense in the corresponding space of all weakly compact operators.*

The set of norm attaining bilinear forms on $L_1[0, 1]$ is not dense in the corresponding space of bilinear forms (we will state this result later). By using the usual identification of bilinear forms on a space with operators from this space to its dual, every norm attaining bilinear form corresponds to a norm attaining operator. But the reversed is not true. In fact, Finet and Payá proved the following result for operators, which was generalized later by Payá and Saleh:

Theorem 20 *([35]). For every σ-finite measure μ, $NA(L_1(\mu), L_\infty[0, 1])$ is dense in $L(L_1(\mu), L_\infty[0, 1])$.*

Theorem 21 *([52, Theorem 1]). For every measure μ and every localizable measure ν, then $NA(L_1(\mu), L_\infty(\nu))$ is dense in $L(L_1(\mu), L_\infty(\nu))$.*

In view of Theorem 17, not every $C(K)$ has the property B. Until now, no characterization is known of the spaces $C(K)$ having such a property. In any case, by the next result, the domain spaces in order to show the lack of property B of $C(K)$ (if possible) cannot be Asplund spaces.

Theorem 22 *([44, Theorem 2]). Let X be an Asplund space, then for every compact and Hausdorff topological space K, the subset $NA(X, C(K))$ is dense in $L(X, C(K))$.*

For the special class of the compact operators, there is no known negative result on denseness of the subset of norm attaining compact operators in this set. Under some strengthening of the 1-approximation property on the domain space, there are just some positive results due to Johnson and Wolfe [44]. Also under some other restrictions on the Banach spaces and on the classes of operators considered, Baker obtained several positive results (see [18]). Diestel and Uhl proved the following result in the case that X is an L_1-space [33, Theorem, p. 6]

Theorem 23 *([44, Theorems 3 and 4]). Assume that X and Y are Banach spaces such that either X or Y is isometric to $C(K)$ (for some compact and Hausdorff topological space K) or to some $L_1(\mu)$ (some positive measure μ). Then the subset of norm attaining finite rank operators from X to Y is dense in the corresponding space of all compact operators.*

The above result also holds if the range space is an isometric predual of some $L_1(\mu)$ (see [44, Proposition 3.3]). Also a certain version of the approximation property (trivially satisfied for spaces having a shrinking and monotone Schauder basis) on X is a sufficient condition in order to obtain the denseness of the subset of norm attaining finite rank operators from X to Y in the subset of all compact operators in $L(X, Y)$ (see [44, Section 3] for details).

Johnson and Wolfe [44, Proposition 4.2] also characterized the pairs of the infinite dimensional $C(K)$ or $L_p(\mu)$ for which the subset of norm attaining operators between them is the whole space of operators. In general, the condition $NA(X, Y) = L(X, Y)$ is quite strong. In fact, Holub proved that for (non trivial) reflexive spaces X and Y such that one of them has the approximation property, that condition is equivalent to the reflexivity of $L(X, Y)$ [39, Theorem 2].

At the beginning of the nineties, for some "simple" Banach spaces such as ℓ_p ($1 \leq p < \infty$) it was not known whether or not they had property B. In the reflexive case, the answer was obtained by Gowers:

Theorem 24 *([37, Theorem, p. 149]). For $1 < p < \infty$ the space ℓ_p does not have the property B of Lindenstrauss.*

In fact, the space X used as the domain to show the previous result was the canonical predual (denoted by $d_*(w, 1)$) of a Lorentz sequence space $d(w, 1)$ for $w = \{1/n\}$. This space shares with c_0 some isometric properties (the unit ball has no extreme points), but not all the operators from this space to ℓ_p are compact. Otherwise, as we mentioned before, there is a positive result of denseness of norm attaining operators in the subset of the compact operators from a Banach space with a shrinking and monotone basis to any Banach space.

Aguirre generalized Theorem 24 by using a result due to N. and V. Gurarii (see for instance [32, Theorem VIII.3]). In fact, he gave also a result by assuming a weaker isometric condition on the range space [12, Corollary 6].

Theorem 25 *([12, Corollary 9]). If Y is any infinite-dimensional uniformly convex Banach space, then Y does not have the property B of Lindenstrauss.*

The presence of a normalized basic sequence $\{y_n\}$ that admits an upper-p-estimate in Y (for some $p > 1$) is used in order to construct an operator from the canonical predual of $d(\{1/n\}, 1)$ to Y that applies the usual vector basis of the domain space on the sequence $\{y_n\}$. This operator cannot be approximated by norm attaining operators.

Acosta showed that ℓ_1 does not have the property B of Lindenstrauss. In fact, it holds the following result:

Theorem 26 *([3, Theorem 2.3]). Every infinite-dimensional $L_1(\mu)$ does not satisfy the property B of Lindenstrauss.*

By considering ℓ_1 as the range space, the domain space used in order to prove that ℓ_p $(1 < p < \infty)$ does not have the property B could not be useful in order to obtain a negative result here. In fact, we do have:

Theorem 27 *([3, Theorem 2.4]). Let $w \in c_0 \backslash \ell_1$ be a decreasing sequence of positive real numbers and μ any positive measure. The following assertions hold:*

i) $\overline{NA(d_*(w,1), L_1(\mu))} = K(d_*(w,1), L_1(\mu))$ *(the space of compact operators).*

ii) *If μ is purely atomic, the set of norm attaining operators from $d_*(w,1)$ to $L_1(\mu)$ is dense.*

iii) *If μ is not purely atomic and σ-finite, then*

$$\overline{NA(d_*(w,1), L_1(\mu))} = L(d_*(w,1), L_1(\mu)) \Leftrightarrow w \notin \ell_2.$$

One can see the definition of the space $d_*(w,1)$ in [43, Section 2], for instance.

The isometric assumption of Theorem 25 can be weakened:

Theorem 28 *([2, Theorem 2.3]). Every infinite-dimensional strictly convex space (or $\mathbb{C}$-rotund in the complex case) lacks the property B of Lindenstrauss.*

For the Lorentz sequence spaces $d(w,1)$, Choi, Kaminska and Lee proved the following result, and they answered to a question posed in [6, Example 7.b].

Theorem 29 *([24, Theorem 4.3]). Assume that $w \in \ell_2 \backslash \ell_1$. For the complex Lorentz sequence space $d(w,1)$ and its canonical predual $d_*(w,1)$, then $NA(d_*(w,1), d(w,1))$ is not dense in $L(d_*(w,1), d(w,1))$.*

One of the facts that the authors used in order to prove the above result is that the space $d(w,1)$ is $\mathbb{C}$-rotund (in the complex case). But $d(w,1)$ is never rotund. In fact, the extreme points of the unit ball are those finite supported sequences x in the unit sphere such that the set $\{|x(n)| : n \in \mathbb{N}\}$ has two elements. Even so, Theorem 29 also holds in the real case [5]. By combining this result with [43, Proposition 2.4 and Theorem 2.6], and by using the arguments in [17, Theorem 2.9], the following result can be deduced (valid both for the real and complex case):

Theorem 30 . *For a sequence of weights $w \in c_0 \backslash \ell_1$, the following conditions are equivalent:*

i) The subset $NA(d_*(w,1), d(w,1))$ is dense in $L(d_*(w,1), d(w,1))$.

ii) $w \notin \ell_2$.

iii) The space $L(d_*(w,1), d(w,1))$ coincides with the space $K(d_*(w,1), d(w,1))$ of the compact operators.

From an isomorphic point of view, there are very general renorming results in order to obtain properties A or B. Schachermayer introduced the property α, a strengthening of the property used by Lindenstrauss to obtain property A (Theorem 5). There is a certain duality between properties α and β that can be seen, for instance, in [58, Proposition 1.4].

Definition 1 *([58, Definition 1.2]). A Banach space X has property α if there is a real number $0 \leq \rho < 1$ and a subset $\{(x_i, x_i^*) : i \in I\} \subset S_X \times S_{X^*}$ such that $x_i^*(x_i) = 1$ for every i, $|x_i^*(x_j)| \leq \rho$ for every $i \neq j$ and the closed unit ball of X is the closed, circled and convex hull of the set $\{x_i : i \in I\}$.*

He proved that property α implies the assumption of Theorem 5. Even so, he gave a direct proof of the result stated below, which is simpler than the proof of Theorem 5.

Theorem 31 *([58, Proposition 1.3]). Property α implies property A. In fact, if X is a Banach space with the property α, Y is any Banach space, and $T \in L(X,Y)$, for every $\varepsilon > 0$, there is a rank-one operator $F \in L(X,Y)$ such that $\|F\| < \varepsilon$ and $T + F \in NA(X,Y)$.*

Theorem 32 *([58, Propositions 2.1 and 3.1]). If X is a superreflexive Banach space or c_0, then for every $\varepsilon > 0$, there is a space Y isomorphic to X such that Y has property α and the Banach-Mazur distance between Y and X is less than $1 + \varepsilon$. The same result also holds for superreflexive spaces or ℓ_1 and the property β.*

Also Schachermayer proved a quite general renorming result (valid for WCG) in order to obtain property α (see [58, Theorems 4.1 and 4.4]). The same assertion also holds for ℓ_∞ [58, Proposition 4.5]. We will state just the following result along the same spirit and quite general, which is due to Godun and Troyanski [36, Theorem 1]:

Theorem 33 . *Every Banach space with a biorthogonal system whose cardinality is equal to the density character of X can be renormed to have property α.*

By considering embeddings in such spaces, there is no restriction at all.

Theorem 34 *([58, Theorem 4.6]). Every Banach space X may be isometrically and 1-complemented embedded into a Banach space Z with property α and having the same density character as X.*

In view of the next result and Theorem 6 property B is not restrictive at all if isomorphisms are allowed.

Theorem 35 *([51, Theorems 1 and 2]). Let $(X, \| \ \|)$ be a Banach space. Then for any $K > 3$ there is a norm $|\!|\!| \ |\!|\!|$ on X such that $\|x\| \leq |\!|\!| x |\!|\!| \leq K\|x\|$, and $(X, |\!|\!| \ |\!|\!|)$ has property β. If X is superreflexive, then the same result holds for every $K > 1$.*

Very recently, Choi and Song defined a property weaker than property α that still implies property A and such that it is stable under finite ℓ_1 sums.

Definition 2 *A Banach space X has property quasi-α if there are sets $A = \{x_i : i \in I\} \subset S_X$, $A^* = \{x_i^* : i \in I\} \subset S_{X^*}$ and a function $\rho : A \longrightarrow [0, 1[$ such that*

i) $x_i^*(x_i) = 1$, *for every* $i \in I$

ii) $|x_i^*(x_j)| \leq \rho(x_j) < 1$, *for every* $i, j \in I, i \neq j$.

iii) For every $e \in \mathrm{Ext}(B_{X^{**}})$, there is a subset $A_e \subset A$ and a scalar t with $|t| = 1$ such that $te \in \overline{A_e}^{w^*}$ and $r_e := \sup\{\rho(x) : x \in A_e\} < 1$ (where we used the weak-$*$ topology of X^{**}).

Theorem 36 *([29, Theorem 1]). Property quasi-α implies property A.*

4. Multilinear mappings and polynomials

For two Banach spaces X and Y, by using the usual (isometric) identification of $\mathcal{L}^2(X \times Y)$ (set of bounded and bilinear forms on $X \times Y$) and $L(X, Y^*)$, given by $B(x, y) = T(x)(y)$, if an element $B \in \mathcal{L}^2(X \times Y)$ attains its norm, then the corresponding operator T also does it. The converse does not hold for every non reflexive Banach space Y if X is non trivial. For instance, any norm attaining operator $T \in L(X, Y^*)$ satisfying that Tx_0 is not a norm attaining functional for every x_0 where T attains its norm, satisfies that the corresponding bilinear form does not attain its norm. Aron, Finet and Werner posed the question whether Bishop-Phelps Theorem held for bilinear or multilinear forms. They also proved the first positive results of denseness for these mappings.

In the following, given $N+1$ normed spaces X_i $(1 \leq i \leq N)$ and Y, we will denote by $\mathcal{L}^N(X_1 \times \ldots \times X_N, Y)$ the space of N-linear and bounded mappings from $X_1 \times \ldots \times X_N$ to Y, endowed with its usual sup norm. If Y is the scalar field, we will just write $\mathcal{L}^N(X_1 \times \ldots \times X_N)$; $NA\mathcal{L}^N(X_1 \times \ldots \times X_N)$ will be the subset of norm attaining N-linear forms. In the case that all the normed spaces X_i coincide (with X), then we will only write $\mathcal{L}^N(X)$ and $NA\mathcal{L}^N(X)$ to denote both sets.

Aron, Finet and Werner proved the following positive result:

Theorem 37 *([15, Theorem 1]). If X has the Radon-Nikodým property, then $NA\mathcal{L}^N(X)$ is dense in $\mathcal{L}^N(X)$.*

Also these authors proved a multilinear version of Theorem 31:

Theorem 38 *([15, Theorem 3]). If a Banach space X satisfies property α, then the subset $NA\mathcal{L}^N(X)$ is dense in $\mathcal{L}^N(X)$.*

By using Theorem 33 they obtained the following renorming result:

Corollary 2 *([15, Corollary 5]). For every Banach space X with a biorthogonal system whose cardinality is equal to the density character of X, there is a Banach space Y isomorphic to X such that $NA\mathcal{L}^N(Y)$ is dense in $\mathcal{L}^N(Y)$.*

In full generality, there cannot be a version of the Bishop-Phelps Theorem for bilinear forms. The space $X = d_*(w, 1)$ for $w = \{1/n\}$ was the first known counterexample [6, Corollary 4]. In view of the next result, also the same result holds for N-linear forms if $N \geq 3$.

Theorem 39 *([11] or [43, Proposition 2.1]). For every $N \geq 2$, if X is a Banach space such that $NA\mathcal{L}^{N+1}(X)$ is dense in $\mathcal{L}^{N+1}(X)$, then the subset $NA\mathcal{L}^N(X)$ is dense in $\mathcal{L}^N(X)$.*

The converse of the above result does not hold for any N ([43, Corollary 2.7]). Jiménez-Sevilla and Payá proved this as a consequence of the result that we state below. In fact, these authors characterized the canonical predual of the Lorentz sequence spaces $d(w, 1)$ that satisfy the denseness of the norm attaining N-linear forms in the corresponding space of all N-linear forms. They proved that in the case that a space of this class satisfies a positive result, then all multilinear forms satisfy that all their partial operators are compact. Since the spaces used have monotone and shrinking basis, these multilinear forms can be easily approximated by norm attaining multilinear forms.

Theorem 40 *([43, Theorem 2.6]). Let $N \geq 2$. Then $NA\mathcal{L}^N(d_*(w, 1))$ is dense in $\mathcal{L}^N(d_*(w, 1))$ if, and only if, $w \notin \ell_N$.*

The following positive answers are due to Choi and Kim:

Theorem 41 *([28, Theorem 2.2]). If a Banach space X has a monotone shrinking basis and the Dunford-Pettis property, then the subset of norm attaining N-linear forms on X is dense in $\mathcal{L}^N(X)$. Also in this case the subset of symmetric norm attaining N-linear forms is dense in the space of all symmetric N-linear forms on X.*

As a consequence, the above results are valid for c_0 ([28, Corollary 2.3]). Also Choi and Kim proved that for every Banach space Y, the subset of norm attaining N-linear mappings from ℓ_1 to Y is dense in $\mathcal{L}^N(\ell_1, Y)$ and the same result holds for the class of the symmetric mappings. In this case the complement of the subset of norm attaining N-linear mappings is not dense in the whole space (see [28, Theorem 2.5]).

Choi was the fist author who showed that a classical space was also a counterexample:

Theorem 42 *([25, Theorem 3]). $NA\mathcal{L}^2(L_1[0,1])$ is not dense in $\mathcal{L}^2(L_1[0,1])$.*

Alaminos, Choi, Kim and Payá obtained a version of Theorem 17.(i) for bilinear forms valid in the real as well as in the complex case. They also showed a vector-valued version of the next result (see [13, Corollary 6]).

Theorem 43 *([13, Theorem 3 and Corollary 5]). If L is a Hausdorff and locally compact topological space, then the subset $NA\mathcal{L}^2(C_0(L))$ is dense in $\mathcal{L}^2(C_0(L))$. If L is scattered, then for every $N \geq 2$, then $NA\mathcal{L}^N(C_0(L))$ is dense in $\mathcal{L}^N(C_0(L))$.*

Whether the above result holds or not for N-linear forms for any compact space K, it is still an open question. In the proof of the previous result, the authors essentially use that the operators from $C(K)$ on its dual are weakly compact, a fact that does not hold for $N \geq 3$, unless K is scattered. Also some denseness results for these spaces and for special classes of multilinear mappings can be deduced from general ones (see [1, Corollary 5]).

Since $C_0(L)$ (for some locally compact space L) represents any commutative C^*-algebra, one reasonable try to extend the previous result may be to the non commutative C^*-algebras. Along this line, Alaminos, Payá and Villena showed a positive result by using a certain technical assumption on the C^*-algebra. As a consequence, they obtained the result that follows.

Theorem 44 *([14, Theorem 3]). For every n, let L_n be some scattered locally compact space and A_n a space of complex valued matrices of finite dimension, endowed with its usual (C*-algebra) norm. Then the (complex) Banach space $X = \oplus_{c_0} C_0(L_n, A_n)$ satisfies that $NA\mathcal{L}^N(X)$ is dense in $\mathcal{L}^N(X)$.*

Choi and Kim proved the following positive result for the vector valued case:

Theorem 45 *([28, Theorem 2.1]). Assume that X and Y are Banach spaces and Y satisfies the property (β). Then for every $N \geq 2$ the following results hold:*

i) If $NA\mathcal{L}^N(X)$ is dense in $\mathcal{L}^N(X)$, then the subset of norm attaining N-linear mappings from X to Y is dense in the space of all N-linear mappings.

ii) The parallel result is also true in the class of the symmetric N-linear mappings.

Bilinear forms on a product of two Banach spaces can be extended to the product of their biduals in a way that the extension preserves the original norm. Hence, it may happen that the original bilinear form does not attain its norm, but its extension does. Since in full generality, there is no a version of the Bishop-Phelps result for bilinear forms, the possible denseness of the subset of those bilinear forms whose extension attain their norms was considered. This can be seen as a bilinear version of Theorem 8 previously obtained by Lindenstrauss for operators.

In the case of bilinear mappings, there are (at least) two possible extensions to the product of the biduals which are due to Arens. One of them can be obtained by using three times the following formula that defines the first Arens transpose:
$A^t : Z^* \times X \longrightarrow Y^*$, $A^t(z^*, x)(y) := z^*(A(x, y))$, where $A \in \mathcal{L}^2(X \times Y, Z)$.
The third Arens transpose of A is an element in $\mathcal{L}^2(X^{**} \times Y^{**}, Z^{**})$, which is an extension of A and has the same norm that A. The other Arens transpose is defined in a similar way, but changing the roles of X and Y.

Theorem 46 *([1, Theorem 1]). The subset of bilinear forms on a product of Banach spaces such that (one of) their Arens transposes attain their norms is dense in the corresponding space of all bilinear forms.*

Here the perturbation used is a product of two functionals. This result was improved by Aron, García and Maestre.

Theorem 47 *([16, Theorem 2.2]). The subset of those bilinear forms on a product of Banach spaces such that both of its Arens extensions to the product of the biduals attain the norm (at the same element) is dense in the corresponding space of all bilinear forms.*

Also these authors gave an example showing that the subset of bilinear forms such that both Arens extensions to the bidual attain the norm is a proper subset of the set of bilinear forms that satisfy that just one Arens extension satisfies that condition (see [16, Example 2]). Under special restrictions on the Banach spaces X and Y, then it happens that the Arens extension of any bilinear form on $X \times Y$ to $X^{**} \times Y^{**}$ always attains its norm (see [1, Proposition 6]).

For N-linear mappings, in general, there are also several ($N!$) ways of extending them (preserving the norm) to the product of the biduals. In fact, for every permutation σ of $\{1, \ldots, N\}$, given elements $x_i^{**} \in X^{**}$ $(1 \leq i \leq N)$ and a net $\{x_{\alpha_i}\}$ in X w^*-convergent to x_i^{**} with $\|x_{\alpha_i}\| \leq \|x_i^{**}\|$, for every i, then

$$A_\sigma(x_1, \ldots, x_N) = \lim_{\alpha_{\sigma(1)}} \ldots \lim_{\alpha_{\sigma(N)}} A(x_{\alpha_1}, \ldots, x_{\alpha_N})$$

gives an extension of A to $X_1^{**} \times \ldots \times X_N^{**}$ that satisfies $\|A_\sigma\| = \|A\|$. These extensions are called Arens extensions of A.

Theorem 48 *([10, Theorem 2.1]). Let X_k be Banach spaces $(1 \leq k \leq N)$. Then the set of N-linear forms on $X_1 \times \ldots \times X_N$ such that all their Arens extensions to $X_1^{**} \times \ldots \times X_N^{**}$ attain their norms at the same N-tuple is dense in the space of all N-linear forms on $X_1 \times \ldots \times X_N$.*

Also the same result holds for some special classes of multilinear forms, like integrals, nuclear and p-summing (see [16, Corollaries 2.5, 2.6 and 2.7]). The vector valued version of the above result also holds [10, Theorem 2.3].

Payá and Saleh generalized Theorem 38 by proving the following:

Theorem 49 *([53, Theorem 5 and Example 7]) Assume that X_i $(1 \leq i \leq N)$ are Banach spaces such that all of them satisfy the property stated in Theorem 5, then the subset of norm attaining N-linear mappings on $X_1 \times X_2 \ldots \times X_N$ is dense in the corresponding space of N-linear forms.*

The same authors also prove a positive result by assuming that $N - 1$ of the spaces has a stronger property (α) that the one assumed in the above result. Very recently, Choi and Song proved the following generalization:

Theorem 50 *([29, Theorem 16]) Assume that X_i $(1 \leq i \leq N)$ are Banach spaces such that (at least) $N-1$ of them have the property quasi$-alpha$, then $NA\mathcal{L}^N(X_1 \times \ldots \times X_N)$ is dense in $\mathcal{L}^N(X_1 \times \ldots \times X_N)$.*

Also under some additional condition, Payá and Saleh obtained the following result:

Theorem 51 *([53, Theorem 8]) Let X be a Banach space satisfying property (α) with constant $\rho < \sqrt{2}-1$. Then the subset of symmetric norm attaining bilinear forms on X is dense in the space of the symmetric bilinear forms on X.*

If stronger restrictions are assumed on the constant ρ (depending on N), then some more positive results can be obtained by the same method on the denseness of the symmetric norm attaining N-linear forms (see [53, Remark 9]). By using Theorem 33, then every Banach space under the assumptions of that result, satisfies that, for each N, it has an equivalent norm (depending on N) such that for this new norm, it is satisfied the denseness of the symmetric norm attaining N-linear mappings [53, Remark 9].

Very recently, by proving that under certain assumptions, every multilinear form admits an integral representation, Grecu and Ryan deduced the following results:

Theorem 52 *([38, Corollaries 2 and 3]) Assume that X_i $(1 \le i \le N)$ are Banach spaces such that (at least) $N-1$ of the duals X_i^* $(1 \le i \le N)$ have the approximation property. Assume also that the injective tensor product $X_1 \otimes_\varepsilon \ldots, \otimes_\varepsilon X_N$ is separable and it does not contain a copy of ℓ_1 and . Then it holds that*

a) $NA\mathcal{L}^N(X_1^* \times \ldots \times X_N^*)$ is dense in $\mathcal{L}^N(X_1^* \times \ldots \times X_N^*)$.

b) In the case that all the spaces X_i coincide with X, then the same result holds for the class of the symmetric N-linear mappings on X^*.

For polynomials, we will list some results, that are the versions of some known results for N-linear mappings. However, this case is usually more involving and there are open problems whose answer is known for multilinear mappings.

We recall that P is an N-homogeneous polynomial from X to Y if there is a bounded and N-linear mapping T from X to Y such that

$$P(x) = T(x, \overset{N)}{\ldots}, x), \quad \forall x \in X.$$

We will denote by $P^N(X,Y)$ the space of such polynomials on X, endowed with its usual sup norm and $NA\mathcal{P}^N(X,Y)$ will be the subset of the norm attaining N-homogeneous polynomials. If Y is the scalar field, we will just write $\mathcal{P}^N(X)$ and $NA\mathcal{P}^N(X)$ to denote both sets.

To begin with, let us note that there is also no a version of the Bishop-Phelps result for polynomials. The first counterexample is essentially contained in [6, Corollary 2], where it is stated for quadratic forms.

Proposition 1 . *If $w \in \ell_2 \backslash \ell_1$ is a decreasing sequence of positive reals, there is a continuous 2-homogeneous polynomial P on the canonical predual of the Lorentz space $d(w,1)$ such P cannot be approximated by norm attaining 2-homogeneous polynomials. In fact, the space $d_*(w,1)$ is a sequence space (with a shrinking basis $\{e_n\}$) and the following polynomial satisfies the stated condition:*

$$P(x) = \sum_{n=1}^{\infty} w_n x(n)^2 \quad (x \in d_*(w,1)) \, .$$

The following positive answers are due to Choi and Kim:

Theorem 53 *([28, Theorem 2.2]). If a Banach space X has a monotone shrinking basis and the Dunford-Pettis property, then the subset of norm attaining N-homogeneous polynomials on X is dense in $\mathcal{P}^N(X)$.*

As it happens for operators and for multilinear mappings, the Radon-Nikodým property is also a sufficient condition for the denseness of the norm attaining polynomials. Here the key idea is the use of Theorem 11:

Theorem 54 *([28, Theorem 2.7]) If X has the Radon-Nikodým property, then $NA\mathcal{P}^N(X,Y)$ is dense in $\mathcal{P}^N(X,Y)$.*

By using that for a symmetric N-linear mapping L on a Hilbert space, the norm of L coincides with the norm of the associated N-homogeneous polynomial, then Choi and Kim obtain one of the few results that is known for denseness of symmetric norm attaining N-linear mappings:

Corollary 3 *([28, Corollary 2.8]) Let H be a real Hilbert space. Then the subset of symmetric norm attaining N-linear forms from H to a Banach space Y is norm dense in the space of all symmetric N-linear mappings from H to Y.*

By using elementary ideas that allow to obtain a general result valid for some spaces whose dual has the 1-approximation property (see [1, Proposition 4]), it can be deduced the following positive result that can be applied to certain classical Banach spaces. For some of the following spaces some other results also can be applied:

Theorem 55 *([1, Proposition 4 and Corollary 5]). For $N \geq 2$, if X is one of the following Banach spaces:*

a) A space with a shrinking and monotone finite-dimensional decomposition.

b) $C_0(L)$ (for some Hausdorff and locally compact topological space L).

c) $L_p(\mu)$, where μ is a finite measure and $1 \leq p \leq \infty$.

Then the subset of norm attaining N-homogeneous polynomials on X which are weakly continuous on bounded sets is dense in the space of all N-homogeneous polynomials weakly continuous on bounded sets.

In the case of the complex space $C(K)$, then Choi, García, Kim and Maestre proved that if K has dimension less or equal to one, then every norm attaining N-homogeneous polynomial on $C(K)$ attains its norm at some extreme point of the unit ball of $C(K)$ [26, Corollary 2.10]. A version of this result for the space $C(K,X)$ can be also found in [4, Corollary 2.1].

The corresponding polynomial version of Theorem 45 for polynomials was proved also by Choi and Kim [28, Theorem 2.1].

Also Jiménez Sevilla and Payá obtained the corresponding version of Theorem 40 for polynomials.

Theorem 56 *([43, Theorem 3.2]). For a sequence of weights $w \in c_0 \backslash \ell_1$ and $N \geq 2$, the following conditions are equivalent:*

i) The subset $NA\mathcal{P}^N(d_*(w,1))$ is dense in $\mathcal{P}^N(d_*(w,1))$.

ii) $w \notin \ell_N$.

iii) *Every element in* $\mathcal{P}^N(d_*(w,1))$ *is weakly sequentially continuous.*

Usually, there is no known description of the subset of norm attaining N-homogeneous polynomials. We will pose just two cases for which the description is known. The next one is also valid for the complex space c_0.

Proposition 2 *([27, Theorem 2]). The subset of norm attaining* 2*-homogeneous polynomials on the complex space* $d_*(w,1)$ *are just those that depend on a finite number of coordinates. That is, if* Q *is a norm-attaining* 2*-homogeneous polynomial on* $d_*(w,1)$*, then there is a natural number* n *such that* $Q = Q \circ P_n$*, where* $\{P_n\}$ *is the sequence of canonical projections associated to the usual Schauder basis* $\{e_n\}$ *of* $d_*(w,1)$*.*

This result is not valid even for 3-homogeneous polynomials (see [27, Remark 4]). Kaminska and Lee obtained the following generalization:

Theorem 57 *([46, Corollary 3.5]). Let* X *be a rearrangement invariant complex Banach sequence space. Suppose that for each* x*, there is* $n \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that* $X^{**} = [e_1, \ldots, e_n] \oplus M$ *and it is satisfied that* $x + \varepsilon B_M \subset B_{X^{**}}$*, then the* 2*-homogeneous norm attaining polynomials on* X *are those* 2*-homogeneous norm attaining polynomials that depend on a finite number of coordinates.*

Grecu and Ryan proved the corresponding version of Theorem 52 for polynomials (see [38, Corollary 5]) and so they obtained a result on denseness of the set of norm attaining N-homogeneous polynomials in the space of all N-homogeneous polynomials on a dual space under certain restrictions. Until now, as far as the author knows, there is no counterexample to the denseness of the norm attaining polynomials defined on a dual space.

Recently it was considered the problem of the denseness of the subset of norm attaining holomorphic mappings. More concretely, for a complex Banach space X, let $\mathcal{A}_u(B_X)$ denote the set of bounded and uniformly continuous functions defined on B_X whose restriction to the open unit ball are holomorphic functions. This space is a Banach space under the sup norm.

The following results are due to Acosta, Alaminos, García and Maestre. Next theorem is an extension of Theorem 54.

Theorem 58 *([8, Theorem 3.1]). Assume that a complex Banach space* X *has the Radon-Nikodým property. Then for every function* $H \in \mathcal{A}_u(B_X)$*, every natural number* N *and every* $\varepsilon > 0$*, there is an* N*-homogeneous polynomial* Q *on* X*, satisfying that* $\|Q\| < \varepsilon$ *and* $H + Q$ *attains its norm. In fact,* Q *can be chosen to be a product of* N *functionals. As a consequence, the set of norm attaining* N*-homogeneous polynomials is dense in the space of all* N*-homogeneous polynomials.*

The next positive result is a version of the ideas appearing in [44, Section 3] valid for compact operators.

Theorem 59 *([8, Corollary 2.4]). Let* X *be a complex Banach space satisfying the following condition: For every finite dimensional space* F*, every (bounded and linear) operator* $T : X \longrightarrow F$ *and* $\varepsilon > 0$ *there is a norm-one projection* $P : X \longrightarrow X$ *with finite dimensional range such that* $\|T - TP\| \leq \varepsilon$*. Then, the subset of norm attaining holomorphic functions in* $\mathcal{A}_u(B_X)$ *that are weakly continuous on the unit ball is dense in the space of all functions in* $\mathcal{A}_u(B_X)$ *that are weakly continuous on the unit ball.*

As a consequence, the parallel version of Theorem 55 also holds for the class of those holomorphic mappings that are weakly continuous on the unit ball. It was also obtained a positive result valid for some $C(K)$ spaces.

Corollary 4 *([8, Corollary 3.5]). If K is a scattered compact set, then the set of elements in $\mathcal{A}_u(C(K))$ which attain their norm on extreme points of $B_{C(K)}$ is dense in $\mathcal{A}_u(C(K))$.*

There is a negative result of denseness of the norm attaining holomorphic mappings valid in the space $d_*(w,1)$ under some weak additional assumption (see [8, Theorem 2.2]). As a consequence of that result, it was obtained the following strengthening of Proposition 1, which is valid for the complex case.

Theorem 60 *Let $X = d_*(w,1)$ (complex) for $w \in \ell_2 \backslash \ell_1$, then there is a 2-homogeneous polynomial on X that cannot be approximated by norm attaining (non-necessarily homogeneous) polynomials on X.*

5. Open problems

Different techniques were used to obtain the results on this topic, and several interesting results were obtained. Even so, up to now, there are many aspects that are far from being clear. Here are some of the problems that still remained opened:

- Which classical (complex) Banach spaces satisfy the complex version of the Bishop-Phelps result (denseness of modulus support functionals for any bounded, closed and convex non-empty set)?
- Characterize property A (even for special classes of Banach spaces).
- Characterize property B for finite-dimensional normed spaces.
- Can a compact operator be approximated by norm attaining compact operators?
- Is the subset of the norm attaining 3-linear forms dense in the space of all the 3-linear forms on $C(K)$?
- Is it true that the subset of symmetric N-linear forms on a Banach space whose Arens extensions to the bidual attains the norm is dense in the corresponding space of symmetric N-linear forms?
- Is the subset of norm attaining 2-homogeneous polynomial on $L_1[0,1]$ dense in $\mathcal{P}^2(L_1[0,1])$?
- Is there a version of Theorem 48 for N-homogeneous polynomials (for $N \geq 3$)?

Acknowledgement. I would like to thank Jerónimo Alaminos for his comments. Thanks also to the Department of Mathematical Sciences of the University of Memphis for the hospitality and the friendly atmosphere.

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