

## The Riemann Curvature Through History\*

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**Abstract.** The concept of curvature is very common in Differential Geometry. In this article we try to show its evolution along history, as well as some of its applications. This survey is limited both in number of topics dealt with and the extent with which they are treated. Some of them, like minimal submanifolds, Kähler manifolds or Morse Theory are completely omitted. Though in an implicit way, the curvature is already present in the Fifth Euclid's Postulate. However it does not emerge explicitly in Mathematics until the appearance of the theory of curves and surfaces in the euclidean space. Taking basically the work of Gauss's as a starting point, Riemann defines the curvature tensor in an abstract and rigorous way. The introduction of multilinear algebra in the second half of the XIX century allowed a better analytic formulation and its further development. It is worth stressing its fundamental role in the development of the Theory of Relativity. Besides, the curvature is present, not only in riemannian manifolds, but also in many other geometric structures, like homogeneous and symmetric spaces, the theory of connections, characteristic classes, etc. Having in mind that the physical world cannot be explained in a linear way, the curvature also arises in the theories of Mathematical Physics. Likewise, it seems interesting to note its presence in applied sciences, like Estereology.

### La curvatura de Riemann a través de la historia

**Resumen.** El concepto de curvatura es muy familiar en la Geometría Diferencial. En este artículo se procura mostrar tanto la evolución de su concepto a lo largo de la historia como alguna de sus aplicaciones. En esta exposición existe una limitación tanto en la presentación de algunos tópicos como en la ausencia de otros que son básicos en la Geometría de Riemann. Entre estos últimos cabría destacar, entre otros, las variedades minimales y las kählerianas o la teoría de Morse. Aunque de manera implícita, la curvatura ya está presente en el quinto postulado de Euclides, no se detecta explícitamente su presencia en las Matemáticas hasta la aparición de la teoría de curvas y superficies en el espacio euclídeo. Tomando fundamentalmente como base la obra geométrica de Gauss, Riemann define de una manera abstracta, pero rigurosa, el tensor curvatura. El desarrollo del álgebra multilineal durante la segunda mitad del siglo XIX permitió su comprensión analítica y su desarrollo posterior. Cabe destacar su papel fundamental en el desarrollo de la teoría de la relatividad. Además, la curvatura no sólo está presente en las variedades de Riemann sino también en muchas otras estructuras geométricas tales como espacios simétricos y homogéneos, teoría de conexiones, clases características, etc. Teniendo en cuenta que el mundo físico no se puede explicar de una forma lineal, la curvatura aparece también en todas las teorías Físico-Matemáticas. Igualmente, parece interesante hacer notar su presencia en ciencias aplicadas, entre las que cabe señalar la Estereología.

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## 1 Introduction

The world we live in, and the mathematical models describing the geometrical and physical objects, cannot be properly explained with only linear constructions. In order to obtain an adequate description of Nature, it is necessary to introduce models in which the relations between parameters go beyond the linear ones. That is why the concept of curvature appears in a natural way.

According to Osserman, [36], the notion of curvature is one of the main concepts of differential geometry; it could be argued that it is indeed the central one, by distinguishing the geometrical core of the subject from those aspects that are analytic, algebraic or topological. According to Berger, [3], the curvature is the most important invariant of Riemann's Geometry, and the most natural one. In [24], Gromov writes: "the curvature tensor of a Riemann manifold is a little monster of multilinear algebra whose complete geometrical meaning remains obscure".

Thus, for Riemannian manifolds without additional structures, the curvature is a complicated magnitude. Its properties in the simplest manifolds were the first to be studied. Later, the situation in a more general manifold could be compared to that in the simplest ones. The latter are often called "model spaces".

The curvature also plays a fundamental role in Physics and other experimental sciences. For example, the force required to move an object at a constant speed is, according to Newton's laws, a constant multiple of the curvature of its trajectory; and the movement of a body in a gravitational field is determined, according to Einstein, by the curvature of the space-time.

## 2 A remark on Euclid's Elements

Plato, a disciple of Socrates, established his school, "The Academy", in a sacred area of Athens. It was like a small university where the philosopher and his friends taught their students. Plato was very fond of Mathematics, especially Geometry. The legend says that the inscription engraved at the doorstep of the Academy read "Let no one ignorant of geometry enter".

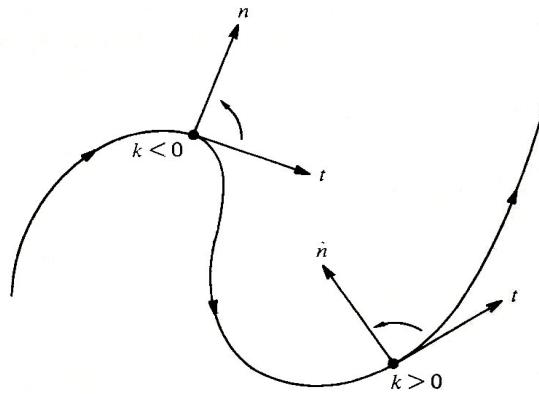
When Euclid (or maybe some members of his school) wrote the "Elements", the first five postulates seemed so evident that they should be accepted without proof. The curvature is already present therein, though not explicitly. "Elements" rivals, in its diffusion, with the most famous books of universal literature: The Bible, The Divine Comedy, Faustus or The Quixote. This is very remarkable, since it is a scientific work, not accessible to the general public. But its logical rigor -which in part contributed to lay the foundations of the modern mathematical thought- and the unity of its contents make it a unique knowledge corpus, which should be a compulsory reading for every geometry student.

However, Euclid's work is not easy to understand. For example, in the XVIII century, the members of the Academy of Science of Paris MM. Delambre and Prony wrote: "Nobody would agree if we proposed to begin the study of mathematics with the Elements, but it is right to assert that any geometer should read them at least once in his life".

Regarding its importance, Professor Dou says: "The geometry in the Elements is such that it could be considered nowadays as physical geometry, since for Euclid and Aristotle, the terms of their propositions are exactly meant for the natural fields of the reality in the physical world, with a unique frame which is, at the same time, immediate and utmost. It is a geometry that tries to study the structure of the physical space".

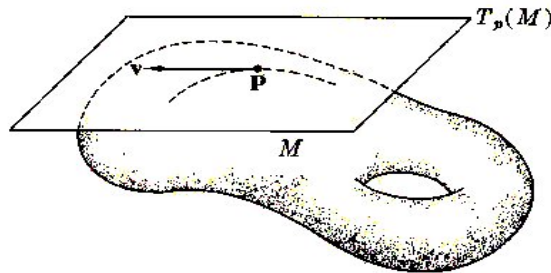
## 3 The birth of the Differential Geometry of curves and surfaces

The study of curves begins with Infinitesimal Analysis. Newton already studied the curvature of plane curves. For a curve, the curvature at a point measures the deviation from its tangent.



The theory of surfaces in the euclidean space was developed mainly in the XVIII and XIX centuries, and the first half of the XX century.

In the early XIX century, Young and Laplace proved that, for a spherical surface, the inner pressure is always higher than the outer one, and that the difference increases when the radius decreases. The laws of Physics dictate that liquids tend to minimize their surfaces. In the interior of a drop or a bubble in equilibrium, the inner pressure is bigger than the outer one. This difference of pressure is due to the curvature of the boundary surface. Intuitively, it can be concluded that the curvature of a surface at a point measures its deviation from its tangent plane.



The contributions of Euler, Monge and Dupin were of great importance. Yet, it is Gauss's famous paper "Disquisitiones generales circa superficies curvas", which became essential to establish the concept of space. In this article, the concept of "curvature" of a surface at a point was introduced, which happens to be intrinsic. To do this, Gauss studies the intrinsic properties of the geometry of a surface, by using the first fundamental form as a starting point.

"Gauss's Theorema Egregium" can be stated: "At a point of the surface, the curvature is an isometric invariant".

The tag of "Theorema Egregium" was ascribed by Gauss himself, due to its exceptional geometrical meaning. This is the prototype of a universal theorem. As Chern asserted: "Mathematics are there, you just have to discover them and bring them to the light".

According to Berger, there are no simple geometrical proofs of the "Theorema Egregium". Many have been produced along history, among which we can point out those by Hilbert and Cohn-Vossen, [26], Chern, [15], Do Carmo, [16], O'Neill, [34], Stoker, [42], Sternberg, [41], Klingenberg, [27], and Boothby, [10], among others.

Apart from the aforementioned proofs, I find very interesting (because of its pedagogical nature) the one given by Thorpe, [43]. It makes use of the theory of unit principal bundles on a surface without explicitly

mentioning them . In [3], two proofs are presented. That of Bertrand, Diguët and Puiseux, [7], gives the key for the Riemann’s generalization of the curvature tensor to an arbitrary dimension. Furthermore, the paper by Bertrand, Diguët and Puiseux was a starting point for the development of the theory of volumes of tubes in Riemann manifolds, extended mainly by A. Gray, [21].

As an application of Gauss’s “Theorema Egregium”, we find one of the most profound and difficult formulas of Differential Geometry and Algebraic Topology: “The Gauss-Bonnet theorem for surfaces”. Gauss proves it for geodesic polygons and Bonnet extended it for polygons with edges of non vanishing geodesic curvature. There is no a simple proof of it. Due to its methodological nature, I would choose again the one given by Thorpe, [43]. This theorem was generalized to an arbitrary dimension nearly a century later, by Allendoerfer and Weyl, [1].

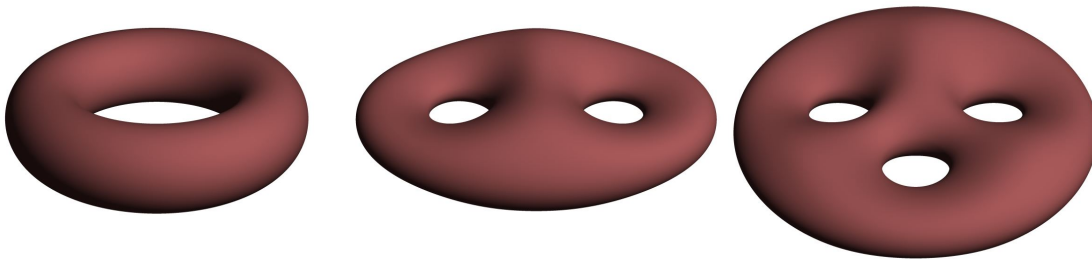
In the XVIII century, Euler proved his famous formula for polyhedra:

$$\text{Faces} - \text{Edges} + \text{Vertices} = 2$$

Despite its simplicity, it seems that this property was not known by Archimede nor Descartes. Perhaps the reason is that mathematicians prior to Euler were not able to think about geometrical properties that were not measurable. The way initiated by Euler was followed by Lhuillier, who remarked that Euler’s formula was false for bodies with  $g$  handles. He proved that

$$\text{Faces} - \text{Edges} + \text{Vertices} = 2 - 2g$$

This is the first known example of a topological invariant.



For a plane domain, in which the boundary consists of straight lines, the most classical and simple result says that “the sum of the inner angles of a triangle is  $\pi$ ”. In general, if  $D$  is a domain in a surface, with a piecewise differentiable boundary  $\partial D$ , for any simplicial subdivision, its Euler-Poincaré characteristic  $\chi(D) = \text{Faces} - \text{Edges} + \text{Vertices}$  is given by the Gauss-Bonnet formula:

$$\sum_i (\pi - \alpha_i) + \int_{\partial D} (ds/\rho_\sigma) + \iint_D K dA = 2\pi\chi(D),$$

where the first summand is the sum of the outer angles at the corners, the second one is the integral of the geodesic curvature, and the last one is the integral of the Gauss curvature. They are, respectively, the point, linear and surface curvature of the domain. The Gauss-Bonnet formula can be interpreted by saying that the Euler-Poincaré characteristic is a total curvature.

For closed surfaces we have:

$$\int_M K dA = 2\pi\chi(M),$$

which can be considered as a characteristic class.

## 4 The birth of Riemann Geometry

In 1854, Riemann generalizes Gauss’s studies to spaces of arbitrary dimension. He defined, in a not very rigorous way, the concept of a differentiable manifold as an  $n$ -dimensional set on which the usual computa-

tions of analysis can be performed. Thus, a geometry on a manifold would be a positive-definite quadratic form on each of its tangent spaces. This definition of Riemann allows to generalize much of Gauss's work. Gauss himself had suggested this topic for Riemann's habilitation thesis. Riemann's famous essay "On the hypotheses which lie at the bases of Geometry" was published after his death. It is obvious that Riemann's spaces of variable curvature include, as particular cases, the space forms, which historically gave rise to the non-euclidean geometries, that are as consistent as the euclidean one. I will come back to this topic later.

The birth of the curvature tensor as we know it today can be ascribed to the publication of Riemann's essay. Its properties are quite complex; however, its basic ideas are deep and simple, as happens with all the great concepts in Science.

In the second half of the XIX century the theory of tensor algebra was developed, mainly by the Italian school. This tool, though cumbersome, allowed a great advance of the Riemann geometry. It allowed especially the formulation of the Levi-Civita parallel displacement and Einstein's Theory of Relativity. Since the publication of the latter, Riemann spaces have caught the attention of many philosophers, physicists and mathematicians.

The concept of Riemann curvature of a surface arises in a natural way in the Riemann manifolds of dimension greater than two, since it is possible to consider the germ of a totally geodesic surface tangent to a two-dimensional subspace of the tangent space at a point. The Gauss curvature of such a surface is defined as the "sectional curvature" of that plane at that point. In general, the curvature tensor of a riemannian manifold depends on four arguments, while the sectional curvature depends only on two. It is well known that the knowledge of the sectional curvature of a riemannian manifold at a point determines that of the curvature tensor, [2, 21, 28].

Besse, [8], says that "the Ricci curvature is pretty difficult to perceive". Historically, Ricci introduced the curvature bearing his name due to the following reason: If  $M$  is a hypersurface embedded in the euclidean space, it has an inherent second fundamental form, which is a quadratic differential form. Its eigenvalues are the principal curvatures, and its eigenvectors define the curvature lines. In a riemannian manifold such form does not exist, and there are no special vectors or directions. However, by means of a tensor contraction of the curvature and metric tensors, Ricci defined a covariant symmetric tensor of order two. It is then possible to compute the eigenvalues and eigenvectors. the importance of Ricci curvature in riemannian geometry is extraordinary.

A further contraction of Ricci and metric tensors yields the "scalar curvature". Its geometric interpretation is very interesting, since it is a multiple of the coefficient of the quadratic term of the asymptotic expansion of the volume of a geodesic ball, [8, p. 15] and [23].

## **5 The birth of non-euclidean geometries. Their models**

At the end of the XVIII century, the structure of geometry was not clear. As Berger says, perhaps it was not clear for Euclid himself. At that time, it was believed that the fifth postulate could be a consequence of the first four, possibly with some additional condition.

Because of its importance, both for itself and for its influence in the historical development of the curvature, it is worth reminding the fifth postulate of the "Elements", which can be stated as:

*Through a point not in a line a unique parallel line can be drawn.*

Humans have always used the intuitive fact that, in the ordinary space, the shortest path between two points is the straight line. It has also been known for centuries that the shortest path between two points on a sphere is an arc of meridian, which can be called "line". Given one of these "lines" on the sphere and a point not on it, there is no other "line" through that point that does not cut the former. So, in this case, the fifth postulate does not apply.

The study of Trigonometry goes back to the beginning of Christian era, perhaps even some centuries earlier. This was motivated by the study of Astronomy. Intuitively, it is easy to guess that, on a sphere, the

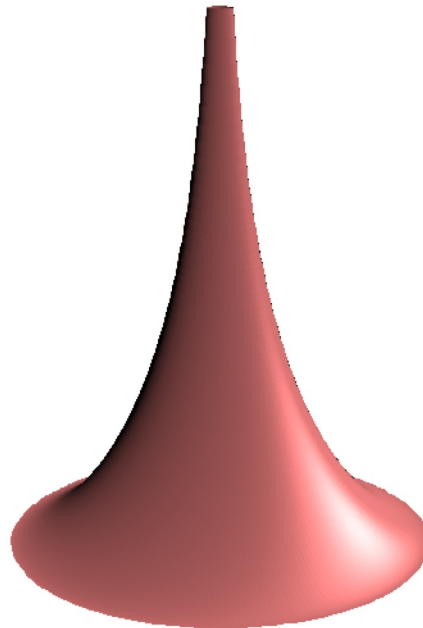
sum of the angles of a triangle is greater than two right angles, and that the excess seems to be due to its “curvature”.

Riemann showed how, for a given sphere, it is possible to assign to it a quadratic form with functions of the coordinates as coefficients, and with positive curvature. He explicitly established that the geometry everyone was looking for, “the hyperbolic geometry”, was defined by the same quadratic form with negative curvature. The interpretation could hardly be simpler. These geometries appear in almost any field of mathematics: algebraic geometry, theory of numbers, differential geometry, complex analysis, dynamical systems, mathematical physics, etc.

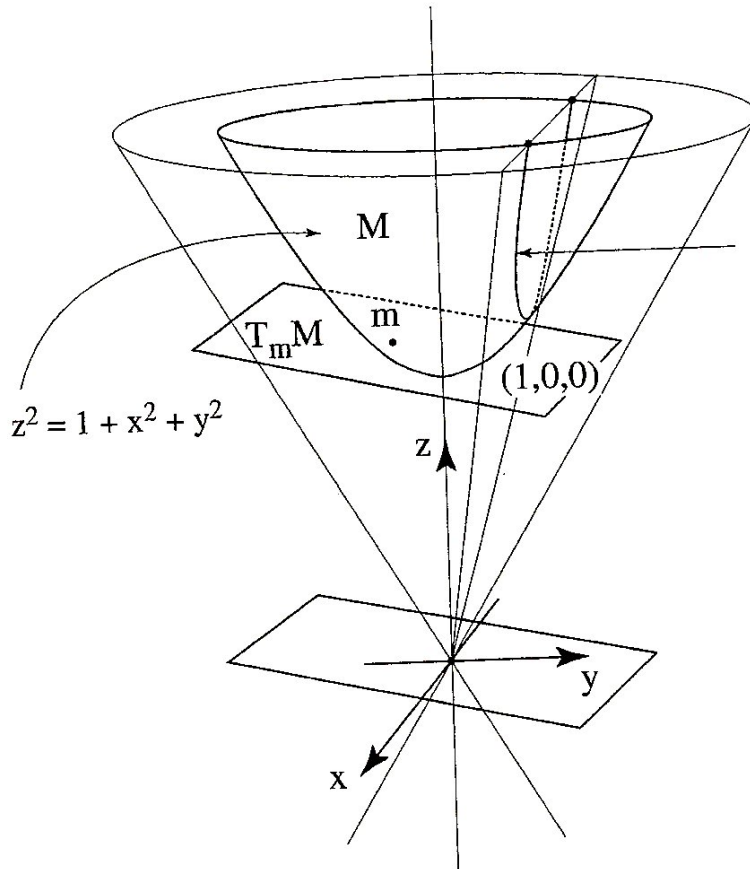
At the beginning of XIX century, the works of Lobatchevski and Bolyai were published. Both authors, independently, discovered the “hyperbolic geometry”. Gauss himself was convinced that there could exist other geometries satisfying the first four postulates, but not the fifth. He did not publish these ideas, and had a pretty convincing reason for this: He wanted to keep his image as a perfectionist in front of his friends, therefore it did not seem logic to invent “new geometries”.

The constructions of new geometries given by Lobatchevski and Bolyai were rather rudimentary, since, as it can be seen in Hilbert’s works, non-euclidean geometries require abstraction and there are no models for them in the euclidean one. In fact, the isometries of a geometric object are not necessarily included among those of the euclidean space. The consistence of the Hyperbolic Geometry is fully justified in the literature. See, for instance, the works by Klein, Poincaré and the lecture by José María Montesinos in this Real Academia, [33].

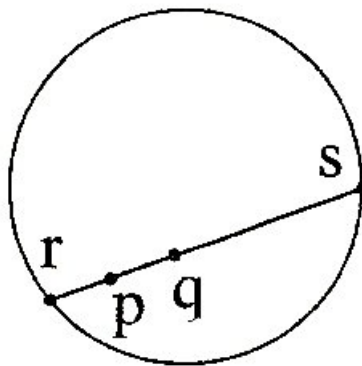
It is possible to define the “Hyperbolic Geometry” as the one satisfying all the trigonometric formulas of a spherical geometry in which the radius would be pure imaginary. The hyperbolic plane cannot be globally isometrically embedded in the ordinary space. This would only be possible if the problem is considered locally. Indeed, Minding discovered the “pseudosphere” (revolution surface of the tractrix) which, locally, has the properties of the hyperbolic plane. This surface has been extensively studied by Beltrami.



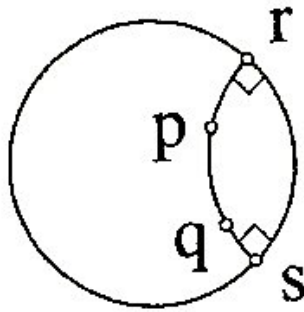
It is well known that Minkowski space can be realized as  $\mathbb{R}^3$  with the metric  $\langle v, v \rangle = x^2 + y^2 - z^2$ . Thus, the imaginary sphere  $\mathbb{S}^2(i)$  is exactly the hyperboloid  $x^2 + y^2 - z^2 = -1$ .



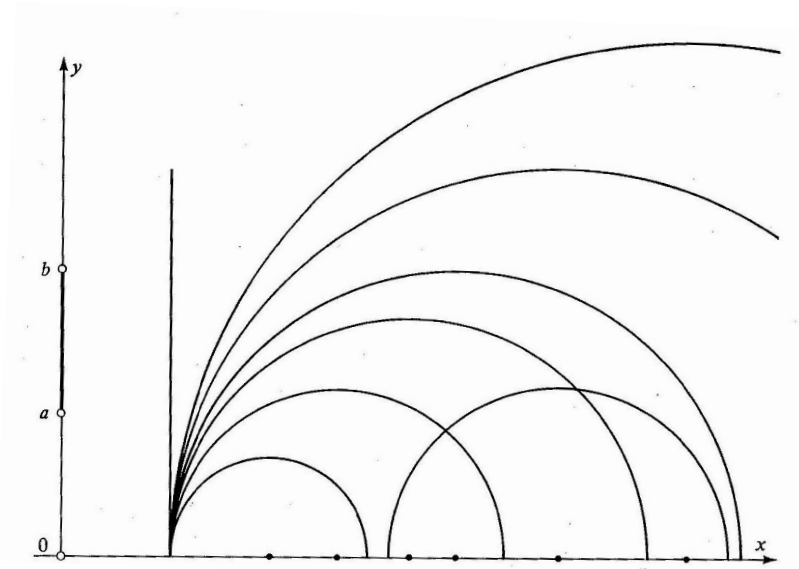
The Klein model is the metric representation in which the manifold is the unit open disk  $x^2 + y^2 < 1$  and the model geodesics are straight lines.



The third model is that of Poincaré, defined in the unit open disk. The geodesics are circles orthogonal to the boundary.



The fourth one is built on the Poincaré half-plane, and its geometric representation is:



It is an interesting exercise to prove that these four geometries are isometric to one another. Geometrically, it is possible to realize the “space forms” as the manifolds with constant Riemann curvature. They are, locally,

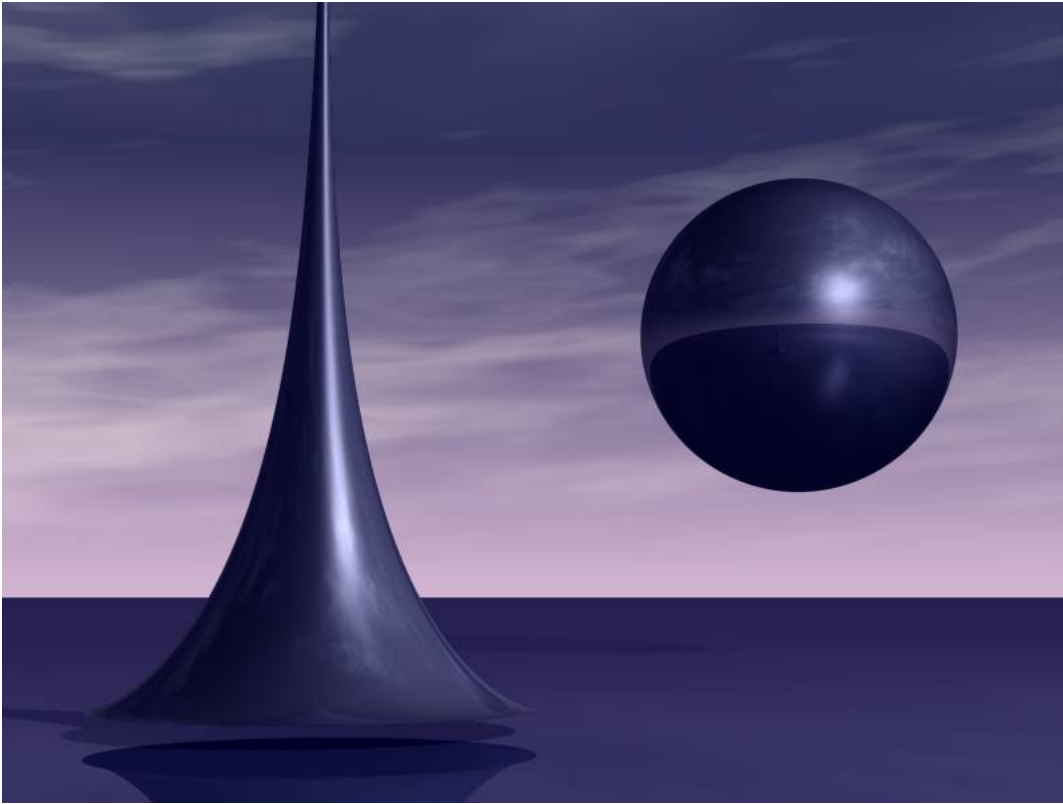
- The euclidean space  $\mathbb{R}^n$ ;
- The sphere  $S^n$ ;
- The hyperbolic space  $\mathbb{H}^n$ .

## 6 Two major historical research lines

As a consequence of the impact of both Riemann’s essay and Klein’s Erlangen program, two major research lines in Geometry arose:

- (a) The study of geometric and topological properties of riemannian manifolds of low dimension.
- (b) The study of riemannian manifolds with the greatest possible number of isometries.





With regard to the first of these lines, Poincaré's conjecture was formulated more than ninety years ago. This problem happened to be of an extreme difficulty in dimension three. During a good part of the XX century, several attempts of solving it were made. Now, Perelman seems to have found a satisfactory solution, [37] and [38].

The first interesting family of 3-manifolds that were classified was that of the "flat" Riemannian manifolds, which are locally isometric to the euclidean space. In order to accomplish this, the following result of Bieberbach was essential: "A flat compact Riemann manifold is characterized, up to an affine diffeomorphism, by its fundamental group".

My teacher, Professor Vidal Abascal, who contributed in such an outstanding way to the development of Mathematics in Spain, told me that, in the decade of the nineteen forties, he began to get interested in the theory of curves and surfaces when he read Bieberbach's book, [9]. When studying some of its theorems, Vidal thought over its generalization to a greater dimension. Thus, he was able to publish some interesting papers such as the one devoted to Steiner's formula for spaces of constant curvature, [44].

## 7 The curvature of homogeneous spaces

The theory of Lie groups has been extensively studied, and has multiple applications in other branches of Mathematics, in particular in Theoretical Physics. The structure of Lie groups is particularly rich. Their curvature is defined in a canonical way. A classical result asserts that every subgroup of a Lie group  $G$  defines a foliation on  $G$ . If it is closed, then the set of equivalence classes admits a structure of a differentiable manifold. This is an elegant and safe method of finding non trivial examples of a class of differentiable manifolds, known as homogeneous spaces. Among them, the symmetric spaces are those with a better

resemblance with the euclidean space, especially those of rank one or two-point homogeneous spaces, [25, p. 164]; that is, the sphere, the hyperbolic space, and the real, complex and quaternionic projective spaces. Symmetric spaces were classified by Cartan in [12, 13, 14] and their geometrical properties have been widely studied; see, for example, the encyclopedic book by Helgason, [25]. The curvature of symmetric spaces is pretty simple, and they have the fundamental property that their curvature tensor is parallel.

During the second half of the XX century, many mathematicians have devoted their efforts to the study and classification of the non-symmetric homogeneous spaces. As a consequence, several families of spaces have arisen, such as the naturally reductive homogeneous spaces and the s-symmetric spaces. The latter have been studied by Kowalski and Vanhecke, [30], Berger, [4], and Wolf and Gray, [46]. It is important to note that these two families of homogeneous spaces are not disjoint. For example, the naturally reductive homogeneous space  $M = \frac{U(3)}{U(1) \times U(1) \times U(1)}$  is also 3-symmetric. Some naturally reductive homogeneous spaces of the flag type have been used by Penrose to study the correspondences bearing his name, [45].

## 8 Some remarks on Integral Geometry

In my opinion the “Integral Geometry” emerges with “Buffon’s needle problem” at the end of the XVIII century. Its development is due to Crofton, first, and later to Blaschke and his school. Though at first they did not appear explicitly, the homogeneous spaces are the fundamental pillar of this theory. This was observed by the renowned Professor Santaló, [69].

Santaló, “Académico Correspondiente” of this Real Academia and, at my proposal, Honorary Member of the RSME, has left us many publications on different fields of mathematical research and teaching. Santaló went to Hamburg in 1931 to work with Blaschke. There, he met Chern, who taught him the basics of the method of Cartan’s moving frame. As a curiosity, Santaló told me that Blaschke encouraged him to attend a seminar given by a young mathematician, that he considered interesting. The young mathematician was Kähler. For many years, Chern and Santaló produced a series of generalizations of theorems of Integral Geometry on kinematic density, many of which are now at the core of Classical Integral Geometry.

The curvature is present through all the geometric work of Santaló. Unfortunately, many of his publications are scattered in libraries, with poor diffusion, or are out of print. I think that it would be very interesting to compile all his scientific work, on the occasion of the 100th anniversary of his birth, to make it accessible to the scientific world. Indeed, this work is not only important for itself, but also because of its applications to Estereology.

The contributions of Professors Cruz-Orive and Gual-Arnau deserve an especial mention. They have published several papers on Tomography, using techniques of Integral Geometry, with interesting applications to Biomedicine. I consider this a very interesting speciality of Applied Mathematics.

## 9 Why are connections important?

The parallel transport of a vector in the euclidean space is known since Euclid’s times. The definition of a similar concept along a curve in a riemannian manifold is provided by Levi-Civita, [32]. The geometric idea is very simple. Given a curve in a surface and the tangent developable of its tangent planes, we get a developable surface to which we can apply the euclidean geometry. Going back to the tangent spaces, we get a definition of the concept of parallel displacement in a natural way. Obviously, it depends on the curve. This concept was generalized by Cartan, using the method of the moving frame. The curvature arises in a natural way in this theory and, in fact, Cartan uses this word in the title of his paper.

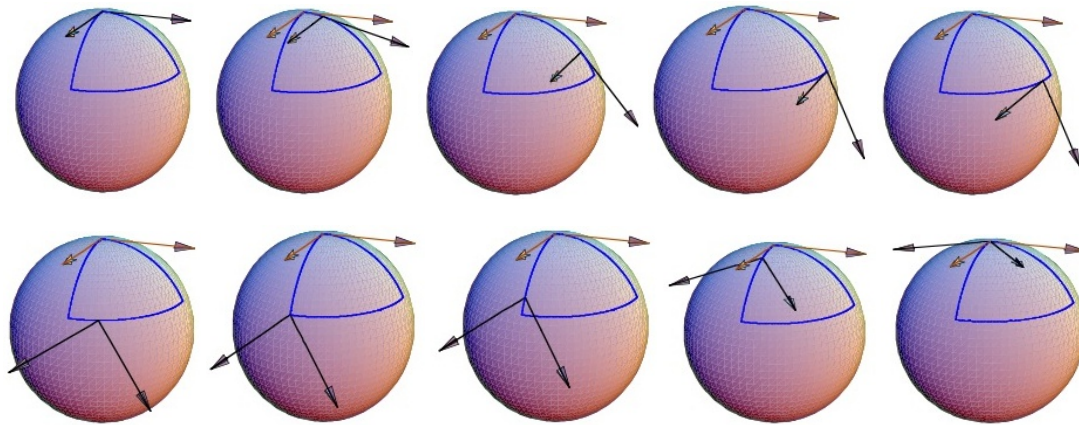
Levi-Civita and his teacher, Ricci-Curbastro, [39], built the theory of absolute differential calculus, which happened to be essential for the theory of relativity. Levi-Civita himself helped Einstein to understand the tensor calculus.

In the 1930’s, several generalizations of the concept of product space appeared. The principal bundles and the vector bundles are perhaps the most natural of them. In the 1950’s, Koszul, [29], and Ehres-

mann, [17], defined the same concept of “connection” for vector bundles in an axiomatic way, and using forms for principal bundles. The importance of this concept lies in the fact that it is a natural generalization of the parallel displacement. In the works of Spivak, [40], and Kobayashi-Nomizu, [28], among others, the theory of connections is extensively developed. This theory has been very useful, not only in Differential Geometry, but also in Theoretical Physics.

Given the importance of the structure of group in Mathematics, it seems natural to find out if there is some groups that carry information about the geometry of a riemannian manifold. The notion of holonomy of a connection can be formulated using the notion of parallel transport and the theory of principal bundles. The holonomy group is defined by means of the parallel transport along loops. Cartan expected that the holonomy would be able of faithfully describing the geometric structure of manifolds, thus allowing their classification. In fact, the holonomy group was basic for Cartan’s classification of symmetric spaces, [8]. This study fell into oblivion until Borel and Lichnerowicz got interested in the topic, [11]. Since then, its study has been very popular.

The phylosophy is as follows: a manifold possesses a structure that is invariant under parallel displacement if and only if that structure is invariant, at any point, under the holonomy group. If we look at the curvature, from a geometrical viewpoint, it is described by the parallel transport along an infinitesimal parallelogram, [47].



It is a surprising fact that, on classifying the holonomy, few irreducible groups appear. Recently, these groups became important because of their applications in Mathematical Physics, [19, 20].

## 10 The isoperimetric problem

The Isoperimetric Inequality in the plane is, perhaps, the oldest global theorem of Differential Geometry. The problem can be stated as follows: *Among all the closed simple curves in the plane having the same given length, which one is that enclosing a domain of greatest area?*

The Greek already knew the solution: The circle. Weierstrass, as a corollary to his theory of the calculus of variations, gave a complete proof of this solution. Later, many other simpler proofs appeared. Do Carmo, [16], reproduced that of Schmidt. The classical isoperimetric inequality is as follows:

$$L^2 \geq 4 \pi A,$$

and the equality holds if, and only if, the curve is a circle.

Apparently, the curvature does not intervene. The explanation is that the ambient manifold is the plane. For a domain on a sphere of curvature  $K$ , the inequality becomes:

$$L^2 \geq 4 \pi A - K A^2, \tag{1}$$

and the equality holds if, and only if, the curve is a parallel, [6].

The isoperimetric problem has been widely studied since the beginning of the XX century, and many results of diverse nature have been obtained. A complete account of them, until the 1980's, can be found in Osserman's essay, [35].

## 11 The importance of the Jacobi operator

Among the most useful tools to study the curvature operator of a Riemann manifold we find the "Jacobi fields". These are solutions of Jacobi's differential equation along a geodesic. The symmetric tensor field

$$R_\gamma = R(\cdot, \gamma')\gamma'$$

is called "Jacobi operator along  $\gamma$ " and plays, via the Jacobi fields, a central role in the study of intrinsic and extrinsic geometry of geodesic spheres, tubes and reflections with respect to points, curves and submanifolds.

In general, it is very difficult to find explicitly the Jacobi fields, except for riemannian manifolds with a simple curvature tensor. But some properties of a riemannian manifold can be analyzed using those of its Jacobi operators. In [5], families of manifolds such that their Jacobi operators have a constant spectrum or parallel eigenvectors were studied. Except for symmetric spaces, these two families are disjoint.

## 12 Conclusion

The facts exposed in this work show that the study of curvature is difficult, yet exciting.

Fridman, in the preface of his book "El mundo como espacio y tiempo", [18], recounts that one night, Descartes was observing the firmament. A man, passing by, asked him: "How many stars are there in the sky?" Descartes answered: "The human intellect is finite, and it is in the nature of finite intellect not to embrace all things".

The man has always had the desire to find an explanation for the laws of Nature. Many have devoted their efforts to explain, with more or less success, the physical laws, depending on their experience and level of knowledge.

The difficulty is not small that supposes the uncertainty of the goal, maybe more in Mathematics than in other branches of Science. To discern which are the objectives worth pursuing is already a step forward to find the solution of the problems, even if we cannot solve them by ourselves.

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