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A One-sided Version of Alexiewicz-Orlicz's Differentiability Theorem

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Abstract. Modifying appropriately the method of a forgotten work [1], we show that if a continuous mapping from a nonempty open subset U of a metrizable separable Baire topological vector space E to a locally convex space is directionally right differentiable on U along a comeager subset of E, then it is generically Gâteaux differentiable on U. Our result implies that every metrizable separable Baire topological vector space is weak Asplund. It partially covers also the similar statements of [8, 16, 23].

Una version lateral del teorema de diferenciabilidad de Alexiewicz-Orlicz

Resumen. Modificando adecuadamente el método de un trabajo olvidado [1], probamos que si una aplicación continua, de un subconjunto abierto no vacío U de un espacio vectorial topológico metrizable separable y de Baire E, en un espacio localmente convexo, es direccionalmente diferenciable por la derecha en U según un subconjunto comagro de E, entonces, es genéricamente Gâteaux diferenciable en U. Nuestro resultado implica que cualquier espacio vectorial topológico, metrizable, separable y de Baire, es débilmente Asplund. También cubre parcialmente similares resultados de [8, 16, 23].

1 Introduction

The main goal of the present article is to show that in a rather general setting the one-sided directional differentiability of a mapping already implies its generic Gâteaux differentiability. A precise formulation of the obtained result and clarification of its connection with other known related statements will be easier to give after recalling the corresponding definitions.

We consider vector spaces over the same field $\mathbb{K}$ of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers and keep the following notations:

— E is a topological vector space, U, H a non-empty subsets of E being U *open*,

— F a Hausdorff topological vector space,

— $f\colon U \to F$ a function.

Moreover, for a fixed scalar $t \in \mathbb{K} \setminus \{0\}$ and $x \in U$ we introduce a mapping $\Delta_{f,x,t}\colon E \to F$, defined at a given $h \in E$ as follows:

$$\Delta_{f,x,t}(h) = \begin{cases} \dfrac{f(x+th) - f(x)}{t}, & \text{when } x + th \in U \\ \theta, & \text{when } x + th \notin U. \end{cases}$$

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We adopt the following definitions (cf. [4, 20, 28]):

Definition 1 *The function f will be caled:*

- Directionally differentiable at x along a vector $h \in E$ *if the function* $t \mapsto \Delta_{f,x,t}(h)$ *has a limit in* F *as* $t \to 0$.
- Directionally differentiable at x along H *if* f *is directionally differentiable at* x *along every* $h \in H$.
- Directionally differentiable at x *if* f *is directionally differentiable at* x *along the whole space* E.
- Directionally right differentiable at x along a vector $h \in E$ *if the function* $t \mapsto \Delta_{f,x,t}(h)$ *has a limit in* F *as* $t \in \mathbb{R}, \quad t > 0$ *and* $t \to 0$.
- Directionally right differentiable at x along H *if* f *is directionally right differentiable at* x *along every* $h \in H$.
- Directionally right differentiable or one-sided directionally differentiable *at* x *if* f *is directionally right differentiable at* x *along the whole space* E.

If f is directionally differentiable at x along H, then the mapping $D_{f,x} \colon H \to F$ defined for $h \in H$ by the equality

$$D_{f,x}(h) = \lim_{t \to 0} \Delta_{f,x,t}(h)$$

is called the *directional derivative of f at x along H*.

Clearly, if f is directionally differentiable at x then its directional derivative at x is defined everywhere on E, i.e. $D_{f,x} \colon E \to F$.

If f is directionally right differentiable at x along H, then the mapping $D^+_{f,x} \colon H \to F$ defined for $h \in H$ by the equality

$$D^+_{f,x}(h) = \lim_{t \in \mathbb{R},\ t>0,\ t \to 0} \Delta_{f,x,t}(h)$$

is called the *directional right derivative or one-sided directional derivative of f at x along H*.

If f is directionally right differentiable at x, then its directional right derivative at x is defined everywhere on E, i.e. $D^+_{f,x} \colon E \to F$.

Definition 2 *The function f is said to be Gâteaux differentiable at x if f is directionally differentiable at x and its directional derivative $D_{f,x} \colon E \to F$ is a continuous linear mapping.*

Definition 3 *Let E, F be normed spaces. The function f is said to be Fréchet differentiable at x if f is directionally differentiable at x,*

$$\sup_{h \in E,\ \|h\|=1} \|\Delta_{f,x,t}(h) - D_{f,x}(h)\| \to 0 \quad as \quad t \to 0$$

and the directional derivative $D_{f,x}$ is a continuous linear mapping.

If f is Fréchet differentiable at x, then f is continuous and Gâteaux differentiable at x. A Gâteaux differentiable at x function may not be continuous at x; also, a continuous Gâteaux differentiable at x function may not be Fréchet differentiable at x (even if $U = \mathbb{R}^2$ and $F = \mathbb{R}$).

Let us recall that a set in a topological space which is a countable union of nowhere dense sets is called *meager* (or *of the first category*) and the complement of a meager set is called *comeager* (or *residual*). A topological space X is called a *Baire space* if the intersection of any sequence of open dense subsets of X is dense. If X is a non-empty Baire space, then the dense G_δ subsets of X are not meager and any comeager subset of X contains a dense G_δ set. A comeager subset of a Baire space is a Baire space as well (see, e.g., [10, Ch.IX, §5, Proposition 5]). The Baire-Hausdorff theorem asserts that if X is either a completely metrizable or a locally compact regular topological space, then X is Baire. There are also metrizable separable Baire topological spaces which are neither completely metrizable, nor locally compact.

The mapping $f\colon U \to F$ is called *generically Gâteaux differentiable* on U if it is Gâteaux differentiable at every point of a dense G_δ subset of U.

A topological vector space F will be called *dually separated* if for every $y \in F \setminus \{0\}$ there exists a continuous linear functional $u\colon F \to \mathbb{K}$ such that $u(y) \neq 0$. Every Hausdorff locally convex space is dually separated. If $0 < p < 1$, then the sequence space l_p with its standard topology presents an example of a non-locally convex complete separable metrizable dually separated space.

Our starting point was the following remarkable statement obtained in [1].

(AO) *If E is a **real separable** Banach space, F is a Banach space, $f\colon U \to F$ is continuous and f is directionally differentiable at every $x \in U$, then f is generically Gâteaux differentiable* on U (see [1, Theorem]).

In a huge amount of the literature dedicated to the differentiability questions of mappings between Banach and more general spaces, seemingly only [30] contains [1] in its referencee's list. However, even in [30] the statement (AO) is not mentioned at all.

In the given article we show that, modifying appropriately the method of [1], it is possible to obtain the next one-sided version of (AO).

Theorem 1 *Let E be a real separable metrizable Baire topological vector space, $U \subset E$ be an open set, F be a metrizable topological vector space and $f\colon U \to F$ be a continuous mapping.*

*(a) If F is locally convex and f is directionally right differentiable at every $x \in U$ **along a comeager subset** H of an open neighborhood of zero $V \subset E$, then f is generically Gâteaux differentiable on U.*

*(b) If F is a (not necessarily locally convex) dually separated space and f is directionally right differentiable at every $x \in U$ **along whole** E, then f is generically Gâteaux differentiable on U.*

It is clear that Theorem 1 contains and refines (AO). In the next remark we discuss a consequence and some results related with Theorem 1 in case when $E = \mathbb{R}$.

Remark 1 *Let U be an open subset of $\mathbb{R}$. We note:*

— *f is directionally differentiable at x along 1 iff f is differentiable at x in ordinary sense. Moreover, if f is directionally differentiable at x along 1, then $D_{f,x}(1) = f'(x)$, where $f'(x)$ stands for the ordinary derivative of f at x.*

— *f is directionally right differentiable at x along 1 iff f is right differentiable at x in ordinary sense. Moreover, if f is directionally right differentiable at x along 1, then $D^+_{f,x}(1) = f'_+(x)$, where $f'_+(x)$ stands for the ordinary right derivative of f at x.*

— *f is directionally right differentiable at x along -1 iff f is left differentiable at x in ordinary sense. Moreover, if f is directionally right differentiable at x along -1, then $D^+_{f,x}(-1) = -f'_-(x)$, where $f'_-(x)$ stands for the ordinary left derivative of f at x.*

Taking into account these observations, from Theorem 1 we get the next statement:

(I) *If F is a metrizable locally convex space and $f\colon U \to F$ is a continuous function which is right and left differentiable at each point $x \in U$ and f'_+, f'_- are continuous functions on U, then there exists some comeager subset $C \subset U$, such that f is differentiable at every point $x \in C$.*

Note, however, that the following stronger version of (I) (which is not a direct consequence of Theorem 1) is true:

(II) *If F is a Banach space and $f\colon U \to F$ is left and right differentiable at each point $x \in U$, then there exists some finite or countable subset $S \subset U$, such that f is differentiable at every point $x \in U \setminus S$*

(see [14, Ch.VIII, §.4, Probleme 2 (b)]; in [2, p. 159, Lemma 1] a similar statement is proved for real-valued functions and it is attributed to W. Sierpinski).

In the next remark are collected some other related results and consequences of Theorem 1.

Remark 2

a) Gâteaux differentiability of convex functions

(J1) *If U is an open convex subset of $\mathbb{R}$ and $f\colon U \to \mathbb{R}$ is a convex function, then f is continuous on U, is right and left differentiable at each point of U and the set of points of non-differentiability of f is at most countable (see, e.g.,* [26, p. 9, Th. 1.16]*).*

(J2) *If U is a convex open subset of a real topological vector space and $f\colon U \to \mathbb{R}$ is a convex functional, then f is directionally right differentiable at each point of U (see, e.g.,* [26, p. 2, Lemma 1.2]*).*

(M) *If U is a convex open subset of a real separable Banach space E and $f\colon U \to \mathbb{R}$ is a continuous convex functional, then f is generically Gâteaux differentiable on U (this is the **Mazur's theorem**, see* [24]*; see also* [26, p. 12, Th. 1.20]*).*

A topological vector space E is said to be a weak Asplund space *provided every continuous convex function defined on a nonempty open convex subset U of E is generically Gâteaux differentiable on U (cf.* [26, p. 13, Def. 1.22] *and* [17, Def. 1.01]*).*

Theorem 1 *together with* (J2)*, implies the next generalization of* (M)*:*

(M1) *If E is a real separable metrizable Baire topological vector space, then E is a weak Asplund space.*

In particular, we get that every separable Baire normed space is weak Asplund. Whether every separable weak Asplund normed space is Baire, we do not know.

Note, however, that for some non-locally convex complete separable metrizable spaces E the conclusion of (M1) *may hold trivially (e.g., it is known that if $E = L_p[0,1]$ with $0 < p < 1$ and $f\colon E \to \mathbb{R}$ is a continuous convex functional, then f is a constant functional).*

(M2) *An analogue of* (M) *may fail in general: the norm of $E = L_\infty[0,1]$ is nowhere Gâteaux differentiable (this was noticed already in* [24]*).*

From (M2)*, since the Banach spaces $E = L_\infty[0,1]$ and l_∞ are isomorphic, we get that l_∞ admits an equivalent norm which is nowhere Gâteaux differentiable (see also* [26, Example 1.21] *for a simple example of a nowhere Gâteaux differentiable continuous seminorm given on l_∞).*

It follows from (J2) *and* (M2) *that Theorem* 1(b) *may fail when E is a non-separable Banach space and $F = \mathbb{R}$ (we do not know whether or not some version of* (AO) *remains true when E is a non-separable Banach space).*

In connection with (M2) *let us mention that the class of weak Asplund spaces is much wider than the class of separable Banach spaces as the next generalization of Mazur's theorem shows:*

(As1) *If E is a (not necessarily separable) closed subspace of a WCG Banach space, then E is a weak Asplund space (*[3, Th. 2]*; see also* [26, p. 37, Th. 2.45] *and* [17, Th. 1.3.4]*).*

Whether every WCG Baire normed space is weak Asplund, we do not know.

b) Fréchet differentiability of convex functions

It is known that if a continuous convex $f\colon U \to \mathbb{R}$ *given an open convex subset* U *a* finite-dimensional Banach space E *is Gâteaux differentiable at a point* $x \in U$*, then it is Fréchet differentiable at* x*. A similar assertion is not true in general as the next result shows.*

(BF) *If* E *is an infinite-dimensional real Banach space, then there exists an equivalent norm* $f : E \to \mathbb{R}$ *and a point* $x \in E$*, such that* f *is Gâteaux differentiable at* x*, but* f *is not Fréchet differentiable at* x *(see* [9, Theorem 1]*).*

A normed space E *is said to be an* Asplund space *provided every continuous convex function defined on a nonempty open convex subset* U *of* E *is Fréchet differentiable at each point of some* G_δ *subset of* U *(see* [26, p. 13, Def. 1.22]*; cf. also* [17, Def. 1.01]*).*

(As2) *A Banach space* E *is an Asplund space iff every separable closed subspace of* E *has a separable dual (cf.* [3, Th. 1]*; see also* [26, p. 32, Th. 2.34] *and* [17, Th. 1.1.1]*).*

We refer to [7, 13, 17, 18, 26] *for an extensive study of the differentiability problems of convex and Lipschitz functions given on a Banach space.*

c) Differentiability in complex spaces

(Sukh) *If* E *is a* ***complex*** *Banach space and* $f\colon U \to \mathbb{C}$ *is locally bounded and is directionally differentiable at every* $x \in U$*, then* f *is Fréchet differentiable at every* $x \in U$ *(see* [28, Th. 10]*).*

(Z1) *If* E*,* F *are* ***complex*** *Banach spaces and* $f\colon U \to F$ *is directionally differentiable at every* $x \in U$*, then for every* $x \in U$ *the directional derivative* $D_{f,x} : E \to F$ *is linear (see* [32, (2.3)]*).*

When E *is infinite-dimensional, in* (Z1)*, in general, it cannot be asserted the continuity of* $D_{f,x}\colon E \to F$ *(because, if* $f\colon E \to \mathbb{C}$ *is a non-continuous linear functional, then* $D_{f,x} = f$*,* $\forall x \in E$*).*

(Z2) *If* E, F *are* ***complex*** *Banach spaces, the restriction of* $f\colon U \to F$ *on some comeager subset of* U *is continuous and the function* f *is directionally differentiable at every* $x \in U$*, then* f *is Fréchet differentiable at every* $x \in U$ *(see* [32, (4.10)]*).*

An analogue of (Sukh) *(and hence, of* (Z2) *too) is not true for real spaces: a continuous function* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *may be Gâteaux differentiable at every* $x \in \mathbb{R}^2$*, but may not be Fréchet differentiable at every* $x \in \mathbb{R}^2$.

We can consider (AO) *and Theorem* 1 *as a real continuous separable and category version of* (Z2).

d) Other results related with Theorem 1

The next statement can be considered as a refinement of (AO)

(LW) *If* E *is real separable Banach space,* F *is a Banach space,* f *is continuous on* U *and there is a dense* G_δ *subset* B *of* U *such that at every* $x \in B$ *the function* f *is directionally differentiable along a* G_δ *subset* $H \subset E$ *which is dense in a non-empty open subset of* E*, then there exists a dense* G_δ *subset* A *of* B *such that* f *is Gâteaux differentiable at every* $x \in A$ *(see* [23, Th. 3.1]*; cf also,* [8, Coroll. 2 to Th. 1]*).*

In [16] *it was proved the next one-sided version of* (LW)*:*

(Fa) *Let* E *be a separable Banach space,* F *a Banach space,* U*,* V *be nonempty open subsets of* E *and* $f\colon U \to F$ *be a continuous mapping. Suppose that there exists a dense* G_δ *subset* S *of* $U \times V$ *such that* $\forall (x,h) \in S$ *the function* $t \mapsto \Delta_{f,x,t}(h)$ *has a limit in* F *as* $t > 0$ *and* $t \to 0$.

Then there exists a dense G_δ *subset* A *of* U *such that* f *is Gâteaux differentiable at every* $x \in A$ *(see* [16, Th. 3.1]*).*

In [31] *is contained the next version of* (Fa)*:*

(Zh) *If E is a (not necessarily separable) WCG Banach space, $f : U \to \mathbb{R}$ is continuous and there is a dense G_δ subset B of U such that f is directionally right differentiable at every $x \in B$, then there exists a dense G_δ set $A \subset B$ such that f is Gâteaux differentiable at every $x \in A$ (see* [31, Th. 4.4]*).*

Our Theorem 1 *partially covers* (LW) *and* (Fa). □

The one sided directional differentiability was considered in [2, 19, 27] and many other works.

2 Auxiliary results and proofs

The following easily provable observations will be used later.

(S1) If f is directionally differentiable at x along $h \in E$ and $t \in \mathbb{K}$ is a fixed scalar, f is directionally differentiable at x along the vector th and $D_{f,x}(th) = tD_{f,x}(h)$.

(S2) If f is directionally right differentiable at x along $h \in E$ and $t \in \mathbb{R},\ t \geq 0$ is a fixed scalar, then f is directionally right differentiable at x along the vector th and $D^+_{f,x}(th) = tD^+_{f,x}(h)$.

(S3) If f is directionally differentiable at x along $h \in E$, then f is directionally right differentiable at x along the vector h and $D_{f,x}(h) = D^+_{f,x}(h)$.

(S4) If $h \in E$ is a vector such that f is directionally right differentiable at x along the set $\{\zeta h : \zeta \in \mathbb{K},\ |\zeta| = 1\}$ and $D^+_{f,x}(\zeta h) = \zeta D^+_{f,x}(h) \quad \forall \zeta \in \mathbb{K},\ |\zeta| = 1$, then f is directionally differentiable at x along h and $D_{f,x}(h) = D^+_{f,x}(h)$.

(S5) If E, F are vector spaces over $\mathbb{R}$, the set H is an additive subgroup of E, f is directionally right differentiable at x along H and $D^+_{f,x}(h_1 + h_2) = D^+_{f,x}(h_1) + D^+_{f,x}(h_1) \quad \forall h_1, h_2 \in H$, then f is directionally differentiable at x along H and $D_{f,x}(h) = D^+_{f,x}(h) \quad \forall h \in H$.

(S6) If f is directionally (right) differentiable at x along $h \in E$ and u is a continuous linear mapping from F to a topological vector space G, then $u \circ f : U \to G$ is directionally (right) differentiable at x along h and $D_{u \circ f,x}(h) = u(D_{f,x}(h))$ ($D^+_{u \circ f,x}(h) = u(D^+_{f,x}(h))$).

To present a proof Theorem 1, we need several statements which may have an independent interest. The next Lemma is a one-sided version of Lagrange's mean value theorem.

Lemma 1 *Let $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$ and $\varphi : [t_1, t_2] \to \mathbb{R}$ be a function satisfying the conditions:*

(c1) φ is bounded and attains its supremum and infimum on $[t_1, t_2]$.

(c2) φ is right differentiable at every point of $]t_1, t_2[$.

(c3) φ is continuous at t_1.

Then:

(a) There exist $\lambda \in [0, 1]$ and $s_1, s_2 \in]t_1, t_2[$ such that

$$\frac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} = \lambda \varphi'_+(s_1) + (1 - \lambda)\varphi'_+(s_2)$$

(b) There exists $s \in]t_1, t_2[$ *such that*

$$\frac{|\varphi(t_2) - \varphi(t_1)|}{t_2 - t_1} \le |\varphi'_+(s)|.$$

(c) If $\varphi'_+(t) \ge 0, \quad \forall t \in]t_1, t_2[$ *(resp. if* $\varphi'_+(t) \le 0, \quad \forall t \in]t_1, t_2[$*), then* φ *is increasing (resp. is decreasing) on* $[t_1, t_2]$.

PROOF. We will prove first the next

Step 1. *If* φ *satisfies (c1) and is right differentiable at every point of* $[t_1, t_2[$*, then*

(a') There exist $\lambda \in [0, 1]$ *and* $s_1, s_2 \in [t_1, t_2[$ *such that* $\dfrac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} = \lambda\varphi'_+(s_1) + (1 - \lambda)\varphi'_+(s_2)$.

(c') If $\varphi'_+(t) \ge 0, \quad \forall t \in [t_1, t_2[$ *(resp. if* $\varphi'_+(t) \le 0, \quad \forall t \in]t_1, t_2[$*), then* φ *is increasing (resp. is decreasing) on* $[t_1, t_2]$.

Proof of Step 1. (a'). Let $t_1 = 0$, $t_2 = 1$ and $\varphi(1) - \varphi(0) = 0$. In this case *(a')* will be proved if we can see that

$$\exists s_1,\ s_2 \in [0, 1[\quad \varphi'_+(s_1) \le 0 \le \varphi'_+(s_2). \tag{1}$$

Since $\varphi(1) = \varphi(0)$, it follows easily from *(c1)* that

$$\exists \alpha, \beta \in [0, 1[\quad \varphi(\alpha) \le \varphi(t) \le \varphi(\beta), \quad \forall t \in [0, 1]. \tag{2}$$

If we take now any $\alpha, \beta \in [0, 1[$ for which (2) holds, then (1) will be true with $s_2 = \beta$ and $s_1 = \alpha$.

The conclusion of *(a')* when $[t_1, t_2] \ne [0, 1]$ follows from the already proved step, applied to the function $\psi\colon [0, 1] \to \mathbb{R}$ given for a $t \in [0, 1]$ by equality:

$$\psi(t) := \frac{\varphi(t_1 + t(t_2 - t_1))}{t_2 - t_1} - \frac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} t.$$

Since *(c')* follows from *(a')*, Step 1 is proved.

Step 2. *Proof of (a).* Let $t_1 = 0$, $t_2 = 1$ and $\varphi(1) - \varphi(0) = 0$. In this case *(a')* will be proved if we can see that

$$\exists s_1,\ s_2 \in [0, 1[\quad \varphi'_+(s_1) \le 0 \le \varphi'_+(s_2). \tag{3}$$

Take any $\alpha, \beta \in [0, 1[$ for which (2) holds.

If $\alpha, \beta \in]0, 1[$, then (3) will be true with $s_2 = \beta$ and $s_1 = \alpha$.

It remains to prove (3) in remaining cases:

Case 1: $\alpha = \beta = 0$,

Case 2: $\alpha = 0$ and $0 < \beta < 1$,

Case 3: $\beta = 0$ and $0 < \alpha < 1$.

In case 1 the function φ is identically zero, and so, (3) holds trivially. Therefore we can exclude this case and can suppose that $\varphi(\alpha) < \varphi(\beta)$.

Consider Case 2. For $s_1 := \beta$, we have $\varphi'_+(s_1) \le 0$. So, we need to find some $s_2 \in]0, 1[$ such that $\varphi'_+(s_2) \ge 0$. Suppose this is not possible; then we shall have $\varphi'_+(s) \le 0\ \forall s \in]0, 1[$. Fix a number $r \in]0, \beta[$. Since *(c2)* is satisfied, φ is right differentiable at each point of $[r, \beta[$. An application of Step 1 *(c')* for the restriction of φ to $[r, \beta]$, together with $\varphi'_+(s) \le 0 \quad \forall s \in [r, \beta[$, gives that φ is decreasing on $[r, \beta]$. Hence, $\varphi(r) \ge \varphi(\beta),\ \forall r \in]0, \beta[$. From this, *since (c3) is satisfied at* $t_1 = 0$, we get

$$\varphi(0) = \lim_{r > 0,\, r \to 0} \varphi(r) \ge \varphi(\beta),$$

but this contradicts to the inequality $\varphi(0) = \varphi(\alpha) < \varphi(\beta)$. Consequently, the relation $\varphi'_+(s) \le 0 \quad \forall s \in]0, 1[$ cannot hold, hence some $s_2 \in]0, 1[$ we have $\varphi'_+(s_2) \ge 0$ and (3) is proved in Case 2.

The validity of (3) in Case 3 can be verified in a similar way.

The conclusion of *(a)* when $[t_1, t_2] \neq [0, 1]$ follows from the already proved step, applied to the function $\psi\colon [0, 1] \to \mathbb{R}$ given for a $t \in [0, 1]$ by equality:

$$\psi(t) := \frac{\varphi(t_1 + t(t_2 - t_1))}{t_2 - t_1} - \frac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} t.$$

(b) and *(c)* follow from *(a)*. ■

Remark 3 *(1) Lemma* 1*(a) in case of a continuous* φ *is contained in* [15, 16.3]*; as L. Maligranga informed us, it dues, probably, to D. Kurepa.*

(2) The ordinary mean value theorem is not a consequence of Lemma 1*(a), because, even if* $\varphi\colon [0, 1] \to \mathbb{R}$ *is continuous and it is right differentiable at every point of* $[0, 1[$*, there may not exist a point* $s \in [0, 1[$ *with property:* $\varphi(1) - \varphi(0) = \varphi'_+(s)$*.*

(3) Lemma 1*(c) in case of a continuous* φ *is known, see* [12, Cor. 3.2.1]*. In* [11, Ch. I, §2, Prop. 2] *is contained the next more general result:*

(PrBo) If for a ***continuous*** $\varphi\colon [t_1, t_2] \to \mathbb{R}$ *there exists an at most countable set* $A \subset [t_1, t_2]$ *such that* φ *is right differentiable at each point of* $[t_1, t_2] \setminus A$ *and* $\varphi'_+(s) \geq 0$ *for every* $s \in [t_1, t_2] \setminus A$*, then* $\varphi(t_1) \leq \varphi(t_2)$*.*

In (PrBo) "an at most countable set $A \subset [t_1, t_2]$*" cannot be replaced by "a nowhere dense set* $A \subset [t_1, t_2]$*" (see* [11, Ch. I, §2, p. 22, Rem. 2]*).*

Lemma 2 *Let* $(F, \|\cdot\|)$ *be a normed space and* $\varphi\colon [0, 1] \to F$ *be a continuous function which is right differentiable at every point of* $[0, 1[$*.*

(a) (cf. [12, Cor. 3.2.2]*) There exists* $s \in]0, 1[$ *such that* $\|\varphi(1) - \varphi(0)\| \leq \|\varphi'_+(s)\|$*.*

(b) For every $r \in [0, 1[$ *there exists* $s \in]0, 1[$ *such that*

$$\|\varphi(1) - \varphi(0) - \varphi'_+(r)| \leq \|\varphi'_+(s) - \varphi'_+(r)\|$$

PROOF. *(a)*. By Hahn-Banach theorem for some continuous linear functional $u\colon F \to \mathbb{R}$ with $\|u\| \leq 1$ we have $\|\varphi(1) - \varphi(0)\| = u(\varphi(1) - \varphi(0))$. An application of Lemma 1*(b)* to the function $u \circ \varphi$ gives *(a)*.

(b). Fix $r \in [0, 1[$ and apply *(a)* to the new function $t \mapsto \psi(t) := \varphi(t) - \varphi'_+(r)t$. ■

Remark 4 *The following stronger version of Lemma* 2*(a) is known: if a continuous* $\varphi\colon [0, 1] \to F$ *there exists some finite or countable subset* $A \subset [0, 1]$*, such that* φ *is right differentiable at every point* $t \in [0, 1[\setminus A$*, then there exists* $s \in]0, 1] \setminus A$ *such that* $\|\varphi(1) - \varphi(0)\| \leq \|\varphi'_+(s)\|$ *(see* [11, Ch. I, §2, exer. 14]*, or* [14, Ch. VIII, §5, Prob. 7]*).*

Lemma 3 *Let* E *be a real Hausdorff topological vector space,* $U \subset E$ *be an open set,* F *be a normed space,* $f\colon U \to F$ *be a mapping and the elements* $b \in U$*,* $h \in E$ *be such that*

$$\{b + th : t \in [0, 1]\} \subset U. \tag{4}$$

If either

(crd) *the mapping* f *is* ***continuous*** *and it is directionally right differentiable at every* $x \in U$ *along the vector* h*,*

or

(dd) *the mapping f is **directionally differentiable** at every $x \in U$ along the vector h, then there exists $s \in]0,1[$ such that*

$$\|f(b+h) - f(b) - D^+_{f,b}(h)\| \le \|D^+_{f,b+sh}(h) - D^+_{f,b}(h)\|. \tag{5}$$

PROOF. Write $A = \{t \in \mathbb{R} : b + th \in U\}$. Since U is open in E, the set A is an open in $\mathbb{R}$ and since (4) is satisfied, $[0,1] \subset A$. Consider $\varphi\colon A \to F$ given for $t \in A$ by the equality: $\varphi(t) = f(b+th)$.

If **(crd)** is satisfied, then $\varphi\colon A \to F$ is continuous, right differentiable at every $t \in A$ and

$$\varphi'_+(t) = D^+_{f,b+th}(h), \qquad \forall t \in A.$$

Now an application of Lemma 2*(b)* for the restriction of φ to $[0,1] \subset A$ gives (5).

If **(dd)** is satisfied, then $\varphi\colon A \to F$ **differentiable** at every $t \in A$ and

$$\varphi'(t) = D_{f,b+th}(h), \qquad \forall t \in A.$$

Since the differentiability of φ implies its continuity on A, again an application of Lemma 2*(b)* for the restriction of φ to $[0,1] \subset A$ gives (5). ■

The next statement is a one-sided version of [1, Lemma].

Proposition 1 *Let E be a real Hausdorff topological vector space, $n > 1$ a natural number, $h_1, \ldots, h_n \in E$, $U \subset E$ be an open set, $x_0 \in U$, F be a Hausdorff topological topological vector space and $f\colon U \to F$ be a continuous mapping.*

(a) If F is locally convex, f is directionally right differentiable at every $x \in U$ along the set $\{h_1, \ldots, h_n\}$ and the maps $x \to D^+_{f,x}(h_i)$, $i = 1, \ldots, n-1$ are continuous at x_0, then f is directionally right differentiable at x_0 along the vector $h_1 + \cdots + h_n$ and

$$D^+_{f,x_0}(h_1 + \cdots + h_n) = D^+_{f,x_0}(h_1) + \cdots + D^+_{f,x_0}(h_n). \tag{6}$$

*(b) If F is a (not necessarily locally convex) dually separated space, f is directionally right differentiable at every $x \in U$ **along whole** E and the maps $x \to D^+_{f,x}(h_i)$, $i = 1, \ldots, n-1$ are continuous at x_0, then equality* (6) *holds.*

*(a',b') The conclusion of (a) (resp. of (b)) remains true for a not necessarily continuous $f\colon U \to F$ provided f is **directionally differentiable** at every $x \in U$ **along** the set $\{h_1, \ldots, h_n\}$ and the maps $x \to D^+_{f,x}(h_i)$, $i = 1, \ldots, n-1$ are continuous at x_0 (resp. f is **directionally differentiable** at every $x \in U$ **along whole** E and the maps $x \to D^+_{f,x}(h_i)$, $i = 1, \ldots, n-1$ are continuous at x_0).*

PROOF. *(a,a')* First we shall prove:

(a_1) Proposition 1*(a,a') is true when F is a normed space*.

Write $a_k := \sum_{j=k}^n h_j$, $k = 1, 2, \ldots, n$. It is needed to show that

$$\lim_{t>0,\, t\to 0} \left\| \Delta_{f,x_0,t}(a_1) - \sum_{k=1}^n D^+_{f,x_0}(h_k) \right\| = 0. \tag{7}$$

Fix $\varepsilon > 0$. Let us find a $\delta > 0$ so that

$$\lim_{t>0,\, t\to 0} \left\| \Delta_{f,x_0,t}(a_1) - \sum_{k=1}^n D^+_{f,x_0}(h_k) \right\| < \delta, \qquad \forall t \in]0, \delta[. \tag{8}$$

Since the maps $x \to D^+_{f,x}(h_i)$, $i = 1, \ldots, n-1$ are continuous at x_0, there is a neighborhood V of zero in E such that $x_0 + V \subset U$ and

$$\|D^+_{f,x}(h_k) - D^+_{f,x_0}(h_k)\| < \varepsilon/4(n-1), \qquad \forall x \in V, \quad k = 1, \ldots, n-1. \tag{9}$$

Let V_0 be a balanced neighborhood of zero in E such that $V_0 + V_0 \subset V$. Since V_0 is absorbing, there is a $\delta_1 > 0$ such that

$$ta_k \in V_0, \text{ and } th_k \in V_0 \qquad \forall t \in [0, \delta_1], \quad k = 1, 2, \ldots, n.$$

Since f is directionally right differentiable at x_0 along h_n, there is a $0 < \delta_2 < 1$ such that

$$\left\| \Delta_{f,x_0,t}(h_n) - D^+_{f,x_0}(h_n) \right\| < \varepsilon/4 \qquad \forall t \in]0, \delta_2]. \tag{10}$$

Write $\delta := \min(\delta_1, \delta_2)$ and fix a number $t \in]0, \delta]$.

Since

$$f(x_0 + ta_1) - f(x_0) = \sum_{k=1}^{n-1} [f(x_0 + ta_k) - f(x_0 + ta_{k+1})] + [f(x_0 + ta_n) - f(x_0)],$$

we have

$$\begin{aligned} \frac{f(x_0 + ta_1) - f(x_0)}{t} - \sum_{k=1}^{n} D^+_{f,x_0}(h_k) = \\ \sum_{k=1}^{n-1} \frac{f(x_0 + ta_k) - f(x_0 + ta_{k+1}) - D^+_{f,x_0+ta_{k+1}}(th_k)}{t} \\ + \sum_{k=1}^{n-1} \left[D^+_{f,x_0+ta_{k+1}}(h_k) - D^+_{f,x_0}(h_k) \right] + \left[\Delta_{f,x_0,t}(h_n) - D^+_{f,x_0}(h_n) \right]. \end{aligned}$$

Since V_0 is balanced, for a fixed $k < n$ we have

$$x_0 + ta_k + sth_k \in x_0 + V_0 + V_0 \subset x_0 + V \subset U, \qquad \forall s \in [0, 1],$$

an application of Lemma 3 gives the existence of numbers $s_k \in]0, 1[$ such that

$$\begin{aligned} &\|f(x_0 + ta_k) - f(x_0 + ta_{k+1}) - D^+_{f,x_0+ta_{k+1}}(th_k)\| \\ &\qquad \le \|D^+_{f,x_0+ta_{k+1}+s_k th_k}(th_k) - D^+_{f,x_0++ta_{k+1}}(th_k)\|, \qquad k = 1, \ldots n - 1. \end{aligned}$$

From this and preceding relation we obtain:

$$\begin{aligned} \left\| \Delta_{f,x_0,t}(a_1) - \sum_{k=1}^{n} D^+_{f,x_0}(h_k) \right\| \le \sum_{k=1}^{n-1} \left\| D^+_{f,x_0+ta_{k+1}+s_k th_k}(h_k) - D^+_{f,x_0+ta_{k+1}}(h_k) \right\| \\ + \sum_{k=1}^{n-1} \left\| D^+_{f,x_0+ta_{k+1}}(h_k) - D^+_{f,x_0}(h_k) \right\| + \left\| \Delta_{f,x_0,t}(h_n) - D^+_{f,x_0}(h_n) \right\|. \end{aligned}$$

Since $ta_{k+1} + s_k th_k \in V$, $k = 1, \ldots n - 1$ and $ta_{k+1} \in V$, $k = 1, \ldots n - 1$ from (9) and triangle inequality we can write:

$$\sum_{k=1}^{n-1} \left\| D^+_{f,x_0+ta_{k+1}+s_k th_k}(h_k) - D^+_{f,x_0+ta_{k+1}}(h_k) \right\| < \varepsilon/2$$

Again from (9) we have:

$$\sum_{k=1}^{n-1} \left\| D^+_{f,x_0+ta_{k+1}}(h_k) - D^+_{f,x_0}(h_k) \right\| < \varepsilon/4$$

From the last two relations and (10) we get:

$$\Big\|\Delta_{f,x_0,t}(a_1) - \sum_{k=1}^{n} D^+_{f,x_0}(h_k)\Big\| < \varepsilon, \qquad \forall t \in]0,\delta].$$

Since $\varepsilon > 0$ is arbitrary, (8) (and hence (a_1) too), is proved.

Let now F be an arbitrary Hausdorff locally convex space. Then there exists a family $(Y_j)_{j\in J}$ of normed spaces and a family $(u_j)_{j\in J}$ of continuous linear mappings $u_j\colon F \to Y_j$, such that the topology of F is determined by $(u_j)_{j\in J}$. Taking into account this fact and observation (S6), for F Prop. 1*(a)* will be proved if we can show:

(a_2) *For a normed space Y and for a continuous linear mapping $u\colon F \to Y$*

$$\lim_{t>0,\, t\to 0} \Big\| u\Big(\Delta_{f,x_0,t}\Big(\sum_{k=1}^{n} h_k\Big) - \sum_{k=1}^{n} D^+_{f,x_0}(h_k)\Big)\Big\|_Y = 0.$$

Clearly, (a_2) follows from the already proved (a_1), applied to the mapping $u \circ f\colon U \to Y$.

(b, b') Fix a continuous linear functional $u : F \to \mathbb{R}$. An application of *(a)* to the function $u\circ f : E \to \mathbb{R}$ gives that $u \circ f$ is directionally right differentiable (is directionally differentiable) along $h_1 + \cdots + h_n$ and

$$D^+_{u\circ f,x_0}(h_1 + \cdots + h_n) = D^+_{u\circ f,x_0}(h_1) + \cdots + D^+_{u\circ f,x_0}(h_n). \tag{11}$$

Since, by assumption, f itself is directionally right differentiable (is directionally differentiable) along $h_1 + \cdots + h_n$, from (11) we get:

$$u(D^+_{f,x_0}(h_1 + \cdots + h_n)) = u(D^+_{f,x_0}(h_1) + \cdots + D^+_{f,x_0}(h_n)). \tag{12}$$

Since (12) holds for an arbitrary continuous linear functional u and F is dually separated, we get that (6) holds as well. ■

To simplify formulations, for an element h and a nonempty subset W_0 of an additive topological Abelian group E we write $h - W_0 := \{h - x : x \in W_0\}$ and call W_0 **an AO-set with respect to** $V \subset E$ if

$$W_0 \cap (h_1 - W_0) \cap (h_2 - W_0) \neq \emptyset \qquad \forall h_1, h_2 \in V. \tag{13}$$

Proposition 2 *Let E be a real Hausdorff topological vector space, $U \subset E$ be an open set, F be a Hausdorff topological vector space, $f\colon U \to F$ be a continuous mapping and $W_0 \subset E$ be a AO-set with respect to a neighborhood V of zero in E.*

(a) If F is locally convex, f is directionally right differentiable at every $x \in U$ ***along*** *W_0 and for every $h \in W_0$ the map $x \mapsto D^+_{f,x}(h)$ is continuous at $x = x_0 \in U$, then f is directionally differentiable at x_0 and the directional derivative $D_{f,x_0}\colon E \to F$ is a linear mapping.*

(b) If F is a (not necessarily locally convex) dually separated space, f is directionally right differentiable at every $x \in U$ ***along whole*** *E and for every $h \in W_0$ the map $x \to D^+_{f,x}(h)$ is continuous at $x = x_0 \in U$, then the directional derivative $D_{f,x_0}\colon E \to F$ is a linear mapping.*

(a',b') The conclusion of (a) (resp. of (b)) remains true for a not necessarily continuous $f\colon U \to F$ provided f is ***directionally differentiable*** *at every $x \in U$* ***along*** *W_0 and for every $h \in W_0$ the map $x \mapsto D^+_{f,x}(h)$ is continuous at $x = x_0 \in U$ (resp. f is* ***directionally differentiable*** *at every $x \in U$* ***along whole*** *E and for every $h \in W_0$ the map $x \to D^+_{f,x}(h)$ is continuous at $x = x_0 \in U$).*

(c) In (a) and (b) in general it may happen that the linear mapping $D_{f,x_0}\colon E \to F$ ***is not continuous****.*

PROOF. Fix a neighborhood V of zero in E for which (13) holds.

Item *(a,a')*. Thanks to observations (S1) and (S5), the directional differentiability of f at x_0 and the linearity of $D_{f,x_0}\colon E \to F$ will be proved if we can show the validity of the next statement:

(**V**) For every $h_1, h_2 \in V$ the mapping f is directionally right differentiable at x_0 along the set $\{h_1, h_2, h_1 + h_2\}$ and $D^+_{f,x_0}(h_1 + h_2) = D^+_{f,x_0}(h_1) + D^+_{f,x_0}(h_2)$.

Let $h_1, h_2 \in V$. Then, since W_0 is an AO-set with respect to V, we have: $W_0 \cap (h_1 - W_0) \cap (h_2 - W_0) \neq \emptyset$. The last relation implies that there are elements $h', h'', h''' \in W_0$ with $h_1 = h''' + h'$, $\quad h_2 = h''' + h'$. Hence, $h_1 + h_2 = 2h''' + h' + h''$.

Since f is directionally right differentiable at x_0 along the set $\{h', h'', h'''\}$ and the maps $x \mapsto D^+_{f,x}(h')$, $x \mapsto D^+_{f,x}(h'')$, $x \mapsto D^+_{f,x}(h''')$ are continuous at x_0, by Proposition 1*(a)* we conclude:

(1) f is directionally right differentiable at x_0 along the vector $h''' + h' = h_1$ and

$$D^+_{f,x_0}(h_1) = D^+_{f,x_0}(h''' + h') = D^+_{f,x_0}(h''') + D^+_{f,x_0}(h').$$

(2) f is directionally right differentiable at x_0 along the vector $h''' + h'' = h_2$ and

$$D^+_{f,x_0}(h_2) = D^+_{f,x_0}(h''' + h'') = D^+_{f,x_0}(h''') + D^+_{f,x_0}(h'').$$

(3) f is directionally right differentiable at x_0 along the vector $2h''' + h' + h'' = h_1 + h_2$ and

$$D^+_{f,x_0}(h_1 + h_2) = D^+_{f,x_0}(2h''' + h' + h'') = D^+_{f,x_0}(2h''') + D^+_{f,x_0}(h') + D^+_{f,x_0}(h'').$$

Clearly, from (1), (2) and (3) (since $D^+_{f,x_0}(2h''') = 2D^+_{f,x_0}(h''')$) we have also that $D^+_{f,x_0}(h_1 + h_2) = D^+_{f,x_0}(h_1) + D^+_{f,x_0}(h_2)$ and (**V**) is proved.

Item *(b,b')*. In this case the proof is analogous, the only difference is that instead of Proposition 1*(a)*, it is needed to use Proposition 1*(b)*.

Item *(c)*. It remains to show that in either of cases the directional derivative $D_{f,x_0}\colon E \to F$ may not be continuous. This is shown in [6] by means of an example of **a uniformly continuous** real valued functional f given on a countably dimensional real inner product space E, with properties:

— f is directionally differentiable at every $x \in E$,

— for every $h \in E$ the mapping $x \mapsto D_{f,x}(h)$ is continuous on E,

— for every $x \in E$ the directional derivative $D_{f,x} : E \to \mathbb{R}$ is a **discontinuous linear** functional. ■

Proposition 2 is applicable, e.g., when $W_0 = E$. Some versions of Proposition 2*(a')* for $W_0 = E$ and $F = \mathbb{R}$ are contained in [20, Th. 24] and [29, Th. 2.1].

Lemma 4 *Let E be a Baire topological group (written additively), V an open neighborhood of zero in E, $W_0 \subset V$ a set which is comeager in V. Then W_0 is an AO-set with respect to V.*

PROOF. Fix $h_1, h_2 \in V$ and denote $V_3 = V \cap (h_1 - V) \cap (h_2 - V)$. Then $0 \in V_3$, hence, V_3 is a non-empty open set. Since W_0 is comeager in V, we have that $h_1 - W_0$ is comeager in $h_1 - V$ and $h_2 - W_0$ is comeager in $h_2 - V$. Consequently, $W_0 \cap V_3$, $(h_1 - W_0) \cap V_3$ and $(h_2 - W_0) \cap V_3$ are comeager subsets of the non-empty open set V_3, which is a Baire space as well. This implies that

$$W_0 \cap (h_1 - W_0) \cap (h_2 - W_0) \cap V_3 \neq \emptyset,$$

hence, (13) is satisfied too. ■

The proof of Theorem 1 we need two more results, the first of which is a version of *the Baire theorem.*

Proposition 3 *Let X, F be metrizable topological spaces, $p_n\colon X \to F$, $n = 1, 2, \dots$ be continuous mappings, $p\colon X \to F$ a mapping such that $p(x) = \lim_n p_n(x)$, $\forall x \in X$. Then the set $\boldsymbol{C}(p)$ of all continuity points of p is comeager in X.*

For a proof see [21, §31.X, Remark 5 and §34. VII].

The second needed result is *the Kuratowski-Ulam theorem* obtained in [22], (see also [21, §22.V, Th. 1], [25, p. 56, Th. 15.1] and [23, Lemma 2.2]):

Proposition 4 *Let O and Y be separable metric spaces and W be a comeager subset of $O \times Y$. Then there exists a comeager subset $C \subset O$, such that for every $x \in C$ the set $\{y \in Y : (x, y) \in W\}$ is comeager in Y.*

Let us underline that the separability and the metrizability of E in the proof of Theorem 1 will be used only through Proposition 4.

PROOF OF THEOREM 1

Case *(a)*. Fix an open neighborhood V_0 of zero in E such that H is comeager in V_0. First we will prove the next assertion:

(AA) If the assumptions of Theorem 1 are satisfied for an open set U having the form $U = O + V$, where O is a non-empty open subset of E and $V \subset V_0$ is a balanced open neighbourhood of zero in E, then there exists a set $R \subset O$, comeager in O, such that for every $x \in R$ the function f is Gâteaux differentiable at x.

Proof of **(AA)**. Let $Y = H \cap V$. Since H is comeager in V_0 and V is open, we get that Y is comeager in V. In particular, $Y \neq \emptyset$.

Then since f is directionally right differentiable at every $x \in O \subset U$ along $h \in Y \subset H$, we have:

$$(x, h) \mapsto p(x, h) := D^+_{f,x}(h) = \lim_n \Delta_{f,x,1/n}(h), \qquad \forall (x, h) \in O \times Y.$$

Since V is balanced we have that $x + (1/n) \cdot h \in U$, $\forall x \in O$, $h \in V$, $n \in \mathbb{N}$. The last relation and the continuity of f on U imply that the maps $(x, h) \mapsto \Delta_{f,x,1/n}(h)$, $n = 1, 2, \dots$ are continuous from $O \times Y$ to F; consequently, since the spaces $O \times Y$ and F *are metrizable*, we can apply to the mapping $p: O \times Y \to F$ Proposition 3 and get that the set $W := \mathbf{C}(p)$ of all continuity points of p is comeager in $O \times Y$. Now, since E *is a separable metrizable space*, by Proposition 4 there exists a set C comeager in O, such that

(KU) for every $x \in C$ the set $W_x = \{h \in Y : (x, h) \in W\}$ is comeager in Y.

Since O is a Baire space, its comeager subset C contains a dense G_δ-subset of O; in particular $C \neq \emptyset$. Now we fix $x_0 \in C$. It is sufficient to prove that:

(A) *The map f is directionally differentiable at x_0* and $D_{f,x_0}: E \to F$ is linear.

(C) $D_{f,x_0}: E \to F$ *is continuous.*

Proof of (A). Denote $W_0 = W_{x_0}$. From (KU) we have that W_0 is comeager in Y. *Since Y is comeager in V, we obtain that W_0 is comeager in V*. From Lemma 4 we conclude that W_0 is an AO-set with respect to V and write:

(1) $W_0 \cap (h_1 - W_0) \cap (h_2 - W_0) \neq \emptyset, \quad \forall h_1, h_2 \in V$.

By assumption, f is directionally right differentiable at every $x \in U$ along the set H. From this, since $W_0 \subset H$ and and $x_0 \in C$, we obtain:

(2) f is directionally right differentiable at every $x \in U$ along the set W_0 and for every $h \in W_0$ the map $x \mapsto D^+_{f,x}(h)$ is continuous at x_0.

Taking into account (1) and (2), we can apply Proposition 2*(a)* and conclude that (A) is true.

Proof of (C). According to (A) for each given $h \in E$ the function f is directionally differentiable at x_0 along h; hence,

$$D_{f,x_0}(h) = \lim_n \Delta_{f,x_0,1/n}(h), \qquad \forall h \in E. \tag{14}$$

Since $h \mapsto \Delta_{f,x_0,1/n}(h)$, $n = 1, 2, \dots$ are continuous mappings from a metrizable space V to a metrizable space F, it follows from (14) and Proposition 3 that the set $\mathbf{C}(D_{f,x_0}) \subset V$ of all continuity points of $D_{f,x_0}: V \to F$ is comeager in V. Since V is a Baire space, $\mathbf{C}(D_{f,x_0})$ is not empty. From this and the linearity of $D_{f,x_0}\colon E \to F$ (which we have again by (A)) it follows that $D^+_{f,x_0}\colon E \to F$ is a continuous linear mapping. Therefore f is Gâteaux differentiable at x_0 and **(AA)** is proved.

Now we show that Theorem 1 for the general case when $U \neq E$ is an arbitrary open set can be derived from **(AA)**.

Let $(V_n)_{n\in\mathbb{N}}$ be a fundamental sequence of balanced open neighbourhoods of zero in E such that $V_n + V_n \subset V_{n-1} \subset V_0$, $n = 2, 3, \ldots$. For each natural number n denote by O_n the complement in E of the closure of $(E \setminus U) + V_n$. Then $O_n \subset O_{n+1} \subset U$, $n = 1, 2, \ldots$ and $U = \cup_{n=1}^{\infty} O_n$. Fix a natural number k so that $O_k \neq \emptyset$. Then we have also $O_n + V_n \subset O_{n+2} \subset U$, $n = k, k+1, \ldots$. Fix a natural number $n \geq k$. We can apply **(AA)** to function $f\colon O_n + V_n \to F$ and get the existence of a set $R_n \subset O_n$, comeager in O_n, such that for every $x \in R_n$ the function f is Gâteaux differentiable at x. Then the set $C = \cup_{n=k}^{\infty} R_n$ will be comeager in U and will have the property that for every $x \in C$ the function f is Gâteaux differentiable at x.

Case *(b)*. In this case the proof is analogous, the only difference is that instead of Proposition 2*(a)*, it is needed to use Proposition 2*(b)*. ■

Remark 5 *In case when E is a complete metrizable space, in the above given proof the continuity of the directional derivative can be established also by means of well-known Banach's theorem (see* [5, Ch. I, Th. 4]*).*

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