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Spaces of Continuous Functions Taking Their Values in the ε-Product

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Abstract. For a nuclear b-space N and a b-space E, we prove that if X is a compact space then the b-spaces $C\left(X, N\varepsilon E\right)$ and $N\varepsilon C\left(X, E\right)$ are isomorphic. Also the same result holds if X is a locally compact space that is countable at infinity.

Espacios de funciones continuas con valores en el ε-producto

Resumen. Para un b-espacio nuclear Ny un b-espacio E demostramos que si X es un espacio compacto entonces los b-espacios $C\left(X, N\varepsilon E\right)$ y $N\varepsilon C\left(X, E\right)$ son isomorfos. El mismo resultado se verifica también si X es un espacio localmente compacto que es numerable en el infinito.

1 Introduction and notations

We will show that if N is a nuclear b-space and X is a compact space then the exact functors $C\left(X, N\varepsilon.\right)$ and $N\varepsilon C\left(X, \cdot\right)$ are isomorphic on the category of b-spaces of L. Waelbroeck [9]. To prove this, we shall consider the nuclear b-space N as an union of Banach spaces N_B, where each N_B is isometrically isomorphic to the $\mathcal{L}_\infty$-space c_0 (i.e. the space of sequences which converge to 0). As a consequence, we will deduce that if E is a b-space, F is a bornologically closed subspace of E, N a nuclear b-space and X is a compact space then the b-spaces $C(X, N\varepsilon(E/F))$ and $N\varepsilon C(X, E/F)$ are isomorphic. Finally, we will show that if Y is a locally compact space that is countable at infinity then for any nuclear b-space N, the b-spaces $C(Y, N\varepsilon(E/F))$ and $N\varepsilon C(Y, E/F)$ are isomorphic.

To state our results, we need to fix some notations and recall some definitions.

1- Let E be a real or complex vector space, and let B be an absolutely convex set of E. Let E_B be the vector space generated by B, i.e. $E_B = \cup_{\lambda>0}\lambda B$. The Minkowski functional of B is a semi-norm on E_B. It is a norm, if and only if B does not contain any nonzero subspace of E. The set B is completant if its Minkowski functional is a Banach norm.

A bounded structure β on a vector space E is defined by a set of "bounded" subsets of E with the following properties:

1) Every finite subset of E is bounded; 2) every union of two bounded subsets is bounded; 3) every subset of a bounded subset is bounded; 4) a set homothetic to a bounded subset is bounded; 5) each bounded subset is contained in a completant bounded subset.

A b-space (E, β) is a vector space E with a boundedness β. A subspace F of a b-space E is bornologically closed if the subspace $F \cap E_B$ is closed in E_B for every completant bounded subset B of E.

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Given two b-spaces (E, β_E) and (F, β_F), a linear mapping $u\colon E \longrightarrow F$ is bounded, if it maps bounded subsets of E into bounded subsets of F. The mapping $u\colon E \longrightarrow F$ is bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B'$.
We denote by $\mathbf{b}$ the category of b-spaces and bounded linear mappings. For more information about b-spaces we refer the reader to [3] and [9].

2- The ε-product of two Banach spaces E and F is the Banach space $E\varepsilon F$ of linear mappings $E' \longrightarrow F$ whose restrictions to the closed unit ball $B_{E'}$ of E' are continuous for the topology $\sigma(E', E)$. If E_i and F_i are Banach spaces and $u_i\colon E_i \longrightarrow F_i$ are bounded linear mappings, $i = 1, 2$, the ε-product of u_1 and u_2 is the bounded linear mapping $u_1\varepsilon u_2\colon E_1\varepsilon E_2 \longrightarrow F_1\varepsilon F_2$, $f \longmapsto u_2 \circ f \circ u_1'$, where u_1' is the dual mapping of u_1. It is clear that $u_1\varepsilon u_2$ is injective when u_1 and u_2 are injective. If G is a Banach space and F is a closed subspace of a Banach space E, then $G\varepsilon F$ is a closed subspace of $G\varepsilon E$. See [5] and [8] for more information about the ε-product.

2 Main results

The ε-product of a b-space G and a Banach space E is the space $G\varepsilon E = \cup_B G_B\varepsilon E$, where B ranges over the bounded completant subsets of the b-space G. On $G\varepsilon E$ we define the following bornology of b-space: a subset C of $G\varepsilon E$ is bounded if there exists a completant bounded disk A of G such that C is bounded in the Banach space $G_A\varepsilon E$. It is clear that if F is a bornologically closed subspace in G, the subspace $F\varepsilon E$ is a bornologically closed subspace in $G\varepsilon E$.

Now, if G and E are two b-spaces, the ε-product of G and E is the space $G\varepsilon E = \cup_{A,B} G_A\varepsilon E_B$, where A (resp. B) ranges over the bounded completant subsets of the b-space G (resp. E). We endow $G\varepsilon E$ with the following bornology of b-space: a subset C of $G\varepsilon E$ is bounded if there exists a completant bounded disk A of G (resp. B of E) such that C is bounded in the Banach space $G_A\varepsilon E_B$.

Let E be a Banach space, F a closed subspace of E and $(\varphi_i)_{i\in I}$ a set of continuous linear functionals on E such that $F = \{y \in E : \text{for all } i \in I, \varphi_i(y) = 0\}$. If G is a Banach space, then $G\varepsilon F = \{f \in G\varepsilon E : \text{for all } i \in I, Id_G\varepsilon\varphi_i(f) = 0\}$ (indeed, $f \in G\varepsilon F$ iff for all $i \in I$ and for all $x \in G'$, $\varphi_i(f(x)) = (\varphi_i \circ f)(x) = 0$ and $\varphi_i \circ f = (Id_G\varepsilon\varphi_i)(f)$).

As application, we have $c_0\varepsilon E \simeq c_0(E)$, in fact, the Banach space $C(\mathbb{N}_\infty)$ is isomorphic to the space of convergent sequences c, where $\mathbb{N}_\infty$ is the Alexandroff compactification of $\mathbb{N}$. As $c_0\varepsilon E$ is isomorphic to a closed subspace of $c\varepsilon E$, containing the sequences of elements of E which converge to 0, and $c\varepsilon E \simeq c(E)$, the subspace $c_0(E)$ is isomorphic to the subspace of $c(E)$ containing the sequences of elements of E which converge to 0.

In [6], W. Kaballo introduced the class of locally convex spaces which are ε-spaces. For us, a b-space G is an εb-space if the bounded linear mapping $Id_G\varepsilon u\colon G\varepsilon E \longrightarrow G\varepsilon F$, $f \longmapsto u \circ f$ is bornologically surjective when $u\colon E \longrightarrow F$ is a surjective bounded linear mapping between Banach spaces.

Recall that a Banach space E is an $\mathcal{L}_{\infty,\lambda}$-space, $\lambda \geq 1$, if and only if every finite-dimensional subspace F of E is contained in a finite-dimensional subspace F_1 of E such that $d(F_1, l_n^\infty) \leq \lambda$, where $n = \dim F_1$, l_n^∞ is $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the norm $\sup_{1\leq i\leq n} |x_i|$, and $d(X, Y) = \inf\{\|T\|\|T^{-1}\| : T\colon X \longrightarrow Y \text{ isomorphism}\}$ is the Banach-Mazur distance of the Banach spaces X and Y. A Banach space E is an $\mathcal{L}_\infty$-space if it is an $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$. For more information about $\mathcal{L}_\infty$-spaces we refer the reader to [7].

It is clear that any $\mathcal{L}_\infty$-space is an εb-space, the ε-product of two εb-spaces is an εb-space and a bornologically complemented subspace of an εb-space is εb-space.

Also, it is easy to show that a b-space G is an εb-space if and only if, for every bounded linear mapping $u\colon X \longrightarrow Y$ that is bornologically surjective, the bounded linear mapping $Id_G\varepsilon u\colon G\varepsilon X \longrightarrow G\varepsilon Y$, $f \longmapsto u \circ f$ is bornologically surjective, where X and Y are b-spaces. If G is an εb-space, the functor $G\varepsilon.\colon \mathbf{b} \longrightarrow \mathbf{b}$ is exact, and it follows that if E is a b-space and F is a bornologically closed subspace of E, we have $G\varepsilon(E/F) = (G\varepsilon E) / (G\varepsilon F)$.

For examples of εb-spaces, if G is a nuclear b-space (i.e. every bounded completant subset B of G is included in a bounded completant subset A of G such that the inclusion $i_{AB}\colon G_B \longrightarrow G_A$ is a nuclear mapping), by [3], there exists a net $(I,\leq)$ and a base $(B_{0,i})_{i\in I}$ of the bornology of G such that $G_{B_{0,i}}$ is isometrically isomorphic to the Banach space c_0 and $G=\cup_{i\in I}G_{B_{0,i}}$. Since c_0 is an $\mathcal{L}_\infty$-space [7] and the inductive limit is an exact functor on the category $\mathbf{b}$ [4], it follows that every nuclear b-space is an εb-space.

Recall from [1] that if X is a compact topological space and E is a b-space, we defined $C(X,E)$ as the b-space $\cup_B C(X,E_B)$, where B ranges over the bounded completant subsets of E and $C(X,E_B)$ is the space of continuous mappings from X into the Banach space E_B.

Since the Banach space $C(X)$ is an $\mathcal{L}_\infty$-space [7], the functor $C(X,\cdot):\mathbf{Ban}\longrightarrow\mathbf{Ban}$ is exact [6] and since the inductive limit is an exact functor on the category $\mathbf{b}$ [4], it follows that the functor $C(X,\cdot):\mathbf{b}\longrightarrow\mathbf{b}$ is also exact as we showed this in [1]. This implies that, if X is a compact, E is a b-space and F a bornologically closed subspace of E, then $C(X,E/F)=C(X,E)/C(X,F)$.

Now, we are in position to prove our first result.

Theorem 1 *Let X be a compact space, N a nuclear b-space and E a Banach space. Then the b-spaces $C(X,N\varepsilon E)$ and $N\varepsilon C(X,E)$ are isomorphic.*

PROOF. As the functor $C(X,\cdot):\mathbf{b}\longrightarrow\mathbf{b}$ is exact [1], the b-space $C(X,N\varepsilon E)$ is defined as the union of the Banach spaces $C(X,N_i\varepsilon E)$, where $N=\cup_{i\in I}N_i$.

On the other hand, by the definition of the ε-product of a b-space by a Banach space, we have that $N\varepsilon C(X,E)=\cup_i(N_i\varepsilon C(X,E))$.

First we shall prove that the spaces $C(X,N_i\varepsilon E)$ and $N_i\varepsilon C(X,E)$ are isomorphic. As each Banach space N_i is isometrically isomorphic to c_0 [3], we shall construct an isomorphism $C(X,c_0\varepsilon E)\longrightarrow c_0\varepsilon C(X,E)$. Since $c_0\varepsilon E\simeq c_0(E)$ for all Banach spaces E, so we have to construct an isomorphism $C(X,c_0(E))\longrightarrow c_0(C(X,E))$.

Let $f\in C(X,c_0(E))$. For each $x\in X$, $f(x)$ is a sequence $(f_n(x))_n$ of elements of E. We have got a sequence $(f_n)_n$ of continuous functions $X\longrightarrow E$. Let us prove that this sequence is in Banach space $c_0(C(X,E))$, i.e. it converges uniformly to 0 on X.

For all $\varepsilon>0$, and for all $x\in X$, let V_x be an open neighbourhood of x such that for all $x'\in V_x$, we have $\|f(x)-f(x')\|_{c_0(E)}\leq\varepsilon$. We cover X by a finite set of open subsets $\{V_{x_1},\ldots,V_{x_n}\}$. For all $i\in\{1,\ldots,n\}$, $(f(x_i))_i$ is a sequence of elements of E tending to 0. Thus there exists $m\in\mathbb{N}$ such that for all $i\in I$ and for all $n>m$, we have $\|f_n(x)\|\leq\varepsilon$. Clearly this implies that for all $x\in X$, and all $n>m$, $\|f_n(x)\|\leq 2\varepsilon$.

We have a map $C(X,c_0(E))\longrightarrow c_0(C(X,E))$ and it is immediate that this mapping preserves the norm. Let us show that it is surjective. Let $(f_n)_n$ be a sequence of continuous functions $X\longrightarrow E$ which converges uniformly to 0 on X. We define a function $X\longrightarrow c_0(E)$ by $f(x)=(f_n(x))_n$. It remains to prove its continuity. First, we can find an integer m such that for all $n>m$, and for all x, $\|f_n(x)\|\leq\varepsilon/2$. Then for $x_0\in X$, we choose neighbourhoods $V_1,\ldots,V_{m-1}$ of x_0 such that for all $x\in V_k$, $\|f_k(x)-f_k(x_0)\|\leq\varepsilon$ with $k=1,\ldots,m-1$. In the intersection of these neighbourhoods, we get $\|f(x)-f(x_0)\|_{c_0(E)}\leq\varepsilon$.

Thus for all $i\in I$, the Banach spaces $C(X,N_i\varepsilon E)$ and $N_i\varepsilon C(X,E)$ are isomorphic. If we apply the functor inductive limit which is an exact functor on the category of b-spaces [4], we obtain the result. ■

As consequences, we obtain the following results:

Corollary 1 *Let X be a compact space, N a nuclear b-space and E a b-space. Then the b-spaces $C(X,N\varepsilon E)$ and $N\varepsilon C(X,E)$ are isomorphic.*

PROOF. In fact, by definition, we have $C(X,N\varepsilon E)=\varinjlim_B C(X,N\varepsilon E_B)$. Since $C(X,N\varepsilon E_B)=N\varepsilon C(X,E_B)$, we deduce that

$$C(X,N\varepsilon E)=\varinjlim_B(N\varepsilon C(X,E_B))=N\varepsilon(\varinjlim_B C(X,E_B))=N\varepsilon C(X,E)\,. \quad ■$$

Corollary 2 *Let X be a compact space, N a nuclear b-space, E a b-space and F a bornologically closed subspace of E. Then the b-spaces $C(X, N\varepsilon(E/F))$ and $N\varepsilon C(X, E/F)$ are isomorphic.*

PROOF. Since N is an εb-space, we have $N\varepsilon(E/F) = (N\varepsilon E) / (N\varepsilon F)$. On the other hand, the Banach space $C(X)$ is an $\mathcal{L}_\infty$-space and it follows from [1], that

$$\begin{aligned} C(X, N\varepsilon(E/F)) &= C(X, (N\varepsilon E) / (N\varepsilon F)) = C(X, N\varepsilon E)/C(X, N\varepsilon F) \\ &= (N\varepsilon C\,(X, E)) / (N\varepsilon C\,(X, F)) = N\varepsilon(C(X, E)/C(X, F)) = N\varepsilon C(X, E/F). \quad \blacksquare \end{aligned}$$

Recall from [3] that the bornological projective tensor product $E \otimes_{\pi_b} F$ (resp. the bornological injective tensor product $E \otimes_{\varepsilon_b} F$) of two b-spaces E and F is defined as the b-space $\varinjlim_{B,C}(E_B \hat{\otimes}_\pi F_C)$ (resp. $\varinjlim_{B,C}(E_B \hat{\otimes}_\varepsilon F_C)$), where B (resp. C) ranges over the bounded completant subsets of E (resp. F), the inductive limit is taken in the category $\mathbf{b}$ and $E_B \hat{\otimes}_\pi F_C$ (resp. $E_B \hat{\otimes}_\varepsilon F_C$) is the completion of the space $E_B \otimes F_C$ with the projective tensor norm (resp. the injective tensor norm) given by the formula $\|z\|_\pi = \inf\{\sum_{k=1}^n \|x_k\| \|y_k\| : u = \sum_{k=1}^n x_k \otimes y_k\}$ (resp. $\|z\|_\varepsilon = \sup\{|\sum_{k=1}^n x'(x_k)\, y'(y_k)| : x' \in B_{E'}, y' \in B_{F'}\}$) where $z = \sum_{k=1}^n x_k \otimes y_k$ and $B_{E'}$, $B_{F'}$ are the closed unit balls of E', F' respectively.

Note that the complete injective tensor product $E_B \hat{\otimes}_\varepsilon F_C$ induces the same norm on $E_B \otimes F_C$ than the ε-product $E_B \varepsilon F_C$, moreover $E_B \hat{\otimes}_\varepsilon F_C$ is a closed subspace of $E_B \varepsilon F_C$. These two spaces are sometime identical, in fact, the Banach space $E_B \hat{\otimes}_\varepsilon F_C$ is isometrically isomorphic to $E_B \varepsilon F_C$ if E_B or F_C has the approximation property.

Corollary 3 *Let X be a compact space, N a nuclear b-space, E a b-space and F a bornologically closed subspace of E. Then the b-spaces $C(X, N \otimes_{\pi_b} (E/F))$ and $N \otimes_{\pi_b} C(X, E/F)$ are isomorphic.*

PROOF. Since N is a nuclear b-space, the functor $N\varepsilon.$ is exact, and hence the b-spaces $N\varepsilon(E/F)$ and $(N\varepsilon E)/(N\varepsilon F)$ are naturally isomorphic. In other hand, it follows from [3, Theorem 2, p. 78], that $N \otimes_{\pi_b} (E/F)$ and $N \otimes_{\varepsilon_b} (E/F)$ are naturally isomorphic. Next, the b-spaces $N\varepsilon(E/F)$ and $N \otimes_{\varepsilon_b} (E/F)$ are isomorphic because N has the approximation property. Finally, the result follows from Corollary 1 and Corollary 2. $\blacksquare$

Let (X, d) be a metric compact space and E be a Banach space. In [1], we defined the b-space $C(X, E)_e$ as the space $C(X, E)$ that we endow with the equicontinuous boundedness, i.e. a subset A of $C(X, E)$ is bounded if A is uniformly bounded and equicontinuous. We also showed that $C\,(X)_e\,\varepsilon E = C\,(X, E)_e$. The same result rest true when E is a b-space. In fact:

Proposition 1 *Let (X, d) be a metric compact space and let E be a b-space. Then the b-spaces $C\,(X)_e \varepsilon E$ and $C\,(X, E)_e$ are isomorphic.*

PROOF. By the definition of the ε-product of two b-spaces, we have $C\,(X)_e \varepsilon E = \cup_B (C\,(X)_e \varepsilon E_B)$. Since $C\,(X, E_B)_e = C\,(X)_e \varepsilon E_B$ and the functor $C\,(X, \cdot)_e : \mathbf{b} \longrightarrow \mathbf{b}$ is exact [1], we obtain that $\cup_B C\,(X, E_B)_e = C\,(X, \cup_B E_B)_e$. If follows that $C\,(X)_e \varepsilon E = C\,(X, E)_e$. $\blacksquare$

Proposition 2 *Let (X, d) be a metric compact space, N a nuclear b-space, E a b-space and F a bornologically closed subspace of E, then the b-spaces $C(X, N\varepsilon(E/F))_e$ and $N\varepsilon C(X, E/F)_e$ are isomorphic.*

PROOF. The functor $C\,(X, \cdot)_e : \mathbf{b} \longrightarrow \mathbf{b}$ is exact [1], then $C(X, E/F)_e = C(X, E)_e/C(X, F)_e$. It follows that

$$\begin{aligned} C(X, N\varepsilon(E/F))_e &= C(X, N\varepsilon E)_e/C(X, N\varepsilon F)_e = C\,(X)_e \varepsilon(N\varepsilon E)/C\,(X)_e \varepsilon(N\varepsilon F) \\ &= N\varepsilon(C\,(X)_e \varepsilon E)/N\varepsilon(C\,(X)_e \varepsilon F) = N\varepsilon(C(X, E)_e/C(X, F)_e) \\ &= N\varepsilon C(X, E/F)_e. \quad \blacksquare \end{aligned}$$

Finally, we will prove an analogue result of Theorem 1 for locally compact topological spaces X which are countable at infinity.
Let $(E_n)_n$ be a family of b-spaces. We endow the direct product $\prod_{n=0}^{\infty} E_n$, with the product boundedness i.e. a subset B of $\prod_{n=0}^{\infty} E_n$ is bounded if $p_n(B) = \{p_n(x) : x \in B\}$ is bounded in E_n for all $n \in \mathbb{N}$, where $p_m\colon \prod_{n=0}^{\infty} E_n \longrightarrow E_m$ is the canonical projection. It is clear that all the canonical projections $p_m\colon \prod_{n=0}^{\infty} E_n \longrightarrow E_m$ are bounded whenever we endow the space $\prod_{n=0}^{\infty} E_n$ with the product boundedness.

To prove the next Theorem (Theorem 2), we need to recall the following result which comes from [2, Proposition 3.11].

Proposition 3 *Let N be a nuclear b-space, and for all $n \in \mathbb{N}$, let E_n be a b-space. Then the b-spaces $N\varepsilon(\prod_{n=0}^{\infty} E_n)$ and $\prod_{n=0}^{\infty} (N\varepsilon E_n)$ are isomorphic.*

Theorem 2 *Let N be a nuclear b-space, E a b-space and U a locally compact space which is countable at infinity. Then the b-spaces $C\,(U, N\varepsilon E)$ and $N\varepsilon C\,(U, E)$ are isomorphic.*

PROOF. The space U is an union of a sequence of open sets U_n, each relatively compact in the interior of the following one. We consider the "disjoint union" V of the relatively compact sets U_n. For any b-space E, there exists a bounded linear mapping $\Psi\colon C\,(U, E) \longrightarrow C\,(V, E)$, which maps a function $f \in C\,(U, E)$ onto its composition with the obvious mapping $V \longrightarrow U$.

We find next a bounded linear mapping $\Psi'\colon C\,(V, E) \longrightarrow C\,(U, E)$. The locally compact space U is paracompact. We have a partition of the unity, $(\varphi_n)_n$ such that, $\sup p(\varphi_n) \subset \dot{U}_n$ for all n where $\dot{U}_n$ is the interior of U_n. Then Ψ' is defined by the formula $\Psi'((f_n)_n) = \sum_{n\in\mathbb{N}} \varphi_n f_n$.

The mapping Ψ' is a left inverse of Ψ so it is bornologically surjective. Since the nuclear b-space N is an εb-space, it follows that the bounded linear mapping $Id_N\varepsilon\Psi' : N\varepsilon C\,(V, E) \longrightarrow N\varepsilon C\,(U, E)$ is bornologically surjective. There exists also a bornologically surjective mapping $C\,(V, N\varepsilon E) \longrightarrow C\,(U, N\varepsilon E)$.

By the Proposition 3 and Corollary 2, $N\varepsilon C\,(U, E) \simeq N\varepsilon(\prod_{n=0}^{\infty} C\,(U_n, E)) \simeq \prod_{n=0}^{\infty} N\varepsilon C\,(U_n, E)$ and $C\,(U, N\varepsilon E) \simeq \prod_{n=0}^{\infty} C\,(U_n, N\varepsilon E) \simeq \prod_{n=0}^{\infty} N\varepsilon C\,(U_n, E)$, we see that $C\,(U, N\varepsilon E) \simeq N\varepsilon C\,(U, E)$. Moreover, by this isomorphism, the kernel of the mapping $N\varepsilon C\,(V, E) \longrightarrow N\varepsilon C\,(U, E)$ correspond to the kernel of the mapping $C\,(V, N\varepsilon E) \longrightarrow C\,(U, N\varepsilon E)$. ■

Corollary 4 *Let U be a locally compact space that is countable at infinity, N a nuclear b-space, E a b-space and F a bornologically closed space of E. Then the b-spaces $C(U, N\varepsilon(E/F))$ and $N\varepsilon C(U, E/F)$ are isomorphic.*

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