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A Sharp Estimate for Bilinear Littlewood-Paley Operator

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Abstract. We establish a sharp estimate for bilinear Littlewood-Paley operator. As application, we obtain the weighted norm inequalities and $L \log L$ type estimate for the bilinear operator

Una estimación fina del operador bilineal de Littlewood-Paley

Resumen. Se establece una estimación fina para el operador bilineal de Littlewood-Paley. Como aplicación se obtienen desigualdades para la norma ponderada y estimaciones del tipo $L \log L$ para el operador bilineal.

1 Introduction

It is well known that the singular integral operators and their commutators are of importance in many applications (see [5, 9, 16]). As the development of the singular integral operators, their commutators and multilinear operators have been well studied (see [2–7, 12–15]. Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rochberg and Weiss (see [5]) states that the commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in BMO(\mathbb{R}^n)$) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In [2–4], Cohen and Gosselin study the L^p $(p > 1)$ boundedness of the multilinear singular integral operator T^A defined by

$$T^A(f)(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(A; x, y)}{|x-y|^m} K(x, y) f(y)\, dy.$$

However, it has known that the commutator $[b, T]$ is not bounded, in general, from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. In [13], C. Pérez proves that the commutator $[b, T]$ satisfies a $L \log L$ type estimate. In [10], Hu and Yang obtain a variant sharp estimate for the multilinear singular integral operators. The main purpose of this paper is to establish a sharp estimate for the bilinear operator associated to the Littlewood-Paley operator and $BMO(\mathbb{R}^n)$ function.

2 Preliminaries and Theorems

In this paper, we will study a class of bilinear operators related to Littlewood-Paley operators, whose definitions are the following.

Let ψ be a function on $\mathbb{R}^n$ which satisfies the following properties:

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1. $\int \psi(x)\,dx = 0$;
2. $|\psi(x)| \le C(1+|x|)^{-(n+1)}$;
3. $|\psi(x+y)-\psi(x)| \le C\,|y|\,(1+|x|)^{-(n+2)}$ when $2|y| < |x|$.

Let m be a positive integer and let A be a function on $\mathbb{R}^n$. The bilinear Littlewood-Paley operator is defined by

$$g_\psi^A(f)(x) = \left[\int_0^\infty |F_t^A(f)(x)|^2 \,\frac{dt}{t}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)\,\psi_t(x-y)}{|x-y|^m}\, R_{m+1}(A;x,y)\,dy,$$

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t>0$. Set $F_t(f)(x) = f * \psi_t(x)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |f*\psi_t(x)|^2\,\frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley operator (see [16]).

Let H be the Hilbert space $H = \left\{h : \|h\| = \left(\int_0^\infty |h(t)|^2\,\frac{dt}{t}\right)^{1/2} < \infty\right\}$. Then for each fixed $x \in \mathbb{R}^n$, $F_t^A(f)(x)$ and $F_t(f)(x)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x) = \|F_t^A(f)(x)\|, \qquad g_\psi(f)(x) = \|F_t(f)(x)\|.$$

Note that when $m=0$, g_ψ^A is just the commutator of the Littlewood-Paley operator (see [11]). While when $m>0$, it is non-trivial generalizations of the commutators. It is well known that the Littlewood-Paley operator, as the vector-valued singular integral operators, is of great interest in harmonic analysis (see [15]). The purpose of this paper is to establish a sharp estimate for the bilinear operator, then the weighted norm inequalities and $L\log L$ type estimate for the bilinear operator are obtained by using the sharp estimate. We point out that some of our ideas in this paper come from the paper [1] of Álvarez and Pérez.

First, let us introduce some notation (see [8, 9, 13]).

For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy,$$

where, and in what follows, Q will denote a cube with sides parallel to the axes, and $f_Q = |Q|^{-1}\int_Q f(x)\,dx$. It is well-known that

$$f^\#(x) = \sup_{x\in Q}\inf_{c\in C} \frac{1}{|Q|}\int_Q |f(y)-c|\,dy.$$

We say that f belongs to $BMO(\mathbb{R}^n)$ if $f^\#$ belongs to $L^\infty(\mathbb{R}^n)$. For $0<r<\infty$, we denote $f_r^\#$ by

$$f_r^\#(x) = \left[(|f|^r)^\#(x)\right]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

We write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. For $k \in \mathbb{N}$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \qquad \text{when } k \geq 2.$$

Let B be a Young function and $\tilde{B}$ be the complementary associated to B. Set, for a function f,

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\right\}.$$

The maximal function associated to $\|f\|_{B,Q}$ is defined by

$$M_B(f)(x) = \sup_{x\in Q} \|f\|_{B,Q}.$$

The main Young function to be using in this paper is $B(t) = t\,(1 + \log^+ t)$ and its complementary $\tilde{B}(t) \approx \exp t$, the corresponding maximal functions denoted by $M_{L\log L}$ and $M_{\exp L}$. From [13], we have the generalized Hölder's inequality

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|\,dy \leq \|f\|_{B,Q}\,\|g\|_{\tilde{B},Q}$$

and the following equivalence, for any $x \in \mathbb{R}^n$,

$$M_{L\log L}(f)(x) \approx CM^2(f)(x).$$

From the John-Nirenberg inequality (see [9]), we have the following inequalities, for all cube Q and any $b \in BMO(\mathbb{R}^n)$,

$$\|b - b_Q\|_{\exp L,Q} \leq C\|b\|_{BMO}$$

and

$$|b_{2^{k+1}Q} - b_{2Q}| \leq 2k\|b\|_{BMO}.$$

We denote the Muckenhoupt weights by A_p for $1 \leq p < \infty$ (see [9]).

Now we are in position to state our results.

Theorem 1 *Let $D^\alpha A \in BMO(\mathbb{R}^n)$, $|\alpha| = m$. Then for any $0 < r < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(\mathbb{R}^n)$,*

$$(g_\psi^A(f))_r^{\#}(x) \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M^2(f)(x).$$

Theorem 2 *Let $1 < p < \infty$ and $D^\alpha A \in BMO(\mathbb{R}^n)$, $|\alpha| = m$, $\omega \in A_p$. Then g_ψ^A is bounded on $L^p(w)$, that is*

$$\|g_\psi^A(f)\|_{L^p(w)} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^p(w)}.$$

Theorem 3 *Let $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$ and $w \in A_1$. Then there exists a constant $C > 0$ such that for each $\lambda > 0$,*

$$w(\{x \in \mathbb{R}^n : g_\psi^A(f)(x) > \lambda\}) \leq C\Phi\Big(\sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}\Big) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)\,dx,$$

where $\Phi(t) = t(1 + \log^+ t)$.

As in [13, 15], Theorem 2 and 3 follow from Theorem 1. So we only need to prove Theorem 1.

3 Some lemmas

We begin with some preliminary lemmas.

Lemma 1 (Kolmogorov, [9, p. 485]) *Let $0 < p < q < \infty$ and for any function $f \geq 0$. Set*

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^n : f(x) > \lambda\}\right|^{1/q},$$

$$N_{p,q}(f) = \sup_E \frac{\|f\chi_E\|_{L^p}}{\|\chi_E\|_{L^r}}, \qquad (1/r = 1/p - 1/q),$$

where the sup *is taken for all measurable sets E with $0 < |E| < \infty$. Then*

$$\|f\|_{WL^q} \leq N_{p,q}(f) \leq \left(\frac{q}{q-p}\right)^{1/p} \|f\|_{WL^q}.$$

Lemma 2 ([2, p. 448]) *Let A be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q \, dz\right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}\,|x-y|$.

Lemma 3 ([13, p. 165]) *Let $w \in A_1$. Then there exists a constant $C > 0$ such that for any function f and for all $\lambda > 0$,*

$$w(\{y \in \mathbb{R}^n : M^2(f)(y) > \lambda\}) \leq C\lambda^{-1} \int_{\mathbb{R}^n} |f(y)|\,(1 + \log^+(\lambda^{-1}|f(y)|))\, w(y)\, dy.$$

Lemma 4 *Let $1 < p < \infty$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$, $1 < r \leq \infty$, $1/q = 1/p + 1/r$. Then g_ψ^A is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, that is*

$$\|g_\psi^A(f)\|_{L^q} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{L^r} \|f\|_{L^p}.$$

PROOF. By Minkowski inequality and the condition of ψ, we have

$$\begin{aligned} g_\psi^A(f)(x) \leq & \int_{\mathbb{R}^n} \frac{|f(y)|\,|R_{m+1}(A;x,y)|}{|x-y|^m} \left(\int_0^\infty |\psi_t(x-y)|^2 \frac{dt}{t}\right)^{1/2} dy \\ \leq & C \int_{\mathbb{R}^n} \frac{|f(y)|\,|R_{m+1}(A;x,y)|}{|x-y|^m} \left(\int_0^\infty \frac{t^{-2n}}{(1+|x-y|/t)^{2(n+1)}} \frac{dt}{t}\right)^{1/2} dy \\ \leq & C \int_{\mathbb{R}^n} \frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n}} |f(y)|\, dy, \end{aligned}$$

thus, the lemma follows from [6, 7]. ■

4 Proof of Theorems

We only need to prove Theorem 1.

PROOF OF THEOREM 1. For $x \in \mathbb{R}^n$, let $Q = Q(x_0, d)$ be a cube centered at x_0 and having side length d such that $x \in Q$. It is sufficient to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0, that the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |g_\psi^A(f)(x) - C_0|^r dx\right)^{1/r} \le CM^2(f)(\tilde{x}).$$

Set $\tilde{Q} = 5\sqrt{n}\,Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus \tilde{Q}}$,

$$\begin{aligned} F_t^A(f)(x) = &\int_{\mathbb{R}^n} \frac{\psi_t(x-y)R_{m+1}(\tilde{A}; x, y)}{|x-y|^m} f_2(y)\,dy \\ &+ \int_{\mathbb{R}^n} \frac{\psi_t(x-y)R_m(\tilde{A}; x, y)}{|x-y|^m} f_1(y)\,dy \\ &- \sum_{|\alpha|=m} \frac{1}{\alpha!}\int_{\mathbb{R}^n} \frac{\psi_t(x-y)(x-y)^\alpha D^\alpha \tilde{A}(y)}{|x-y|^m} f_1(y)\,dy, \end{aligned}$$

then

$$\begin{aligned} \left|g_\psi^A(f)(x) - g_\psi^{\tilde{A}}(f_2)(x_0)\right| &= \left|\|F_t^A(f)(x)\| - \|F_t^{\tilde{A}}(f_2)(x_0)\|\right| \\ &\le \left|\|F_t^A(f)(x) - F_t^{\tilde{A}}(f_2)(x_0)\|\right| \\ &\le \left\|F_t\left(\frac{R_m(\tilde{A}; x, \cdot)}{|x-\cdot|^m} f_1\right)(x)\right\| + \sum_{|\alpha|=m}\frac{1}{\alpha!}\left\|F_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1\right)(x)\right\| \\ &\quad + \|F_t^{\tilde{A}}(f_2)(x) - F_t^{\tilde{A}}(f_2)(x_0)\| \\ &\equiv I(x) + II(x) + III(x), \end{aligned}$$

thus,

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \left|g_\psi^A(f)(x) - g_\psi^{\tilde{A}}(f_2)(x_0)\right|^r dx\right)^{1/r} \\ &\le \left(\frac{C}{|Q|}\int_Q I(x)^r\,dx\right)^{1/r} + \left(\frac{C}{|Q|}\int_Q II(x)^r dx\right)^{1/r} + \left(\frac{C}{|Q|}\int_Q III(x)^r\,dx\right)^{1/r} \\ &\equiv I + II + III. \end{aligned}$$

Now, let us estimate I, II and III, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, using Lemma 2, we get

$$R_m(\tilde{A}; x, y) \le C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO},$$

thus, by Lemma 1 and the weak type (1,1) of g_ψ (see [11, 16]), we obtain

$$\begin{aligned} I &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}\, |Q|^{-1} \frac{\|g_\psi(f_1)\chi_Q\|_{L^r}}{\|\chi_Q\|_{L^{r/(1-r)}}} \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}\, |Q|^{-1} \|g_\psi(f_1)(f_1)\|_{WL^1} \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}\, |\tilde{Q}|^{-1} \int_{\tilde{Q}} |f(y)|\, dy \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}\, M(f)(\tilde{x}); \end{aligned}$$

For II, similar to the proof of I, we get

$$\begin{aligned} II &\le C \sum_{|\alpha|=m} |Q|^{-1} \frac{\|g_\psi(D^\alpha \tilde{A} f_1)\chi_Q\|_{L^r}}{\|\chi_Q\|_{L^{r/(1-r)}}} \\ &\le C \sum_{|\alpha|=m} |Q|^{-1} \|g_\psi(D^\alpha \tilde{A} f_1)\|_{WL^1} \\ &\le C \sum_{|\alpha|=m} |Q|^{-1} \int_{\tilde{Q}} |D^\alpha \tilde{A}(y)| |f(y)|\, dy \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\exp L, \tilde{Q}} \|f\|_{L \log L, \tilde{Q}} \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M_{L \log L} f(\tilde{x}) \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M^2(f)(\tilde{x}); \end{aligned}$$

To estimate III, we write,

$$\begin{aligned} &F_t^{\tilde{A}}(f_2)(x) - F_t^{\tilde{A}}(f_2)(x_0) \\ &\quad = \int_{\mathbb{R}^n} \left[\frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x_0-y)}{|x_0-y|^m} \right] R_m(\tilde{A}; x, y) f_2(y)\, dy \\ &\qquad + \int_{\mathbb{R}^n} \frac{\psi_t(x_0-y) f_2(y)}{|x_0-y|^m} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)]\, dy \\ &\qquad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left(\frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m} - \frac{\psi_t(x_0-y)(x_0-y)^\alpha}{|x_0-y|^m} \right) D^\alpha \tilde{A}(y) f(y)\, dy \\ &\quad = III_1 + III_2 + III_3. \end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in \mathbb{R}^n \setminus \tilde{Q}$. By Lemma 3 and the following inequality (see [9])

$$|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|)\, \|b\|_{BMO}, \qquad \text{for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}$,

$$\begin{aligned} |R_m(\tilde{A}; x, y)| &\le C|x-y|^m \sum_{|\alpha|=m} (\|D^\alpha A\|_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|) \\ &\le Ck|x-y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}; \end{aligned}$$

By the condition of ψ, and similar to the proof of Lemma 4, we obtain

$$\begin{aligned}\|III_1\| \leq & C\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{m+n+1}}|R_m(\tilde{A};x,y)\||f(y)|\,dy\\ \leq & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|\,dy\\ \leq & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=1}^\infty k2^{-k}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|dy\\ \leq & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=1}^\infty k2^{-k/2}M(f)(\tilde{x})\\ \leq & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}M(f)(\tilde{x});\end{aligned}$$

For III_2, by the formula (see [2]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|\leq C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}\|D^\alpha A\|_{BMO},$$

similar to the estimates of III_1, we get

$$\begin{aligned}\|III_2\| \leq & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|\,dy\\ \leq & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}M(f)(\tilde{x});\end{aligned}$$

For III_3, similar to the estimates of III_1, we get

$$\begin{aligned}\|III_3\| \leq & C\sum_{|\alpha|=m}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}|D^\alpha\tilde{A}(y)\||f(y)|dy\\ \leq & C\sum_{|\alpha|=m}\sum_{k=1}^\infty k2^{-k}(\|D^\alpha A\|_{\exp L,2^k\tilde{Q}}\|f\|_{LlogL,2^k\tilde{Q}}+\|D^\alpha A\|_{BMO}M(f)(\tilde{x}))\\ \leq & C\sum_{|\alpha|=m}\sum_{k=1}^\infty k(2^{-k}+2^{-k/2})\|D^\alpha A\|_{BMO}M_{L\log L}(f)(\tilde{x})\\ \leq & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}M^2(f)(\tilde{x}).\end{aligned}$$

Thus,

$$III\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}M^2(f)(\tilde{x}).$$

This completes the proof of Theorem 1. ■

From Theorem 1 and the weighted boundedness of g_ψ and M, we may obtain the conclusion of Theorem 2.

From Theorem 1 and Lemma 3, we may obtain the conclusion of Theorem 3.

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