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A result of existence for an original convection-diffusion equation

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Abstract. In this paper, we are interested in the mathematical analysis of a non-classical conservation law. The model describes stratigraphic processes of the geology, taking into account a limited weathering condition. Firstly, we present the physical topic and the mathematical formulation, which lead to an original conservation law. Then, the definition of a solution and some mathematical tools in order to prove the existence of a solution.

Un resultado de existencia para una ecuación original de difusión-convección

Resumen. En este artículo se estudia el análisis matemático de una ley de conservación que no es clásica. El modelo describe procesos estatigráficos en Geología y tiene en cuenta una condición de tasa de erosión limitada. En primer lugar se presentan el modelo físico y la formulación matemática (posiblemente nueva). Tras enunciar la definición de solución, se presentan las herramientas que permiten probar la existencia de soluciones.

1 Introduction

In this paper, we are interested in the mathematical analysis of a stratigraphic model developed by the Institut Français du Pétrole (IFP). The model concerns geologic basin formation by the way of erosion and sedimentation. By taking into account large scale in time and space and by knowing *a priori*, the tectonics, the eustatism and the sediments flux at the basin boundary, the model has to state about the transport of sediments. One may find, on the one hand, in D. Greanjeon *and al.* [10] and R. Eymard *and al.* [7] the physical and the numerical modelling of the multi-lithological case, and, on the other hand, in S.N. Antontsev *and al.* [2], G. Gagneux *and al.* [8], [9] a mathematical analysis of the mono-lithological case.

Let us consider in the sequel a sedimentary basin, denoted by Ξ with base Ω a smooth, bounded domain in $\mathbb{R}^N$ ($N = 1, 2$), assumed horizontal. In order to describe the theoretical topography $u(t, x)$ of the basin for a.e. (t, x) in $Q =]0, T[\times\Omega$, the gravitational model has to take into account:
i) a sediment flux assumed to be proportional to $\overrightarrow{q}_{th} = \nabla[u + \tau\partial_t u]$, where τ is a nonnegative parameter,
ii) an erosion speed, $\partial_t u$ in its nonpositive part, that is underestimated by $-E$, where E is a given nonnegative bounded measurable function in Q (a weathering limited process): i.e. $\partial_t u + E \geq 0$ in Q.

The original aspect of this model is its weather limited condition on the erosion rate.

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In order to join together the constraint and a conservative formulation, D. Greanjeon [10] proposes to correct the diffusive flux $\overrightarrow{q}_{th}$ by introducing a dimensionless multiplier λ : the new flux is then $\overrightarrow{q}_{cor} = -\lambda\, \nabla[u + \tau\partial_t u]$, where λ is an unknown function with values *a priori* in $[0, 1]$. Then, we will have to add a relevant law of state in the form $\lambda = \lambda(\partial_t u)$ for posing satisfactory closure conditions.

Therefore, the mathematical modelling has to express respectively:

- the mass balance of the sediment: $\partial_t u - \mathrm{div}(\lambda\nabla[u + \tau\partial_t u]) = 0 \quad in \quad Q$.
- the boundary conditions on $\partial\Omega = \overline{\Gamma_e} \cup \overline{\Gamma_s}$ specifying the input and output fluxes:

$$\begin{array}{ccc} \overrightarrow{q}_{cor} = f & \text{on} &]0, T[\times\Gamma_e, \\ \partial_t u + E \geq 0, \quad f \geq \overrightarrow{q}_{cor} & \text{and} & (f - \overrightarrow{q}_{cor})(\partial_t u + E) = 0 \text{ on }]0, T[\times\Gamma_s. \end{array}$$

- the weather limited condition (moving obstacle depending on climate, bathymetry, ...):

$$\partial_t u \geq -E \quad \text{in} \quad Q. \tag{1}$$

- the initial condition $u_{|t=0} = u_0$ and the closure of the model by defining the role of the flux limitor λ.

In order to simplify, one considers in the sequel homogeneous Dirichlet conditions on the boundary. G. Gagneux, D. Etienne and G. Vallet [6] have considered boundary conditions of unilateral type with $\tau = 0$. The mathematical analysis is inspired by the chapter 2 of G. Duvaut and J. L. Lions [5] and by the "new problems" of J. L Lions [11, p. 420], dealing of thermic. Then, the problem of the regularity of $\partial_t u$ arises in view of defining the trace of $\partial_t u$ on Γ_s. If $\tau = 0$, $\partial_t u$ belongs *a priori* to $L^2(Q)$ and not to $L^2(0, T, H^1(\Omega))$. What justifies the interest for the study of the case $\tau > 0$.

In order to give a mathematical modelling of λ, Th. Gallouët and R. Masson [7] propose to consider the following global constraint, leading to a non-standard free boundary problem:

$$\partial_t u + E \geq 0, \quad 1 - \lambda \geq 0 \quad \text{and} \quad (\partial_t u + E)(1 - \lambda) = 0 \quad \text{in } Q. \tag{2}$$

It means that the flux has to be corrected because of the constraint (*i.e.* if $\lambda < 1$, then $\partial_t u + E = 0$). If the erosion rate constraint is inactive, the flux is equal to the diffusive one.

Then S. N. Antontsev, G. Gagneux and G. Vallet [2] propose (if $\tau = 0$) the following conservative formulation that contains implicitly (1). See G. Vallet [14] too. If $\mathbf{H}$ denotes the maximal monotone graph of the Heaviside function, then (λ, h) formally solves:

$$0 = \partial_t u - \mathrm{div}(\lambda\nabla[u + \tau\partial_t u]) \quad \text{where} \quad \lambda \in \mathbf{H}(\partial_t u + E) \quad \text{in } Q.$$

This general problem remains an open problem and the aim of this paper is to give some mathematical results when $\mathbf{H}$ is replaced by a continuous function a, an approximation Yosida of $\mathbf{H}$ for example. Another physical motivation can be found in [12] and [13] for describing the hysteresis phenomena of pore water.

2 Presentation of the model

Thus, the problem of the stratigraphic motion may be written:

find u *a priori* in $L^2(0, T, V)$ with $\partial_t u$ in $L^2(0, T, V)$ where $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$ such that, $\partial_t u + E \geq 0$ $a.e.\ in\ Q$ and $\forall v \in V$, for a.e. t in $]0, T[$,

$$\int_\Omega \{\partial_t uv + a(\partial_t u + E)\nabla[u + \tau\partial_t u]\nabla v\}\, dx \quad = \quad 0, \tag{3}$$

together with the initial condition $u_{|t=0} = u_0$ a.e. in Ω.

In the sequel $\tau > 0$, $E : [0, T] \to \mathbb{R}$ is a non negative continuous function and a is a continuous function defined on $\mathbb{R}^+$ that satisfies: $0 \leq a \leq M, \quad a(0) = 0$ and $a(x) > 0$ if $x > 0$, in order to have implicitly $\partial_t u + E \geq 0$ a.e. in Q and $A(x) = \int_0^x a(s)\,ds$. Moreover, and for technical reasons (the motivation is to derive various ways of approximating the Heaviside graph by Yosida regularized functions), one assumes that $a \circ A^{-1}$ has got on $\mathbb{R}^+$ a continuity modulus whose square is an Osgood function; for simplicity, we consider the case where $a \circ A^{-1}$ is Hölder continuous with exponent $\frac{1}{2}$.

Remark 1 *1. The problem can be written in the non-classical form:*

$$\begin{cases} \partial_t u - \tau\Delta A(\partial_t u + E) - \operatorname{div}(a(\partial_t u + E)\nabla u) = 0 \text{ and } \partial_t u + E \geq 0 \quad \text{in } Q, \\ u = 0 \quad \text{and} \quad \partial_t u = 0 \quad \text{in }]0, T[\times\partial\Omega, \quad u_{|t=0} = u_0 \quad \text{in } \Omega. \end{cases}$$

2. If by extension $a(x) = 0$ for any negative x, the suitable test function $v = (\partial_t u + E)^-$ leads to $\partial_t u + E \geq 0$ a.e. in Q. So, one may assumes in the sequel such an hypothesis on the function a.

3. If $u_0 \in V$, then u and $\partial_t u$ belong to $L^\infty(0, T, V)$.
By considering $\varepsilon > 0$ and $v = \int_0^{\partial_t u} \frac{ds}{a(s+E)+\varepsilon}$ in the variational formulation, one gets:

$$\frac{1}{M}||\partial_t u||_H^2 + \int_\Omega \nabla[u + \tau\partial_t u]\nabla\partial_t u\,dx \leq 0, \tag{4}$$

and

$$\frac{1}{M}||\partial_t u||^2_{L^2(0,t,H)} + \frac{1}{2}||\nabla u(t)||^2_{H^N} + \tau||\nabla\partial_t u||^2_{L^2(0,t,H^N)} \leq \frac{1}{2}||\nabla u_0||^2_{H^N}. \tag{5}$$

Note that this is also true for any t since $u \in C_s([0, T], V)$, i.e. $u(t)$ exists in V for any t.
Let us consider again inequality (4) *to obtain with the above estimates the assertion:*

$$\frac{1}{M}||\partial_t u||_H^2 + \frac{\tau}{2}||\nabla\partial_t u||^2_{H^N} \leq C||\nabla u_0||^2_{H^N} \quad \blacksquare$$

3 Existence of a strong solution

The key for understanding compactness properties in the regularizing procedure is the following assertion:

Lemma 1 *Let us consider κ given in $H^1(\Omega)$, E a real number and b a bounded essentially nonnegative continuous function such that $b \circ B^{-1}$ is Hölder continuous with exponent $\frac{1}{2}$, $B' = b$. Then, there exists at most one solution w in V such that for any v in V,*

$$\int_\Omega \{wv + b(w + E)\nabla[\kappa + w]\nabla v\}\,dx = 0. \tag{6}$$

PROOF. If one denotes by w_1 and w_2 two admissible solutions, one gets

$$\int_\Omega \{(w_1 - w_2)v + \nabla[B(w_1 + E) - B(w_2 + E)]\nabla v + [b(w_1 + E) - b(w_2 + E)]\nabla\kappa\nabla v\}\,dx = 0.$$

For a given $\mu > 0$, let us set $p_\mu(t) = 1_{[\mu,+\infty[}(t) + \ln(\frac{e}{\mu}t)1_{[\frac{\mu}{e},\mu[}(t)$ and $v = p_\mu(\xi)$ where $\xi = B(w_1 + E) - B(w_2 + E)$; therefore, it comes

$$\int_\Omega (w_1 - w_2)p_\mu(\xi)\,dx + \int_{\Omega\cap\{\frac{\mu}{e}\leq\xi\leq\mu\}} \frac{1}{2\xi}\nabla^2\xi\,dx \quad \leq \quad C\int_{\Omega\cap\{\frac{\mu}{e}\leq\xi\leq\mu\}} \nabla^2\kappa\,dx,$$

and the solution is unique by considering the limits when μ goes to 0^+. $\blacksquare$

3.1 Semi-discretized degenerated processes

Let us consider $h > 0$, u_0 in V, $E \geq 0$ and for a given positive α, $a_\alpha = \max(\alpha, a)$. We record for later use the following assertion thanks to Schauder-Tychonov fixed point theorem and lemma 1.

Proposition 1 *There exists a unique u_α in V such that, $\forall v \in V$,*

$$\int_\Omega \{\frac{u_\alpha - u_0}{h} v + a_\alpha(\frac{u_\alpha - u_0}{h} + E)\nabla[u_\alpha + \tau \frac{u_\alpha - u_0}{h}]\nabla v\}\, dx = 0. \tag{7}$$

We derive from the proposition 7 and the lemma 1 a version of the degenerated case.

Proposition 2 *Let us consider $h > 0$, u_0 in V, then there exists a unique u in V such that, $\forall v \in V$,*

$$\int_\Omega \{\frac{u - u_0}{h} v + a(\frac{u - u_0}{h} + E)\nabla[u + \tau \frac{u - u_0}{h}]\nabla v\}\, dx = 0, \quad \frac{u - u_0}{h} + E \geq 0 \text{ a.e. in } \Omega. \tag{8}$$

3.2 Semi-discretized degenerated differential inclusion

Proposition 3 *There exists (u, λ) in $V \times L^\infty(\Omega)$ such that, $\lambda \in \mathbf{H}(\frac{u-u_0}{h} + E)$ where $\mathbf{H}$ is the maximal monotone graph of the function of Heaviside, and $\forall v \in V$,*

$$\int_\Omega \left\{\frac{u - u_0}{h} v + \lambda \nabla[u + \tau \frac{u - u_0}{h}]\nabla v\right\} dx = 0. \tag{9}$$

PROOF. Let us assume in this section that for any given positive ε, $a = a_\varepsilon = \max(0, \min(1, \frac{1}{\varepsilon} Id))$.
The corresponding solutions are denoted by u_ε and thanks to the above inequality, a sub-sequence still indexed by ε can be extracted, such that u_ε converges weakly in V towards u, strongly in H and a.e. in Ω. Moreover, $\frac{u-u_0}{h} + E \geq 0$ a.e. in Ω. Furthermore, A_ε converges uniformly towards $(.)^+$ and then $A_\varepsilon(\frac{u_\varepsilon - u_0}{h} + E)$ converges weakly in V towards $\frac{u-u_0}{h} + E$.
Up to a new sub-sequence, let us denote by λ the weak$-*$ limit in $L^\infty(\Omega)$ of $a_\varepsilon(\frac{u_\varepsilon - u_0}{h} + E)$ and remark that inevitably $\lambda = 1$ a.e. in $\{\frac{u-u_0}{h} + E > 0\}$ i.e. $\lambda \in \mathbf{H}(\frac{u-u_0}{h} + E)$.
Passing to the limits in the variational relation stating u_ϵ and since $\nabla \frac{u_\varepsilon - u_0}{h} = 0$ in $\{\lambda \neq 1\}$, one gets (9).
■

3.3 Existence of a strong solution

Inductively, the following result can be proved:
let us consider $N \in \mathbb{N}^*$ with $h = \frac{T}{N}$, u_0 in V and $E^k \geq 0$ for any integer k.

Proposition 4 *For $u^0 = u_0$, there exists a unique sequence $(u^k)_k$ in V such that, $\forall v \in V$,*

$$\int_\Omega \{\frac{u^{k+1} - u^k}{h} v + a(\frac{u^{k+1} - u^k}{h} + E^k)\nabla[u^{k+1} + \tau \frac{u^{k+1} - u^k}{h}]\nabla v\}\, dx = 0.$$

Moreover, $\frac{u^{k+1} - u^k}{h} + E^k \geq 0$ a.e. in Ω,

$$\frac{1}{M}||\frac{u^{k+1} - u^k}{h}||_H^2 + \tau||\frac{u^{k+1} - u^k}{h}||_V^2 + \frac{1}{2h}[||u^{k+1}||_V^2 + ||u^{k+1} - u^k||_V^2 - ||u^k||_V^2] \leq 0. \tag{10}$$

For any sequence $(v_k)_k \subset H$, let us note in the sequel

$$v^h = \sum_{k=0}^{N-1} v^{k+1} 1_{[kh,(k+1)h[} \quad \text{and} \quad \widetilde{v}^h = \sum_{k=0}^{N-1} [\frac{v^{k+1} - v^k}{h}(t - kh) + v^k] 1_{[kh,(k+1)h[}.$$

Then, from (10), one has that $(u^h)_h$ and $(\widetilde{u}^h)_h$ are bounded sequences in $L^\infty(0,T,V)$ and that $(\partial_t\widetilde{u}^h)_h$ is a bounded sequence in $L^2(0,T,V)$. Still coming from (10), one gets

$$\frac{1}{M}||\frac{u^{k+1}-u^k}{h}||_H^2+\tau||\frac{u^{k+1}-u^k}{h}||_V^2\leq\frac{1}{2\tau}||u^{k+1}||_V^2$$

and $(\partial_t\widetilde{u}^h)_h$ is a bounded sequence in $L^\infty(0,T,V)$.

In particular, $(\widetilde{u}^h)_h$ is bounded in $H^1(0,T,V)$ and there exists a sub-sequence, still indexed by h such that for any t, $\widetilde{u}^h(t)\rightharpoonup u(t)$ in V.

Moreover, if $t\in[kh,(k+1)h[$, $||u^h(t)-\widetilde{u}^h(t)||_V=||\widetilde{u}^h(kh)-\widetilde{u}^h(t)||_V\leq\int_{kh}^{(k+1)h}||\partial_t\widetilde{u}^h(s)||_V\,ds\leq Ch$. Then, for any t, $u^h(t)\rightharpoonup u(t)$ in V.

Since $\partial_t\widetilde{u}^h$ is bounded in $L^\infty(0,T,V)$, it follows that for any t of Z where $Z\subset[0,T]$ is a measurable set such that $\mathcal{L}([0,T]\backslash Z)=0$, $\partial_t\widetilde{u}^h(t)$ is a bounded sequence in V.

Therefore, up to a sub-sequence indexed by h_t $\partial_t\widetilde{u}^{h_t}\rightharpoonup\xi(t)$ in V, strongly in L^2 and a.e. in Ω with $\xi(t)+E(t)\geq 0$ a.e. in Ω. Let us note that for any t in $[kh,(k+1)h[$,

$$\forall v\in V,\quad\int_\Omega\{\partial_t\widetilde{u}^hv+a(\partial_t\widetilde{u}^h+E^h)\nabla[u^h+\tau\partial_t\widetilde{u}^h]\nabla v\}\,dx=0.\tag{11}$$

Given that, $a(\partial_t\widetilde{u}^{h_t}+E)\nabla v$ converges towards $a(\xi(t)+E)\nabla v$ in $L^2(\Omega)^N$ and that $\nabla[u^{h_t}+\varepsilon\partial_t\widetilde{u}^{h_t}]$ converges weakly towards $\nabla[u(t)+\varepsilon\xi(t)]$ in $L^2(\Omega)^N$, $\xi(t)$ is a solution to the problem: at time t, find w in V with $w+E(t)\geq 0$ a.e. in Ω, such that for any v in V,

$$\int_\Omega\{wv+a(w+E(t))\nabla[u(t)+\tau w]\nabla v\}\,dx=0.\tag{12}$$

Thanks to Lemma 1 with $b=\tau\,a$ in $\mathbb{R}^+$ and $\kappa=\frac{1}{\tau}u(t)$, the solution $\xi(t)$ is unique and all the sequence $\partial_t\widetilde{u}^h(t)$ converges towards $\xi(t)$ weakly in V. For any f in $H^{-1}(\Omega)$, since $t\mapsto<f,\xi(t)>$ is the limit of the sequence of measurable functions $t\mapsto<f,\partial_t\widetilde{u}^h(t)>$, it is a measurable function thanks to Pettis theorem (K. Yosida [16, p. 131]), since V is a separable set.

For any v in $L^2(0,T,V)$, $(\partial_t\widetilde{u}^h(t),v(t))$ converges a.e. in $]0,T[$ towards $(\xi(t),v(t))$. Moreover, $|(\partial_t\widetilde{u}^h(t),v(t))|\leq C||v(t)||_V$ a.e. since the sequence $(\partial_t\widetilde{u}^h)_h$ is bounded in $L^\infty(0,T,V)$. Thus, the weak convergence in $L^2(0,T,V)$ of $\partial_t\widetilde{u}^h$ towards ξ can be proved. And one gets that $\xi=\partial_tu$.

At last, for t a.e. in $]0,T[$, passing to limits in (11) leads to the existence of a solution.

4 About the differential inclusion

On the one hand, one may remark that the estimations (10) remain for the sequence of solutions given by (9). Moreover, the sequence of associated parameters (λ^k) is bounded in $L^\infty(\Omega)$ and then, one gets:

$$\forall v\in L^2(0,T,V),\quad\int_Q\{\partial_t\widetilde{u}^hv+\lambda^h\nabla[u^h+\tau\partial_t\widetilde{u}^h]\nabla v\}\,dxdt=0,\tag{13}$$

where the same *a priori* estimates hold for the sequences $(\widetilde{u}^h)_h$ and $(u^h)_h$ than the one given in the existence section. Furthermore, (λ^h) is bounded in $L^\infty(Q)$ and $\lambda^h\in\mathbf{H}(\partial_t\widetilde{u}^h+E^h)$.

On the other hand, if $a=a_\varepsilon=\max(0,\min(1,\frac{1}{\varepsilon}Id))$ and if u_ε is a solution to the problem (3), the remarks to the presentation section lead to:
(u_ε) (resp. $(\lambda_\varepsilon)_\varepsilon=(a_\varepsilon(\partial_tu_\varepsilon+E))_\varepsilon$) is a bounded sequence in $W^{1,\infty}(0,T,V)$ (resp. $L^\infty(Q)$) and

$$\forall v\in L^2(0,T,V),\quad\int_Q\{\partial_tu_\varepsilon v+\lambda_\varepsilon\nabla[u_\varepsilon+\tau\partial_tu_\varepsilon]\nabla v\}\,dxdt=0.\tag{14}$$

In both cases, each accumulation point provides a “mild solution” in the sense of Ph. Bénilan *and al.* [4] but, as already mentioned in G. Gagneux *and al.* [8] the double weak convergence in the term $\lambda_\varepsilon\nabla u_\varepsilon$ does not allow us to pass to limits.

5 Conclusion and open problems

A solution has been found in $Lip([0,T], H_0^1(\Omega))$ to the degenerated problem

$$\begin{cases} \partial_t u - \tau\Delta A(\partial_t u + E) - \mathrm{div}(a(\partial_t u + E)\nabla u) = 0 & \text{in } Q, \\ \partial_t u + E \geq 0 & \text{in } Q, \\ u = 0 \quad \text{and} \quad \partial_t u = 0 \quad \text{in }]0,T[\times\partial\Omega, \quad u_{|t=0} = u_0 & \text{in } \Omega. \end{cases}$$

In order to conclude that this problem is well-posed in the sense of Hadamard, one still has to prove that such a solution is unique. This is still an open problem, mainly due to a behaviour of hysteresis type of the equation.

Let us cite a recent paper of Z. Wang *and al.* [15] where the uniqueness of the solution to a weakly similar equation has been proved. The method is based on a Holmgren approach.

Another possibility consists in getting information on the localization and the finite speed propagation of ground *via* the methods of S.N. Antontsev *and al.*'s book [1].

The above solution is a solution to a perturbation of the real problem since a has to be the graph of the Heaviside function in order to satisfy the condition (2). This problem is open too as mentioned in the section: About the differential inclusion.

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