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The Peano curves as limit of α-dense curves

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Abstract. In this paper we present a characterization of the Peano curves as the uniform limit of sequences of α-dense curves contained in the compact that it is filled by the Peano curve. These α-dense curves must have densities tending to zero and coordinate functions with variation tending to infinite as α tends to zero.

Las curvas de Peano como límite de curvas α-densas

Resumen. En este artículo presentamos una caracterización de las curvas de Peano como límite uniforme de sucesiones de curvas α-densas en el compacto que es llenado por la curva de Peano. Estas curvas α-densas deben tener densidades tendiendo a cero y sus funciones coordenadas deben de ser de variación tendiendo a infinito cuando α tiende a cero.

1 Introduction

In a metric space (E, d), given a compact set K and a real number $\alpha \geq 0$, an α-dense curve (more information on these curves may be found in [4]) in K is a continuous mapping $\gamma_\alpha : I \to E$, with $I = [0, 1]$, satisfying

i) the image $\gamma_\alpha(I)$, from now on noted γ_α^*, is contained in K,

ii) for any $x \in K$, the distance $d(x, \gamma_\alpha^*) \leq \alpha$.

Whenever $\alpha = 0$, one has a Peano curve provided that the interior of K to be non-void. The minimal α verifying the two preceding properties is, strictly speaking, the density of the curve in K, which coincides with the Hausdorff distance $d_{\mathcal{H}}(K, \gamma_\alpha^*)$ (see [2]).

A compact subset K in (E, d) is said to be densifiable if it contains α-dense curves for arbitrary $\alpha > 0$. For example, in $\mathbb{R}^N$, $N \geq 1$, any cube $\prod_{i=1}^{N} [a_i, b_i]$ is densifiable. Any Peano Continuum, that is, a connected and locally connected compact set, is also densifiable. However, there exist densifiable sets which are not Peano Continua; for instance

$$K = \left\{ \left(x, \sin \frac{1}{x} \right) : 0 < x \leq 1 \right\} \cup \left\{ (0, y) : -1 \leq y \leq 1 \right\}.$$

Therefore, the α-density concept produces a new class, the densifiable sets, which is strictly between the class of Peano Continua and the class of connected and precompact sets.

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Let f be a function defined on a real interval, for brevity we take the unit interval I, and valued on a metric space (E, d). We recall that the total variation of f, noted $V_I(f)$, is defined as

$$V_I(f) \equiv \sup_{\sigma} \left\{ \sum_{i=1}^{n} d(f(t_i), f(t_{i-1})) : \sigma \equiv \{t_0, t_1, \ldots, t_n\} \subset I \,; t_0 < t_1 < \cdots < t_n \right\}.$$

Whenever $V_I(f) < \infty$, it is well-known that f is called of bounded variation on I (detailed properties of these functions can be found, for instance, in [1] or also in [6, Vol. I]). In particular, given a continuous mapping $\gamma : I \to \mathbb{R}^N$, i.e., a curve γ, the total variation $V_I(\gamma)$ is also called the length, written $L(\gamma)$. Whether $V_I(\gamma)$ is finite, the curve is said to be rectifiable and its length may be determined (see [1, theorem 24-6]) by

$$L(\gamma) = \lim_{|\Pi| \to 0} \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1}))\| \,,$$

Π being the partition

$$\Pi = \{t_0, t_1, \ldots, t_n\}\,; \quad 0 = t_0 < t_1 < \cdots < t_n = 1$$

with norm

$$|\Pi| \equiv \max\{t_i - t_{i-1} \,;\, i = 1, \ldots, n\}\,.$$

The variation of a curve may be infinite even for very regular one, such as the following example shows (see [8, p. 53]).

Example 1 *The coordinate functions γ_1, γ_2 of the spiral $\gamma = (\gamma_1, \gamma_2) : I \to I^2$ defined by*

$$\gamma_1(t) = \begin{cases} t\cos\frac{2\pi}{t} & \text{if } 0 < t \le 1 \\ 0 & \text{if } t = 0 \end{cases} \qquad \gamma_2(t) = \begin{cases} t\sin\frac{2\pi}{t} & \text{if } 0 < t \le 1 \\ 0 & \text{if } t = 0 \end{cases}$$

are both of infinite variation.

2 The theorem of characterization

The Hahn-Mazurkiewicz theorem (see [7]) assures that every Peano Continuum set in a metrizable space is the continuous image of the unit interval, and reciprocally. Since the unit square I^2 is a Peano Continuum, it may be taken as a good prototype of the image of a Peano curve, so we shall state our theorem of characterization in that set.

Theorem 1 *A continuous mapping $\gamma = (\gamma_1, \gamma_2) : I \to I^2$ is a Peano curve filling I^2 if and only γ if is the uniform limit of a sequence of α-dense curves $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)})$ in I^2 with densities $\alpha_n \to 0$, for which there is no constant K such that the variation $V_I(\gamma_i^{(n)}) \le K$, for all n, for some $i = 1, 2$.*

PROOF. First we prove the sufficiency. Let P be an arbitrary point of I^2; because of the density, for each n there exists $t_n \in I$ such that the euclidean distance

$$d(P, \gamma^{(n)}(t_n)) \le \alpha_n.$$

By the Bolzano-Weierstrass theorem, given the sequence $(t_n)_n$ there exists a subsequence, noted in the same way, that converges to some $t \in I$. For arbitrary n, we consider the inequality

$$d(P, \gamma(t)) \le d(P, \gamma^{(n)}(t_n)) + d(\gamma^{(n)}(t_n), \gamma^{(n)}(t)) + d(\gamma^{(n)}(t), \gamma(t))\,. \tag{1}$$

Thus, since $\alpha_n \to 0$ and γ is the uniform limit of γ_n, from the continuity of the curves and taking the limit in (1) when $n \to \infty$, the distance $d(P, \gamma(t)) = 0$. Therefore, the point $P = \gamma(t)$ and so γ is a Peano curve that fills I^2.

For proving the necessity, observe that if $\gamma = (\gamma_1, \gamma_2)$ is a Peano curve filling I^2, then each coordinate function γ_1, γ_2 is necessarily surjective onto I. We assume firstly that $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)})$, $n = 1, 2, \ldots$, is a sequence of curves in I^2 uniformly convergent

$$\lim_{n\to\infty} \gamma^{(n)} = \gamma, \tag{2}$$

and prove that latter. ■

Denoting by α_n the density in I^2 of each curve $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)})$, one has

$$\lim_{n\to\infty} \alpha_n = 0. \tag{3}$$

Indeed, if (3) is not true, then there exists $\epsilon > 0$ such that for any k there is an integer N_k so that $\alpha_{N_k} > \epsilon$. Thus we can select a subsequence of curves of densities $\alpha_{N_k} > \epsilon$ for $k = 1, 2, 3, \ldots$. From (2) the limit of this subsequence is also γ, so denoting the subsequence in the same way, we determine, for each n, a point P_n such that

$$\epsilon < d(P_n, \gamma_n^*) \le \alpha_n. \tag{4}$$

Since $(P_n)_n$ belongs to the compact I^2, there exists a subsequence, noted in the same way, that converges to some point $P \in I^2$. Because of the continuity of the distance function, and taking into account that γ is the uniform limit of γ_n, given $0 < \delta < \epsilon$, there exists a sufficiently large n such that

$$|d(P, \gamma_n^*) - d(P_n, \gamma_n^*)| < \frac{\delta}{2}; \qquad |d(P, \gamma^*) - d(P, \gamma_n^*)| < \frac{\delta}{2}. \tag{5}$$

From (5) and (4), one has

$$d(P, \gamma^*) = d(P, \gamma^*) - d(P, \gamma_n^*) + d(P, \gamma_n^*) - d(P_n, \gamma_n^*) + d(P_n, \gamma_n^*) > -\frac{\delta}{2} - \frac{\delta}{2} + d(P_n, \gamma_n^*) > \epsilon - \delta,$$

which is absurd because $d(P, \gamma^*) = 0$. Therefore (3) is showed.

For each $i = 1, 2$, consider the Banach indicatrix Φ_{γ_i} of each coordinate function γ_i on the interval $[0, 1]$, that is, the function on I defined by

$$\Phi_{\gamma_i}(y) = \begin{cases} +\infty & \text{if } \mathrm{card}(\gamma_i^{-1}(y)) \ge \omega \\ \mathrm{card}(\gamma_i^{-1}(y)) & \text{if } \mathrm{card}(\gamma_i^{-1}(y)) < \omega \end{cases}$$

ω being the first infinite cardinal. Φ_{γ_i} is measurable and satisfies the integral formula

$$\int_0^1 \Phi_{\gamma_i}(y)dy = V_I(\gamma_i) \tag{6}$$

(a proof can be found in [3] or [6]). Nevertheless Φ_{γ_i} is identically equal to $+\infty$ on I, so from (6)

$$V_I(\gamma_i) = \infty, \qquad i = 1, 2. \tag{7}$$

Suppose the existence of a constant K such that $V_I(\gamma_i^{(n)}) \le K$, for all n, for some $i = 1, 2$. Thus, as $0 \le \gamma_i^{(n)}(t) \le 1$ for any $t \in I$, by applying the Helly's first theorem (see [6, Vol. I, p.222]), γ_i would be of finite variation and it contradicts (7).

Now, it only remains to prove that, given a Peano curve $\gamma = (\gamma_1, \gamma_2)$ filling I^2 there exists a sequence $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)})$, $n = 1, 2, \ldots$, of curves in I^2 verifying (2). For that, consider the class $\mathcal{C}$ of all rectangles $C = J_1 \times J_2$ of I^2, where J_1, J_2 are intervals contained in I, and define on this class the set function μ by

$$\mu(C) = \Lambda_1 \left[\gamma_1^{-1}(J_1) \cap \gamma_2^{-1}(J_2)\right], \tag{8}$$

Λ_1 being the Lebesgue measure on the real line $\mathbb{R}$.

One can easily check that formula (8) defines a Borel measure on the unit square, wich will be also denoted μ. This measure, associated to the Peano curve γ, satisfies

a) $\mu(C) > 0$ for any rectangle C with interior non-void,

b) $\mu(I^2) = 1$.

Now, for each $n = 1, 2, \ldots$ consider a partition $\Pi_n = \left\{ C_k^{(n)} : k = 1, 2, \ldots, 2^{2n} \right\}$ formed by 2^{2n} equal and disjoint subsquares of I^2, arranged in such a way that $C_k^{(n)}$ to be adjacent to $C_{k-1}^{(n)}$ for $k = 2, \ldots, 2^{2n}$. Furthermore, inductively, given the partition Π_n, the next one $\Pi_{n+1} = \left\{ C_k^{(n+1)} : k = 1, 2, \ldots, 2^{2(n+1)} \right\}$, obtained by dividing each square $C_k^{(n)}$ into four new squares $C_{k,i}^{(n)}$, $i = 1, \ldots, 4$, is arranged by defining

$$C_{4(k-1)+i}^{(n+1)} = C_{k,i}^{(n)}, k = 1, 2, \ldots, 2^{2n}, i = 1, \ldots, 4.$$

From the properties a), b), the 2^{2n} subintervals

$$\begin{aligned} I_1^{(n)} &= \left[0, \mu(C_1^{(n)})\right) \\ I_2^{(n)} &= \left[\mu(C_1^{(n)}), \mu(C_1^{(n)}) + \mu(C_2^{(n)})\right) \\ &\vdots \\ I_{2^{2n}}^{(n)} &= \left[\mu(C_1^{(n)}) + \mu(C_2^{(n)}) + \cdots + \mu(C_{2^{2n}-1}^{(n)}), 1\right] \end{aligned}$$

define a partition of I.

Given n, for each $k = 1, 2, \ldots, 2^{2n}$, we distinguish an arbitray interior point of each square $C_k^{(n)}$, for instance its center, noted $P_k^{(n)} = (x_k^{(n)}, y_k^{(n)})$, and define on I the functions

$$\begin{aligned} h_1^{(n)}(t) &= x_k^{(n)}, \qquad t \in I_k^{(n)}, \\ h_2^{(n)}(t) &= y_k^{(n)}, \qquad t \in I_k^{(n)}. \end{aligned}$$

Observe that, for each n, $h_1^{(n)}$, $h_2^{(n)}$ are, possibly, discontinuous at the points $t_j = \sum_{i=1}^{j} \mu(C_i^{(n)}$, $j = 1, 2, \ldots, 2^{2n} - 1$. However, the sequences $\left(h_1^{(n)}\right)_n$, $\left(h_2^{(n)}\right)_n$ are uniformly convergent to two continuous functions, say γ_1', γ_2', respectively (consult [5]). Therefore one defines a curve $\gamma' = (\gamma_1', \gamma_2')$ which coincides with $\gamma = (\gamma_1, \gamma_2)$, if we take into account that, for each n, the mapping $\gamma'^{(n)}(t) = \left(h_1^{(n)}(t), h_2^{(n)}(t)\right)$, $t \in I$, coincide with $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $t \in I$, at least at 2^{2n} values for t, corresponding to the 2^{2n} centers of the subsquares $C_k^{(n)}$ of the partition Π_n.

To eliminate the discontinuity of $h_1^{(n)}$, $h_2^{(n)}$, we proceed to make a linear interpolation. Hence, consider a partition of I formed by the subintervals

$$\begin{aligned} J_1^{(n)} &= \left[0, \frac{2^{2n}-1}{2^{2n}} \mu(C_1^{(n)})\right], \\ K_1^{(n)} &= \left[\frac{2^{2n}-1}{2^{2n}} \mu(C_1^{(n)}), \mu(C_1^{(n)}) + \frac{1}{2^{2n}} \mu(C_2^{(n)})\right], \end{aligned}$$

$$J_2^{(n)} = \left[\mu(C_1^{(n)}) + \frac{1}{2^{2n}}\mu(C_2^{(n)}), \mu(C_1^{(n)}) + \frac{2^{2n}-1}{2^{2n}}\mu(C_2^{(n)})\right],$$
$$K_2^{(n)} = \left[\mu(C_1^{(n)}) + \frac{2^{2n}-1}{2^{2n}}\mu(C_2^{(n)}), \mu(C_1^{(n)}) + \mu(C_2^{(n)}) + \frac{1}{2^{2n}}\mu(C_3^{(n)})\right],$$
$$\vdots$$
$$K_{2^{2n}-1}^{(n)} = \Big[\mu(C_1^{(n)}) + \mu(C_2^{(n)}) + \cdots + \frac{2^{2n}-1}{2^{2n}}\mu(C_{2^{2n}-1}^{(n)}),$$
$$\mu(C_1^{(n)}) + \mu(C_2^{(n)}) + \cdots + \mu(C_{2^{2n}-1}^{(n)}) + \frac{1}{2^{2n}}\mu(C_{2^{2n}}^{(n)})\Big],$$
$$J_{2^{2n}}^{(n)} = \left[\mu(C_1^{(n)}) + \mu(C_2^{(n)}) + \cdots + \mu(C_{2^{2n}-1}^{(n)}) + \frac{1}{2^{2n}}\mu(C_{2^{2n}}^{(n)}), 1\right].$$

and define, for each n, the new functions $f_1^{(n)}$, $f_2^{(n)}$ by

$$f_1^{(n)}(t) = h_1^{(n)}(t) \qquad \text{if } t \in J_k^{(n)}, k = 1, 2, \ldots, 2^{2n},$$
$$f_1^{(n)}(t) = x_j^{(n)} + \frac{x_{j+1}^{(n)} - x_j^{(n)}}{s_j^{(n)} - r_j^{(n)}}(t - r_j^{(n)}) \qquad \text{if } t \in K_j^{(n)}, j = 1, 2, \ldots, 2^{2n}-1$$

and

$$f_2^{(n)}(t) = h_2^{(n)}(t) \qquad \text{if } t \in J_k^{(n)}, k = 1, 2, \ldots, 2^{2n},$$
$$f_2^{(n)}(t) = y_j^{(n)} + \frac{y_{j+1}^{(n)} - y_j^{(n)}}{s_j^{(n)} - r_j^{(n)}}(t - r_j^{(n)}) \qquad \text{if } t \in K_j^{(n)}, j = 1, 2, \ldots, 2^{2n}-1$$

where $r_j^{(n)}, s_j^{(n)}$ are the end-points of $K_j^{(n)}$.

From the uniform convergence of $\left(h_1^{(n)}\right)_n$, $\left(h_2^{(n)}\right)_n$ to γ_1, γ_2, it follows easily that the sequences $\left(f_1^{(n)}\right)_n$, $\left(f_2^{(n)}\right)_n$ also converge uniformly to γ_1, γ_2, respectively, if we take into account that $J_k^{(n)} \subset I_k^{(n)}$, for all $k = 1, 2, \ldots, 2^{2n}$, and $K_j^{(n)}$ is a closed neighbourhood of t_j of length $\frac{1}{2^{2n}}\left(\mu(C_j^{(n)}) + \mu(C_{j+1}^{(n)})\right)$ for all $j = 1, 2, \ldots, 2^{2n}-1$. Therefore, by defining, for each n, $\gamma^{(n)} = \left(f_1^{(n)}, f_2^{(n)}\right)$ we have definitely a sequence of curves satisfying (2). Now the proof is complete.

Suppose we apply this last theorem, thus the following is immediate.

Corollary 1 *Let $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)})$ be an arbitrary sequence of cartesian (for all n is $\gamma_1^{(n)} = I_d$, the identity) α-dense curves in I^2 with densities $\alpha_n \to 0$. Thus $\left(\gamma^{(n)}\right)_n$ has no uniform limit.*

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