RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **98 (**1), 2004, pp. 259–271
Ciencias de la Computación / Computational Sciences

On MPT-implication functions for Fuzzy Logic

Enric Trillas, Claudi Alsina and Ana Pradera

Abstract. This paper deals with numerical functions $J : [0,1] \times [0,1] \to [0,1]$ able to functionally express operators $\to$: $[0,1]^X \times [0,1]^Y \to [0,1]^{X \times Y}$ defined as $(\mu \to \sigma)(x,y) = J(\mu(x), \sigma(y))$, and verifying either Modus Ponens or Modus Tollens, or both. The concrete goal of the paper is to search for continuous t-norms T and strong-negation functions N for which it is either $T(a, J(a,b)) \leq b$ (Modus Ponens), or $T(N(b), J(a,b)) \leq N(a)$ (Modus Tollens), or both, for all a, b in $[0,1]$ and a given J. Functions J are taken among those in the most usual families considered in Fuzzy Logic, namely, R-implications, S-implications, Q-implications and Mamdani-Larsen implications. *En passant*, the cases of conditional probability and material conditional's probability are analyzed.

Sobre implicaciones MPT en la Lógica Borrosa

Resumen. Los operadores de implicación borrosos $\to$: $[0,1]^X \times [0,1]^Y \to [0,1]^{X \times Y}$ se suelen expresar por medio de funciones numéricas $J : [0,1] \times [0,1] \to [0,1]$ de acuerdo con la igualdad $(\mu \to \sigma)(x,y) = J(\mu(x), \sigma(y))$. El presente artículo estudia la verificación por parte de estas funciones numéricas de las meta-reglas del Modus Ponens y Modus Tollens. En concreto, dada una de estas funciones J, el objetivo es determinar qué t-normas continuas T y qué funciones de negación fuerte N verifican alguna (o ambas) de las desigualdades $T(a, J(a,b)) \leq b$ (Modus Ponens) y $T(N(b), J(a,b)) \leq N(a)$ (Modus Tollens), para cualesquiera a, b en $[0,1]$. Las funciones J se toman entre las pertenecientes a las familias más habitualmente utilizadas en Lógica Borrosa, esto es, R-implicaciones, S-implicaciones, Q-implicaciones e implicaciones de Mamdani-Larsen.

1. Introduction

As it is well-known, the implication operation $\to$: $B \times B \to B$ in a Boolean Algebra $(B, +, \cdot, ')$ is usually defined by means of the so-called *material implication*, given by $a \to b = a' + b$ for every a, b in B. Nevertheless, the consideration that an implication is not only used to represent conditional statements of the form "If a, then b", but also, and mainly, to perform inferences, allows for a broader definition of this kind of operations. Indeed, since the two main classical inference rules are *Modus Ponens* (MP) and *Modus Tollens* (MT) (rules that allow, respectively, to perform forward and backward inferences), the following definition can be established: an operation $\to$: $B \times B \to B$ is an *implication* if for every $a, b \in B$, it is $a \cdot (a \to b) \leq b$ (MP inequality) and $b' \cdot (a \to b) \leq a'$ (MT inequality). Due to the special properties of Boolean Algebras, and taking into account that in such structures the inequality $x \cdot z \leq y$ is easily shown to be equivalent to $z \leq x' + y$ for any $x, y, z \in B$, the two conditions of the above definition appear to be equivalent, collapsing into $a \to b \leq a' + b$ for any $a, b \in B$. Therefore, despite the material implication is

Presentado por Luis M. Laita.
Recibido: December 5, 2003. *Aceptado*: October 13, 2004.
Palabras clave / Keywords: Modus Ponens; Modus Tollens; Fuzzy Implication; Fuzzy Inference.
Mathematics Subject Classifications: 03B52, 68T27.

the greatest boolean implication, it is not the only one (concretely, [8] showed that there are, actually, seven different classes of boolean implications).

The above discussion was restricted to the particular case of Boolean Algebras, but it can of course be generalized to the case of more general lattices $(L, +, \cdot)$ endowed with a negation operator $'$. In this context, the most general expression for a material implication is, perhaps, $a \to b = a' \cdot (b + b') + a \cdot b$. When the negation is a complement (i.e., it verifies the laws $x \cdot x' = 0$ and $x + x' = 1$ for any x in L), the above expression reduces to $a \to b = a' + a \cdot b$. If, in addition, an orthomodular lattice is taken, it can be proven that the former expression is an implication in the sense that it verifies both the MP and the MT inequalities. Nevertheless, contrary to the boolean case, these inequalities are, in general, no longer equivalent. It then makes sense to establish the following definitions for an implication operation in a lattice structure:

Definition 1 *Let $(L, +, \cdot)$ be a lattice endowed with a negation operator $'$. An operation $\to : L \times L \to L$ will be called:*

- *a MP-implication whenever it is $a \cdot (a \to b) \le b$ for every $a, b \in L$.*

- *a MT-implication whenever it is $b' \cdot (a \to b) \le a'$ for every $a, b \in L$.*

- *a MPT-implication whenever it is both a MP and a MT-implication.*

Note that, as it was proved before, if L is a Boolean Algebra then every MP-implication is a MT-implication, and reciprocally.

The aim of this paper is to apply the above ideas to the context of Fuzzy Logics, thus suggesting a revision of the concepts of fuzzy implication and fuzzy inference. Although the fulfillment of the Modus Ponens or the Modus Tollens inequalities in Fuzzy Logics has been partially considered before (see for example references [4], [9], [10] and [11]), this paper proposes a systematic study of both inequalities and of their joint satisfaction. To such end, the paper is organized as follows: section 2 reviews the concepts of implication and inference as they have traditionally been understood in Fuzzy Logics, and then proposes the revision of the definition of fuzzy implication in the light of the approach that was just exposed. The next sections (3 to 6) study how the traditional families of fuzzy implications fit into this new definition. Finally, section 7 briefly analyzes the case of Probabilistic Logics.

2. Fuzzy Implications and Fuzzy Inference

We will deal with Standard Theories of Fuzzy Sets, where the connectives *not*, *and* and *or* are represented, respectively, by means of the well-known classes of *strong negation functions* and continuous *triangular norms* (t-norms) and *triangular conorms* (t-conorms) (see for example [3] or [5]). In order to fix the notation that will be used in this paper, we recall some basic well-known results on these operations:

- a strong negation is a function $N_\varphi : [0,1] \to [0,1]$ defined as $N_\varphi = \varphi^{-1} \circ (1 - Id) \circ \varphi$, where $\varphi : [0,1] \to [0,1]$, called a generator of N_φ, is an order automorphism of the unit interval (i.e., an strictly increasing function such that $\varphi(0) = 0$ and $\varphi(1) = 1$).

- continuous t-norms may be classified as $\{Min\} \cup \mathcal{F}(Prod) \cup \mathcal{F}(W) \cup \Sigma_t$, where $Prod(x, y) = x \cdot y$ is the product t-norm; $W(x, y) = Max(0, x + y - 1)$ is the Łukasiewicz t-norm; for any t-norm T, $\mathcal{F}(T)$ represents the set $\{T_\varphi : T_\varphi = \varphi^{-1} \circ T \circ (\varphi \times \varphi)\}$, where φ is any order automorphism of $[0,1]$, and, finally, Σ_t is the family of ordinal sums. Remember that only t-norms in $\mathcal{F}(W)$ have zero divisors, and that $W_\varphi(x, y) = 0$ is equivalent to $y \le N_\varphi(x)$.

- with a similar notation, the set of continuous t-conorms may be classified as $\{Max\} \cup \mathcal{F}(Prod^*) \cup \mathcal{F}(W^*) \cup \Sigma_s$, where now $Prod^*(x, y) = x + y - x \cdot y$ and $W^*(x, y) = Min(1, x + y)$ are, respectively, the Product and the Łukasiewicz t-conorms.

In this paper, only those continuous t-norms and continuous t-conorms which are not ordinal sums will be consider. Therefore, in the following, any reference to a continuous t-norm T or a continuous t-conorm S should be understood as $T \in \{Min\} \cup \mathcal{F}(Prod) \cup \mathcal{F}(W)$ and $S \in \{Max\} \cup \mathcal{F}(Prod^*) \cup \mathcal{F}(W^*)$.

Regarding *fuzzy implications functions*, these are defined as operations $J : [0,1] \times [0,1] \to [0,1]$ verifying some basic properties. In general, they are built by means of continuous t-norms, continuous t-conorms and strong negations, and the most usual ones belong to the following four families (details on the first three families, which generalize the classical material implication, may be found, for instance, in [3], [4] or [5]; the fourth class, which includes operators commonly used in fuzzy control, is a generalization of the boolean implication $x \cdot y$, and has been studied in [1]).

1. *Residuated or R-implications*, defined as $J_T(a,b) = sup\{z \in [0,1] : T(a,z) \leq b\}$.
2. *Strong or S-implications*, defined as $J(a,b) = S(N(a),b)$.
3. *Quantum Logic or Q-implications*, defined as $J(a,b) = S(N(a),T(a,b))$.
4. *Mamdani-Larsen or ML-implications*, defined as $J(a,b) = T(\varphi_1(a), \varphi_2(b))$, where φ_1 is an order automorphism on $[0,1]$ and $\varphi_2 : [0,1] \to [0,1]$ is a non-null contractive mapping (i.e., $\varphi_2(x) \leq x$ for all $x \in [0,1]$).

With respect to *fuzzy inference*, this is usually performed by means of the well-known *Generalized Modus Ponens* and *Generalized Modus Tollens* schemes, that generalize to the fuzzy world the two classical inference rules. These inference schemes are described as follows:

Generalized Modus Ponens	*Generalized Modus Tollens*
If x is P, then y is Q	If x is P, then y is Q
x is P^*	y is not Q^*
y is Q^*	x is not P^*

In the above schemes, P, P^* and Q, Q^* are fuzzy statements defined, respectively, on some universes X and Y, and the goal is to compute Q^* (respectively "not P^*") in such a way that the facts "y is Q^*" (respectively, "x is not P^*") can be considered as soundly inferred from the given premises. In order to solve this problem, which was first addressed by Zadeh in [12], the following assumptions are made:

- the fuzzy statements P, P^*, Q and Q^* are represented by means of fuzzy sets $\mu_P, \mu_{P^*} : X \to [0,1]$ and $\mu_Q, \mu_{Q^*} : Y \to [0,1]$, and their negations are modelled using some strong negation N;
- the fuzzy rule "If x is P, then y is Q" is interpreted in terms of a fuzzy relation $R : X \times Y \to [0,1]$, which is normally considered to be functionally expressible as $R(x,y) = J(\mu_P(x), \mu_Q(y))$, where $J : [0,1] \times [0,1] \to [0,1]$ is a fuzzy implication function;
- the conjunction of the premises is frequently performed by means of a continuous triangular norm T.

With the former representations in mind, the two stated problems are solved using the so-called *Compositional Rules of Inference* (CRI), that provide, respectively, the following results:

$$\mu_{Q^*}(y) = \sup_{x \in X} T(\mu_{P^*}(x), J(\mu_P(x), \mu_Q(y))) \text{ for all } y \text{ in } Y$$

and

$$N(\mu_{P^*}(x)) = \sup_{y \in Y} T(N(\mu_{Q^*}(y)), J(\mu_P(x), \mu_Q(y))) \text{ for all } x \text{ in } X.$$

A fuzzy inference framework able to perform both forward and backward approximate reasoning involves therefore the use of three fuzzy connectives: a fuzzy implication J, a fuzzy conjunction T and a fuzzy negation N. Since there are several choices for each of them, the important question of how to select them arises. This is a design problem that has been approached from several points of view (see for example [2], [7]), but for which a general solution is still missing. To that respect, it seems clear that the choice of the fuzzy implication cannot be made independently of the inference rules it is going to be used with. This leads us to propose the following revision regarding the definition of fuzzy implication, which is established accordingly with the general ideas on these operators that were discussed in the introduction:

Definition 2 *Let T be a continuous t-norm and N a strong negation. A function $J : [0,1] \times [0,1] \to [0,1]$ will be called:*

- *a MP-implication for the t-norm T whenever it is $T(a, J(a,b)) \leq b$ for any a, b in $[0,1]$;*

- *a MT-implication for the couple (T, N) whenever it is $T(N(b), J(a,b)) \leq N(a)$ for any a, b in $[0,1]$;*

- *a MPT-implication for the couple (T, N) whenever it is a MP-implication for T and a MT-implication for (T, N).*

Note that the use of MP, MT or MPT-implication functions is important in order to avoid one of the main troubles encountered when using the CRI results, namely, the possibility of having fuzzy inference patterns that do not coincide with the classical ones in the special cases where $P^* = P$ (in the case of the generalized Modus Ponens) or $Q^* = Q$ (when using the generalized Modus Tollens). Indeed, if, for example, J is not a MP-implication for T, there will be values $a_0, b_0 \in [0,1]$ such that $T(a_0, J(a_0, b_0)) > b_0$, and then the inequality $sup_{a \in [0,1]} T(a, J(a,b)) \leq b$ will not be guaranteed. In these cases, the CRI will produce the undesirable situation in which, if $\mu_{P^*} = \mu_P$, μ_{Q^*} is not coincidental with μ_Q. The importance of MT-implications can be shown in a similar way.

The next theorem establishes general characterizations for the concepts of MP, MT and MPT-implications:

Theorem 1 *Let T be a continuous t-norm, N a strong negation, $J_T : [0,1] \times [0,1] \to [0,1]$ the R-implication associated to T and $J_T^\delta : [0,1] \times [0,1] \to [0,1]$ the function defined by $J_T^\delta(a,b) = J_T(b,a)$ for any $a, b \in [0,1]$. The following statements are true:*

(1) *A function $J : [0,1] \times [0,1] \to [0,1]$ is a MP-implication for T if and only if $J \leq J_T$;*

(2) *A function $J : [0,1] \times [0,1] \to [0,1]$ is a MT-implication for (T, N) if and only if $J \leq J_T^\delta \circ (N \times N)$.*

(3) *A function $J : [0,1] \times [0,1] \to [0,1]$ is a MPT-implication for (T, N) if and only if $J \leq Min(J_T, J_T^\delta \circ (N \times N))$.*

PROOF. (1) is obvious since it is well-known that $T(a, J(a,b)) \leq b$ is equivalent to $J(a,b) \leq J_T(a,b)$. This result is also used to obtain (2): indeed, according to the former equivalence, it follows that $T(N(b), J(a,b)) \leq N(a)$ is equivalent to $J(a,b) \leq J_T(N(b), N(a))$, and $J_T(N(b), N(a)) = J_T^\delta \circ (N \times N)(a,b)$, where $J_T^\delta(a,b) = J_T(b,a)$. ■

Note that in the particular case where $T = W_\varphi$ and $N = N_\varphi$, the functions J_T and $J_T^\delta \circ (N \times N)$ are the same. Indeed, it is easy to see that $J_{W_\varphi}(a,b)$ is equal to $W_\varphi^*(N_\varphi(a), b)$ for any $a, b \in [0,1]$. On the other side, $J_{W_\varphi}^\delta \circ (N_\varphi \times N_\varphi)(a,b) = J_{W_\varphi}^\delta(N_\varphi(a), N_\varphi(b)) = W_\varphi^*(N_\varphi(N_\varphi(b)), N_\varphi(a)) = J_{W_\varphi}(a,b)$. The following result can therefore be stated:

Proposition 1 *Let $\varphi : [0,1] \to [0,1]$ be an order automorphism and $J : [0,1] \times [0,1] \to [0,1]$ a given function. The following statements are equivalent:*

(i) *J is a MP-implication for W_φ.*

(ii) J *is a MT-implication for* (W_φ, N_φ).

(iii) J *is a MPT-implication for* (W_φ, N_φ).

On the other hand, the following necessary conditions for the verification of the MP inequality are very simple but also very useful:

Proposition 2 *Let T be a continuous t-norm. If a function $J : [0,1] \times [0,1] \to [0,1]$ is a MP-implication for T, then the two following conditions are necessarily fulfilled:*

(MP1) $J(1,b) \leq b$ *for all* $b \in [0,1]$

(MP2) $T(a, J(a,0)) = 0$ *for all* $a \in [0,1]$

PROOF. (MP1) and (MP2) are obtained by just taking, respectively, the values $a = 1$ and $b = 0$ in the MP inequality. ■

The next result is an immediate consequence of property (MP2), since, as it was recalled before, t-norms in the family of Łukasiewicz are the only ones having zero divisors.

Corollary 1 *Let T be a continuous t-norm and $J : [0,1] \times [0,1] \to [0,1]$ a function such that $J(a,0) \neq 0$ for some $a \neq 0$. If J is a MP-implication for T, then it is necessarily $T = W_\varphi$ for some order automorphism φ, and condition* (MP2) *may be written as $J(a,0) \leq N_\varphi(a)$ for all $a \in [0,1]$.*

Similar considerations may be done for the MT inequality; the results obtained in this case are summarized as follows (their proofs are, as in the previous case, immediate):

Proposition 3 *Let T be a continuous t-norm and N a strong negation. If a function $J : [0,1] \times [0,1] \to [0,1]$ is a MT-implication for (T, N), then the two following conditions are necessarily fulfilled:*

(MT1) $J(a,0) \leq N(a)$ *for all* $a \in [0,1]$

(MT2) $T(N(b), J(1,b)) = 0$ *for all* $b \in [0,1]$

Corollary 2 *Let T be a continuous t-norm and $J : [0,1] \times [0,1] \to [0,1]$ a function such that $J(1,b) \neq 0$ for some $b \neq 1$. If J is a MT-implication for (T,N), where N is any strong negation, then it is necessarily $T = W_\varphi$ for some order automorphism φ, and condition* (MT2) *may be written as $J(1,b) \leq N_\varphi(N(b))$ for all $a \in [0,1]$.*

The following sections make use of all these results in order to study when the four mentioned families of traditional fuzzy implications are MP or MT implications, and, as a consequence, when these functions are MPT-implications.

3. The case of R-implications

In this section we will consider two continuous t-norms T and T_1, a strong negation N, and we will study when the residuated implication J_{T_1} is a MP or a MT implication for the couple (T,N). Let us first of all recall the following characteristics of R-implications:

- The values of J_{T_1} for the three main families of continuous t-norms are the following:
  - If $T_1 = Min$, $J_{T_1}(a,b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{otherwise} \end{cases}$
  - If $T_1 = W_{\varphi_1}$, $J_{T_1}(a,b) = W^*_{\varphi_1}(N_{\varphi_1}(a), b)$

– If $T_1 = Prod_{\varphi_1}$, $J_{T_1}(a,b) = \begin{cases} 1, & \text{if } a \leq b \\ \varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)), & \text{otherwise} \end{cases}$

- $J_{T_1}(1,b) = b$ for any b in $[0,1]$.
- Regarding the value $J_{T_1}(a,0)$, the following distinction has to be made:

  - if $T_1 = Min$ or $T_1 = Prod_{\varphi_1}$, then $J_{T_1}(0,0) = 1$ and $J_{T_1}(a,0) = 0$ for any $a \neq 0$;

  - if $T_1 = W_{\varphi_1}$, then $J_{T_1}(a,0) = N_{\varphi_1}(a)$ for any $a \in [0,1]$.
- With the only condition of T_1 being a left-continuous t-norm, it is well known (see e.g. [10]) that $T_1(a, J_{T_1}(a,b)) = Min(a,b) \leq b$, which means that J_{T_1} will always be a MP-implication for any t-norm T such that $T \leq T_1$.

The next theorem provides a complete characterization of the class of R-implications J_{T_1} which are MP-implications.

Theorem 2 *Let T and T_1 be two continuous t-norms. Then:*

(a) *If $T_1 = Min$, then J_{T_1} is a MP-implication for any t-norm T.*

(b) *If $T_1 = W_{\varphi_1}$, then J_{T_1} is a MP-implication for T if and only if there exists an order automorphism $\varphi : [0,1] \to [0,1]$ such that $T = W_\varphi$, $N_{\varphi_1} \leq N_\varphi$ and $W^*_{\varphi_1}(a,b) \leq W^*_\varphi(N_\varphi(N_{\varphi_1}(a)), b)$ for any $a, b \in [0,1]$.*

(c) *If $T_1 = Prod_{\varphi_1}$, then J_{T_1} is a MP-implication for T if and only if there exists an order automorphism $\varphi : [0,1] \to [0,1]$ such that one of the two following situations holds:*

  (c1) *$T = W_\varphi$ and $\varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)) \leq W^*_\varphi(N_\varphi(a), b)$ for any $a, b \in [0,1]$ such that $a > b$.*

  (c2) *$T = Prod_\varphi$ and $\varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)) \leq \varphi^{-1}(\varphi(b)/\varphi(a))$ for any $a, b \in [0,1]$ such that $a > b$.*

PROOF.

(a) $T_1 = Min$ is the greatest t-norm, and therefore for any other t-norm T it is $T(a, J_{T_1}(a,b)) \leq T_1(a, J_{T_1}(a,b)) = Min(a,b) \leq b$.

(b) If $T_1 = W_{\varphi_1}$, let us suppose that J_{T_1} is a MP-implication for T. Then, since, as it was mentioned before, it is $J_{T_1}(a,0) = N_{\varphi_1}(a)$, corollary 1 implies $T = W_\varphi$ and $N_{\varphi_1} \leq N_\varphi$. Now, the MP inequality, which, due to theorem 1, is equivalent to $J_{T_1} \leq J_T$, becomes $W^*_{\varphi_1}(N_{\varphi_1}(a), b) \leq W^*_\varphi(N_\varphi(a), b)$, or, equivalently, $W^*_{\varphi_1}(a,b) \leq W^*_\varphi(N_\varphi(N_{\varphi_1}(a)), b)$.

(c) When $T_1 = Prod_{\varphi_1}$, the MP inequality is true for any t-norm T whenever the values $a, b \in [0,1]$ are taken such that $a \leq b$: indeed, in these cases it is $J_{T_1}(a,b) = 1$ and, therefore, $T(a, J_{T_1}(a,b)) = a \leq b$. Otherwise, the MP inequality becomes $T(a, \varphi_1^{-1}(\varphi_1(b)/\varphi_1(a))) \leq b$, and its verification depends on the t-norm T:

  - if $T = Min$, the inequality becomes $a \leq b$ whenever it is $\varphi_1(a)^2 \leq \varphi(b)$, and $\varphi_1(b) \leq \varphi_1(a) \cdot \varphi_1(b)$ otherwise. Both cases are false, and, therefore, residuated implications based on t-norms such that $T_1 \in \mathcal{F}(Prod)$ will never be MP-implications for $T = Min$.

  - if $T = W_\varphi$ or $T = Prod_\varphi$, it suffices to use the characterization of the MP inequality given in theorem 1 to obtain the conditions stated in (c1) and (c2).

■

On the other hand, the results obtained when studying the MT inequality can be summarized as follows:

Theorem 3 *Let T and T_1 be two continuous t-norms and N a strong negation function. Then J_{T_1} is a MT-implication for (T, N) if and only if there exists an order automorphism $\varphi : [0,1] \to [0,1]$ such that $T = W_\varphi$, $N \leq N_\varphi$ and one of the following situations holds:*

(a) $T_1 = Min$

(b) $T_1 = W_{\varphi_1}$ *for some order automorphism* φ_1, $N_{\varphi_1} \leq N$ *and* $W^*_{\varphi_1}(a,b) \leq W^*_\varphi(N(N_{\varphi_1}(a)), N_\varphi(N(b)))$ *for any* $a, b \in [0,1]$.

(c) $T_1 = Prod_{\varphi_1}$ *for some order automorphism* φ_1 *and* $\varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)) \leq W^*_\varphi(N(a), N_\varphi(N(b)))$ *for any* $a, b \in [0,1]$ *such that* $a > b$.

PROOF. If we suppose that J_{T_1} is a MT-implication for (T, N), then corollary 2 implies that the conditions $T = W_\varphi$ and $N \leq N_\varphi$ are mandatory. We now distinguish the family that the t-norm T_1 belongs to:

(a) if $T_1 = Min$, the MT inequality $T(N(b), J_{T_1}(a,b)) \leq N(a)$ is always true. Indeed, if it is $a \leq b$, then $T(N(b), J_{T_1}(a,b)) = T(N(b), 1) = N(b)$, and $N(b) \leq N(a)$ is true. Otherwise, it is $T(N(b), J_{T_1}(a,b)) = T(N(b), b) = \varphi^{-1}(Max(0, \varphi(N(b)) + \varphi(b) - 1)) = 0$, (since by hypothesis it is $N \leq N_\varphi$), and the MT inequality becomes $0 \leq N(a)$.

(b) if $T_1 = W_{\varphi_1}$, it is $J_{T_1}(a, 0) = N_{\varphi_1}(a)$, and therefore, by condition (MT1) of proposition 3, it is necessarily $N_{\varphi_1} \leq N$. Now, the MT inequality, which by theorem 1 is equivalent to $J_{T_1} \leq J_T^\delta \circ N \times N$, may be written as $W^*_{\varphi_1}(N_{\varphi_1}(a), b) \leq W^*_\varphi(N(a), N_\varphi(N(b)))$, or, performing a variable's change, as $W^*_{\varphi_1}(a,b) \leq W^*_\varphi(N(N_{\varphi_1}(a)), N_\varphi(N(b)))$.

(c) if $T_1 = Prod_{\varphi_1}$, when $a \leq b$ the MT inequality is true, since in these cases it is $T(N(b), J_{T_1}(a,b)) = T(N(b), 1) = N(b) \leq N(a)$. Otherwise, by theorem 1, it is equivalent to $\varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)) \leq W^*_\varphi(N(a), N_\varphi(N(b)))$.

■

4. The case of S-implications

In this section we will consider a S-implication given as $J(a,b) = S(N_1(a), b)$, and study the MP and MT inequalities with respect to a couple (T, N). Some basic results regarding these implications operators are the following:

- $J(1, b) = b$ for any $b \in [0,1]$.
- $J(a, 0) = N_1(a)$ for any $a \in [0,1]$.
- If the negations N_1 and N are chosen such that $N_1 = N$, then the MP and MT inequalities are equivalent. Indeed, if $T(a, S(N(a), b)) \leq b$ holds for any $a, b \in [0,1]$, the change of variables $a = N(\beta)$ and $b = N(\alpha)$ gives $T(N(\beta), S(\beta, N(\alpha))) \leq N(\alpha)$ for any $\alpha, \beta \in [0,1]$, which is the MT inequality. Analogously, from $T(N(b), S(N(a), b)) \leq N(a)$, and with the same change of variables, the MP inequality is obtained.

The next theorem characterizes the class of S-implications that are MP-implications.

Theorem 4 *Let S be a continuous t-conorm, T a continuous t-norm and N_1 a strong negation. Then the S-implication J defined as $J(a,b) = S(N_1(a), b)$ is a MP-implication for T if and only if there exists an order automorphism $\varphi : [0,1] \to [0,1]$ such that $T = W_\varphi$, $N_1 \leq N_\varphi$ and $S(a,b) \leq W^*_\varphi(N_\varphi(N_1(a)), b)$ for any $a, b \in [0,1]$.*

PROOF. If we suppose that J is a MP-implication, the equality $J(a,0) = N_1(a)$ implies, by corollary 1, that $T = W_\varphi$ and $N_1 \leq N_\varphi$. Now, the MP-inequality is $W_\varphi(a, S(N_1(a), b)) \leq b$, or, equivalently, $W_\varphi(N_1(a), S(a,b)) \leq b$. The later is equivalent to $Max[0, \varphi(N(a)) + \varphi(S(a,b)) - 1] \leq \varphi(b)$, which is true either if it is $S(a,b) \leq N_\varphi(N_1(a))$ or if it is $S(a,b) \leq \varphi^{-1}(Min[1, \varphi(b) + \varphi(N_\varphi(N_1(a)))]) = W^*_\varphi(N_\varphi(N_1(a)), b)$. Since it is always $N_\varphi(N_1(a)) \leq W^*_\varphi(N_\varphi(N_1(a)), b)$, only the second inequality has to be considered. ■

Remark 1 *Note that, in the characterization given in the last theorem, the condition $N_1 \leq N_\varphi$, i.e., $N_1(a) \leq N_\varphi(a)$ for any $a \in [0,1]$, implies that $a \leq N_\varphi(N_1(a))$ for any $a \in [0,1]$. Therefore, in particular, any t-conorm S such that $S \leq W^*_\varphi$ will verify the required conditions, because $S(a,b) \leq W^*_\varphi(a,b) \leq W^*_\varphi(N_\varphi(N_1(a)), b)$ for any $a, b \in [0,1]$.*

Regarding the MT-inequality, the following characterization is obtained:

Theorem 5 *Let S be a continuous t-conorm, T a continuous t-norm and N, N_1 two strong negations. Then the S-implication J defined as $J(a,b) = S(N_1(a), b)$ is a MT-implication for the couple (T, N) if and only if there exists an order automorphism $\varphi : [0,1] \to [0,1]$ such that $T = W_\varphi$, $N_1 \leq N \leq N_\varphi$ and $S(a,b) \leq W^*_\varphi(N(N_1(a)), N_\varphi(N(b)))$ for any $a, b \in [0,1]$.*

PROOF. If J is a MT-implication, then because of condition (MT1) of proposition 3 it is $N_1 \leq N$, and because of corollary 2 it is $T = W_\varphi$ and $b \leq N_\varphi(N(b))$, i.e., $N \leq N_\varphi$. Then, the MT inequality is $W_\varphi(N(b), S(N_1(a), b)) \leq N(a)$, or, equivalently, $W_\varphi(N(b), S(a,b)) \leq N(N_1(a))$. The later is true either if it is $S(a,b) \leq N_\varphi(N(b))$ or $S(a,b) \leq \varphi^{-1}(Min[1, \varphi(N(N_1(a))) + \varphi(N_\varphi(N(b)))])$, and the last expression may be written as $W^*_\varphi(N(N_1(a)), N_\varphi(N(b)))$. ■

Remark 2 *As it was the case for the MP inequality, we can remark that it is sufficient to choose an S-implication such that $S \leq W^*_\varphi$ in order to guarantee the MT inequality for a t-norm $T = W_\varphi$ and a negation N such that $N_1 \leq N \leq N_\varphi$. Indeed, from $N_1 \leq N$ it is $a \leq N(N_1(a))$, and from $N \leq N_\varphi$ we get $b \leq N_\varphi(N(b))$. Then, if $S \leq W^*_\varphi$, it will be $S(a,b) \leq S(N(N_1(a)), N_\varphi(N(b))) \leq W^*_\varphi(N(N_1(a)), N_\varphi(N(b)))$.*

5. The case of Q-implications

Concerning Q-implications of the form $J(a,b) = S(N_1(a), T_1(a,b))$, the following preliminary properties are worth-mentioning:

- $J(1,b) = b$ for any $b \in [0,1]$.
- $J(a,0) = N_1(a)$ for any $a \in [0,1]$.
- From the fact that $S(N_1(a), T_1(a,b)) \leq S(N_1(a), b)$ for all $a, b \in [0,1]$, it follows that a sufficient condition for a Q-implication $S(N_1(a), T_1(a,b))$ to be either a MP-implication or a MT-implication for a couple (T, N) is that the corresponding S-implication $S(N_1(a), b)$ verifies such inequalities for any couple (T_2, N) with $T \leq T_2$.

The next two theorems establish, respectively, which Q-implications verify the MP and the MT inequalities.

Theorem 6 *Let S be a continuous t-conorm, T_1, T two continuous t-norms and N_1 a strong negation. Then the Q-implication J defined as $J(a,b) = S(N_1(a), T_1(a,b))$ is a MP-implication for T if and only if there exists an order automorphism $\varphi : [0,1] \to [0,1]$ such that $T = W_\varphi$, $N_1 \leq N_\varphi$ and $S(a, T_1(N_1(a), b)) \leq W^*_\varphi(N_\varphi(N_1(a)), b)$ for any $a, b \in [0,1]$.*

PROOF. Since $J(a,0) = N_1(a)$, corollary 1 states that $T = W_\varphi$ and $N_1 \leq N_\varphi$ are necessary for J being a MP-implication. With such conditions, the MP inequality becomes $W_\varphi(a, S(N_1(a), T_1(a,b))) \leq b$, or, applying a change of variables, $W_\varphi(N_1(a), S(a, T_1(N_1(a), b))) \leq b$. This inequality will be true whenever it is $S(a, T_1(N_1(a), b)) \leq N_\varphi(N_1(a))$ or when $S(a, T_1(N_1(a), b)) \leq W^*_\varphi(N_\varphi(N_1(a)), b)$. ■

Theorem 7 *Let S be a continuous t-conorm, T_1, T two continuous t-norms and N_1, N two strong negations. Then the Q-implication J defined as $J(a,b) = S(N_1(a), T_1(a,b))$ is a MT-implication for the couple (T, S) if and only if there exists an order automorphism $\varphi : [0,1] \to [0,1]$ such that $T = W_\varphi$, $N_1 \leq N \leq N_\varphi$ and $S(a, T_1(N_1(a), b)) \leq W^*_\varphi(N(N_1(a)), N_\varphi(N(b)))$ for any $a, b \in [0,1]$.*

PROOF. If J is a MT-implication, then condition (MT1) in proposition 3 provides $N_1 \leq N$, and condition (MT2), via corollary 2, implies that $T = W_\varphi$ and $N \leq N_\varphi$. Then the MT inequality is $W_\varphi(N(b), S(a, T_1(N_1(a), b))) \leq N(N_1(a))$, and this is only true whenever it is $S(a, T_1(N_1(a), b)) \leq N_\varphi(N(b))$ or $S(a, T_1(N_1(a), b)) \leq W^*_\varphi(N(N_1(a)), N_\varphi(N(b)))$. ■

Remark 3 *As it was observed at the beginning of this section, the conditions under which a S-implication is a MP or MT-implication are sufficient for the corresponding Q-implication to be a MP or a MT-implication. Therefore, according to remark 1, any Q-implication $J(a,b) = S(N_1(a), T_1(a,b))$ such that $S \leq W^*_\varphi$ will be a MP-implication for $T = W_\varphi$ as long as N_1 is chosen such that $N_1 \leq N_\varphi$. Similarly, following remark 2, any such implication will be a MT-implication for the couple (W_φ, N) by just adding the condition $N \leq N_\varphi$.*

6. The case of ML-implications

Mamdani-Larsen's operators are those represented by a functional $J(a,b) = T_1(\varphi_1(a), \varphi_2(b))$, where T is a continuous t-norm, φ_1 is an order automorphism and $\varphi_2 : [0,1] \to [0,1]$ is a non-null contractive mapping, i.e., it verifies $\varphi_2(x) \leq x$ for any $x \in [0,1]$ and $\varphi_2(x) > 0$ for some $x \neq 0$. When $\varphi_1 = \varphi_2 = Id$ and $T = Min$, it is $J(a,b) = Min(a,b)$, which is the so-called Mamdani's implication; when $\varphi_1 = \varphi_2 = Id$ and $T = Prod$, Larsen's implication, $J(a,b) = a \cdot b$, is obtained.

For any ML-implication J, it is $J(1,b) = \varphi_2(b)$ and $J(a,0) = 0$ for all $a, b \in [0,1]$. In addition, for any t-norm T it is

$$T(a, J(a,b)) = T(a, T_1(\varphi_1(a), \varphi_2(b))) \leq T_1(\varphi_1(a), \varphi_2(b)) \leq \varphi_2(b) \leq b,$$

and these inequalities provide the following result:

Theorem 8 *Let T, T_1 be two t-norms, φ_1 an order automorphism and $\varphi_2 : [0,1] \to [0,1]$ a non-null contractive mapping. Then the ML-implication defined as $J(a,b) = T_1(\varphi_1(a), \varphi_2(b))$ is a MP-implication for the t-norm T.*

Therefore, ML-implications are MP-implications for any t-norm T, and, in particular, for $T = Min$. The next theorem states the conditions under which a ML-implication verifies the MT inequality.

Theorem 9 *Let T, T_1 be two t-norms, φ_1 an order automorphism, $\varphi_2 : [0,1] \to [0,1]$ a non-null contractive mapping and N a strong negation. Then the ML-implication defined as $J(a,b) = T_1(\varphi_1(a), \varphi_2(b))$ is a MT-implication (and a MPT-implication) for the couple (T, N) if and only if there exists an order automorphism $\varphi : [0,1] \to [0,1]$ such that $T = W_\varphi$ and $\varphi_2 \leq N_\varphi \circ N$.*

PROOF. Let us first suppose that J is a MT-implication for (T, N). Since $J(1,b) = \varphi_2(b)$ for any $b \in [0,1]$ and φ_2 is a non-null mapping, it is $J(1,b) \neq 0$ for some b. Then, according to corollary 2, it is necessarily $T = W_\varphi$ for some automorphism φ, and $\varphi_2(b) \leq N_\varphi(N(b))$. On the other hand, it appears that these conditions are sufficient for J to be a MT-implication. Indeed, the MT inequality is equivalent to $Max(0, \varphi(N(b)) + \varphi(T_1(\varphi_1(a), \varphi_2(b))) - 1) \leq \varphi(N(a))$, which is always true, because $T_1(\varphi_1(a), \varphi_2(b)) \leq \varphi_2(b) \leq N_\varphi(N(b))$. ■

Note that, unfortunately, the above characterization is reached at a very high cost: when a ML-implication verifies the MT-inequality, it is because the value of the left-hand side of this inequality is zero, and, therefore, the Modus Tollens inference rule will not provide any useful information.

Table 1 summarizes the results obtained, regarding both MP and MT characterizations, for the four classes of fuzzy implications that have been studied. Table 2, which is the result of intersecting (taking the minimum) the two columns of the first table, provides the necessary and sufficient conditions under which the different fuzzy implications' classes appear to be MPT-implications. Note that the conditions in this last table are significantly simplified whenever the negation N is taken as N_φ, since in this particular case (see Proposition 1) MP and MT inequalities are equivalent.

	MP-implication for T	**MT-implication for (T, N)**
R-implication J_{T_1} with $T_1 = Min$	always	$T = W_\varphi, N \leq N_\varphi$
R-implications J_{T_1} with $T_1 = W_{\varphi_1}$	$T = W_\varphi, N_{\varphi_1} \leq N_\varphi,$ $W^*_{\varphi_1}(a,b) \leq$ $W^*_\varphi(N_\varphi(N_{\varphi_1}(a)), b).$	$T = W_\varphi, N_{\varphi_1} \leq N \leq N_\varphi,$ $W^*_{\varphi_1}(a,b) \leq$ $W^*_\varphi(N(N_{\varphi_1}(a)), N_\varphi(N(b))).$
R-implications J_{T_1} **with** $T_1 = Prod_{\varphi_1}$	$[T = W_\varphi,$ $\varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)) \leq$ $W^*_\varphi(N_\varphi(a), b), a > b]$ OR $[T = Prod_\varphi,$ $\varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)) \leq$ $\varphi^{-1}(\varphi(b)/\varphi(a)), a > b]$	$T = W_\varphi, N \leq N_\varphi,$ $\varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)) \leq$ $W^*_\varphi(N(a), N_\varphi(N(b))), a > b.$
S-implications $S(N_1(a), b)$	$T = W_\varphi, N_1 \leq N_\varphi,$ $S(a,b) \leq$ $W^*_\varphi(N_\varphi(N_1(a)), b).$	$T = W_\varphi, N_1 \leq N \leq N_\varphi,$ $S(a,b) \leq$ $W^*_\varphi(N(N_1(a)), N_\varphi(N(b))).$
Q-implications $S(N_1(a), T_1(a,b))$	$T = W_\varphi, N_1 \leq N_\varphi,$ $S(a, T_1(N_1(a), b)) \leq$ $W^*_\varphi(N_\varphi(N_1(a)), b).$	$T = W_\varphi, N_1 \leq N \leq N_\varphi,$ $S(a, T_1(N_1(a), b)) \leq$ $W^*_\varphi(N(N_1(a)), N_\varphi(N(b))).$
ML-implications $T_1(\varphi_1(a), \varphi_2(b))$	always	$T = W_\varphi, \varphi_2 \leq N_\varphi \circ N.$

Table 1. Characterization of MP and MT-implications

	MPT-implication for (T, N)
R-implication J_{T_1} **with** $T_1 = Min$	$T = W_\varphi, N \leq N_\varphi$
R-implications J_{T_1} **with** $T_1 = W_{\varphi_1}$	$T = W_\varphi, N_{\varphi_1} \leq N \leq N_\varphi,$ $W^*_{\varphi_1}(a,b) \leq W^*_\varphi(N_\varphi(N_{\varphi_1}(a)), b),$ $W^*_{\varphi_1}(a,b) \leq W^*_\varphi(N(N_{\varphi_1}(a)), N_\varphi(N(b))).$
R-implications J_{T_1} **with** $T_1 = Prod_{\varphi_1}$	$T = W_\varphi, N \leq N_\varphi,$ $\varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)) \leq W^*_\varphi(N_\varphi(a), b), a > b,$ $\varphi_1^{-1}(\varphi_1(b)/\varphi_1(a)) \leq W^*_\varphi(N(a), N_\varphi(N(b))), a > b.$
S-implications $S(N_1(a), b)$	$T = W_\varphi, N_1 \leq N \leq N_\varphi,$ $S(a,b) \leq W^*_\varphi(N_\varphi(N_1(a)), b),$ $S(a,b) \leq W^*_\varphi(N(N_1(a)), N_\varphi(N(b))).$
Q-implications $S(N_1(a), T_1(a,b))$	$T = W_\varphi, N_1 \leq N \leq N_\varphi,$ $S(a, T_1(N_1(a), b)) \leq W^*_\varphi(N_\varphi(N_1(a)), b),$ $S(a, T_1(N_1(a), b)) \leq W^*_\varphi(N(N_1(a)), N_\varphi(N(b))).$
ML-implications $T_1(\varphi_1(a), \varphi_2(b))$	$T = W_\varphi, \varphi_2 \leq N_\varphi \circ N.$

Table 2. Characterization of MPT-implications

7. A remark on Probabilistic Logics

There are two main branches in Probabilistic Logic: the Bayesian approach, in which one associates to the rule "If a, then b" (for a, b in a Boolean Algebra) the probability $p(b/a)$, provided it is $p(a) > 0$, and the Nilsson's model, where the assigned probability is $p(a' + b)$.

Concerning Nilsson's logic, as it is $J(a, 0) = p(a')$ and $J(1, b) = p(b)$, corollaries 1 and 2 show that the only continuous t-norms that may be considered for the fulfillment of either the MP or the MT inequality are those in the Łukasiewicz family. In particular, it is easy to check that the Łukasiewicz t-norm W verifies the MP inequality: $W(p(a), J(a,b)) = Max(0, p(a) + p(a' + b) - 1) = Max(0, p(b) - p(a' \cdot b)) = p(b) - p(a' \cdot b) \leq p(b)$. In addition, Proposition 1 shows that in this case, taking $N = 1 - Id$, the MT inequality is equivalent to the former one, and, consequently, Nilsson's interpretation appears to be a MPT-implication for the couple $(W, 1 - Id)$.

Regarding Bayesian's approach, it is clear that $J(a,b) = p(b/a)$ verifies the MP inequality with $T = Prod$, since it is $p(a).p(b/a) = p(a \cdot b) \leq p(b)$, and, as $W \leq Prod$, the same will happen for $T = W$. Therefore, the Bayesian implication is a MP-implication for both $T = Prod$ and $T = W$. Nevertheless, since it is $J(1, b) = p(b)$, corollary 2 implies that, with conditional probabilities, the MT inequality cannot be obtained with $T = Prod$, since only t-norms belonging to the Łukasiewicz family can be considered. According to Proposition 1, this will be the case, in particular, for the Łukasiewicz t-norm, provided the negation $1 - Id$ is chosen. Therefore, the Bayesian's interpretation will be, as in Nilsson's model, a MPT-implication for $(W, 1 - Id)$.

8. Conclusions

Looking at the tables that summarize the paper's results, it can be observed that continuous t-norms in the Łukasiewicz family are ubiquitous among the solutions, mainly in the case of MT-implications, and, *a fortiori*, in that of MPT-implications. One of the problems with this kind of t-norms can be illustrated, for example, with the following MP's situation:

$$\begin{array}{l} \epsilon \leq J(\mu_P(x), \mu_Q(y)) \\ \delta \leq \mu_P(x) \\ \hline W_\varphi(\epsilon, \delta) \leq W_\varphi(\mu_P(x), \mu_Q(y)) \leq \mu_Q(y), \end{array}$$

that is, $\mu_Q(y) \in [W_\varphi(\epsilon, \delta), 1]$. But when $W_\varphi(\epsilon, \delta) = 0$ (i.e., $\varphi(\epsilon) + \varphi(\delta) \leq 1$), the above scheme provides the non-informative conclusion $\mu_Q(y) \in [0, 1]$. Hence, it should be $N_\varphi(\delta) \leq \epsilon$ to actually reach some information on $\mu_Q(y)$.

Consequently, and at least in the case of Modus Tollens where t-norms W_φ appear more frequently, it could be convenient to return to the old ideas ([7], [6]) of Modus Ponens and Modus Tollens generating functions, that is, to functions $M : [0, 1] \times [0, 1] \to [0, 1]$ such that either $M(a, J(a, b)) \leq b$ or $M(N(b), J(a, b)) \leq N(a)$ for all a, b in $[0, 1]$ verifying at least $M(1, 1) = 1$ (to capture the crisp case). The t-norms are only a particular case of these more general functions.

Acknowledgement. This work has been partially supported by the Spanish research projects TIC00-1420, PB98-1379-C02-02, TIC2000-1370-C04 and 07T/0011/2000.

References

[1] del Campo, C. and Trillas, E., (2000). On Mamdani-Larsen's type fuzzy implications *Proc.* 8^{th} *IPMU*, **2**, 712-716.

[2] Dubois, D. and Prade, H., (1984). Fuzzy logics and the generalized modus ponens revisited. *Int. J. of Cybernetics and Systems*, **15**, 293-331.

[3] Dubois, D. and Prade, H. (eds), (2000). *Fundamentals of Fuzzy Sets*, Kluwer, Upper Saddle River, NJ.

[4] Klir, G.J. and Yuan, B., (1995). *Fuzzy Sets and Fuzzy Logic*, Prentice Hall PTR, Upper Saddle River, NJ.

[5] Nguyen, H.T. and Walker, E.A., (1997). *A first course in Fuzzy Logic*, Boca Raton, FL, CRC Press.

[6] Pradera A., Trillas, E. and Cubillo, S., (2000). On Modus Ponens Generating Functions. *Int. J. of Uncertainty, Fuzziness and Knowledge Based Systems*. **8**, 1, 7-19.

[7] Trillas, E. and Valverde, L., (1985). On mode and implication in approximate reasoning, in: M.M. Gupta et al. (eds) *Approximate Reasoning in Expert Systems*, North-Holland, Amsterdam, 157-166.

[8] Trillas, E. and Cubillo, S., (1996). Modus Ponens on Boolean Algebras revisited. *Mathware and Softcomputing* **3**, 105-112.

[9] Trillas, E., Cubillo, S. and Del Campo, C., (1997). A few remarks on some T-conditional functions. *Proc.* 6^{th} *IEEE-FUZZ*, **1**, 153-156.

[10] Trillas, E., de Soto, A.R. and Cubillo, S. (2000). A glance at implication and T-conditional functions, in: V. Novák and I. Perfilieva (eds), *Discovering the world with Fuzzy Logic*, Physica-Verlag, Heidelberg, Germany, 126-147.

[11] Trillas, E., del Campo, C. and Cubillo, S., (2000). When QM operators are implication functions and conditional fuzzy relations?. *Int. J. of Intelligent Systems*, **15**, 647-655.

[12] Zadeh, L.A. (1973). Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Trans. on Systems, Man and Cybernetics*, **3**, 28-44.

E. Trillas
Dpto. de Inteligencia Artificial
Facultad de Informática
Universidad Politécnica de Madrid.
Campus de Montegancedo.
28660 Boadilla del Monte, Madrid.
Spain
etrillas@fi.upm.es

A. Pradera
Dpto. de Informática, Estadística y Telemática
Escuela Superior de Ciencias Experimentales y Tecnología
Universidad Rey Juan Carlos.
Calle Tulipán s/n, Móstoles, 28933 Madrid.
Spain
a.pradera@escet.urjc.es

C. Alsina
Secció de Matemàtiques i Informàtica,
Dept. d'Estructures a l'Arquitectura
Universitat Politècnica de Catalunya.
Avda. Diagonal 649, 08028 Barcelona.
Spain
alsina@ea.upc.es