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On Logical Fiberings and Automated Deduction in Many-valued Logics Using Gröbner Bases

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Abstract. The concept of logical fiberings is briefly summarized. Based on experiences with concrete examples an algorithmic approach is developed which leads to a represention of a many-valued logic as a logical fibering. The Stone isomorphism for expressing classical logical operations by corresponding polynomials can be extended to m-valued logics. On the basis of this, a classical deduction problem can be treated symbolically as a corresponding ideal membership problem using computer algebra support with the method of Gröbner bases. A logical fibering representation in this context provides a parallelization of the original problem and leads to (fiberwise) simpler polynomials and thus to a reduction of complexity.

Dedicated to Professor Bruno Buchberger on the occasion of his 60th birthday

Sobre fibrados lógicos y deducción automática en lógicas multi-valuadas usando bases de Gröbner

Resumen. En este trabajo se presenta en primer lugar una explicación resumida del concepto de fibrado lógico. Basándonos en nuestra experiencia sobre ejemplos concretos, desarrollamos una aproximación algorítmica que nos lleva a representar la lógica multi-valuada como un fibrado lógico. El isomorfismo de Stone que traduce expresiones de la lógica clásica a polinomios puede extenderse a lógicas m-valuadas. Basándonos en este último hecho, problemas de deducción clásicos pueden ser tratados desde la perspectiva del problema de pertenencia a un ideal, basándonos en el álgebra computacional usando Bases de Gröbner. Una representación como fibrado lógico en este contexto proporciona un modelo paralelo al problema original y lleva a polinomios (fiberwise) más simples, y a una reducción en la complejidad.

Dedicado al profesor Bruno Bucherger con motivo de su 60 cumpleaños

1. Introduction

The notion of a logical fibering had been introduced in the framework of an industrial case study on applications of so-called polycontextural logic (PCL). Polycontextural logical systems were part of research in the Biological Computing Laboratory (Urbana, Illinois) in the sixtieth when an interdisciplinary group of scientists started an initiative to work on so-called second order cybernetics. The inventor of polycontexturality and PCL, Gotthard Günther, was a member of that group of researchers. It was the objective to extend classical cybernetics especially in the direction of modeling complex communication systems - in particular living systems - and cooperating autonomous agents. The basic idea behind polycontextural logic was to provide each agent with a local individual logic which was assumed to be a classical 2-valued first

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order logic. All the subsystems are composed in a specific way by describing how they form as a whole a many-valued system (via a so-called “mediation scheme” which imposes constraints on the collection of all the classical truth values of local 2-valued systems which are labeled by the index of the corresponding system).

This typical way to introduce a system of distributed logics forming a PCL, motivated us to interpret such a system mathematically as a specific type of a fiber bundle. The theory of fiber bundles is a far developed very powerful mathematical discipline having important applications in physics and systems theory, among others. Characteristically, the notion of a fiber bundle integrates structures from geometry, topology and algebra. Generally spoken, a fiber bundle consists of a base space B, a total space E and a projection map $\pi : E \longrightarrow B$. The set of preimages over a point b in the base space is called the fiber over b. Thus, the total space is decomposed by all fibers over the base space. In a vector bundle the typical fiber is a vector space of a given dimension. Influenced by this powerful notion we came to the idea to replace the typical fiber in a vector bundle by a logical space and this lead us to the introduction of the concept of a logical fibering. In accordance with PCL systems, in our first definition of a logical fibering the typical fiber is a classical two valued logical structure. In the beginning of our work on logical fiberings we were able to show that PCL systems can be modeled as a special class of logical fiberings. More specific, a given PCL system can be modeled as a logical fibering determined by a specific equivalence relation on the global set of truth values (which describes the corresponding “mediation scheme”). In this sense the logical fiberings provide a framework for a systematic construction of many-valued logics. This was pointed out by Dov Gabbay in interesting discussions (in the frame of the Esprit projects MEDLAR I, II) and he further stated that they lead to a general semantics for his extended theory of labeled deductive systems (LDS). He introduced the notion “fibered semantics”. From his very extensive work we only cite here [9, 8, 10, 11]. Furthermore, he suggested to select special well-known 3-, or 4-valued logical connectives and try to express them in the framework of a suitable logical fibering.

Some experimental studies in this direction (cf. [28]) were very successful in the sense that they are suggesting to think at a general representation approach for many-valued logics by an associated logical fibering such that corresponding bivariate operations of a specific many-valued system can be decomposed into classical operations (corresponding to classical systems as fibers, respectively) and possibly some transjunctions. The advantage of such a representation theory would be a “parallelization” of a many-valued system and the reduction of corresponding operations to “fiberwise” classical ones and some elementary non-classical operations (so-called transjunctions), respectively.

Finally, concerning general semantical modeling aspects the previous considerations are also connected with other work which deals with semantical models for relational structures (using the language of category theory) where sheaf semantics appear in a natural way (cf. [21] for a proposed program of work in this direction).

After these introductory remarks, we come to the main topic of work in this article which we briefly describe as follows. The decomposition approach for many-valued logics can be exploited to simplify deduction problems in a many-valued logic in the sense that reduction of complexity is achievable in an automated deduction problem. Important basis for doing this are interesting results developed in [5] concerning the polynomial representation of many-valued logical operations - generalizing the classical Stone isomorphism. We point to the interesting article [30] being closely related to that work. Thus, methods from symbolic computation (computer algebra power) can be applied to automated deduction problems in many-valed logic. Actually, a deduction problem can be translated into a corrsponding ideal membership problem which then is tractable using Buchberger’s algorithm (Gröbner bases method). With the help of the representation of a many-valued system as a logical fibering, an original deduction problem can be parallelized leading to (fiberwise) simpler polynomials and thus to an overall reduction of complexity.

Concluding, we give some prospective comments on the role of logical fiberings as systems of distributed logics suitable for logical modeling of multiagent systems and, especially, cooperating robots scenarios. First steps in this directions were part of our work in the MEDLAR project mentioned above. These aspects form the basis of a fruitful cooperation contact with our colleague Bernhard Mitterauer, director of the institute of forensic neuropsychiatry at the university of Salzburg. He is an expert in brain research and cybernetics. Currently, our cooperation work focuses on developing mathematical models for some notions from his brain research being of basic relevance for agent models and multiagent system behaviour and for modeling connectionist networks and learning systems (cf. our work [25]). B.Mitterauer has a profound knowledge of G.Günther's theory of polycontexturality. Based on this, he proposed a new brain theory with the aim of technical implementations in robotics, cf. his work [18]. This brain model may also be explanatory for so-called mental disorders, like schizophrenia (cf. [19].

Besides these rather far reaching aspects of future work, we see immediate applications of the results subsequently presented concerning symbolic computation treatment of consequence relation problems using computer algebra techniques. This will give us technical support to construct a logical reasoning module for an agent exploiting computational power.

2. Logical Fiberings: Origins and Basic Notions

In this section we briefly recall some remarks on the origins of the concept of logical fiberings and we provide the basic notions and notation. Subsequently, we recall some comments concerning basic ideas and principles in PCL and point to references (cf. for example, the corresponding sections in [20], cf. also [24]).

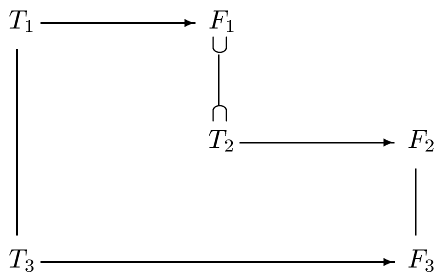
The original motivation for introducing the concept of logical fiberings grew up in a case study on so-called "Polycontextural Logic" (PCL) where two university groups and an industrial company were involved. PCL with all the original ideas was introduced by Gotthard Günther and later studied and continued in some directions by Rudolf Kaehr and coworkers. The work of G.Günther was strongly based on philosophical ideas and he was always interested in technical applications (cybernetics) of his ideas. An important aim of his so-called "transclassical logic" was to form a suitable logical basis for modeling living communicating systems. Actually, G.Günther developed parts of his theory at the Biological Computing Laboratory (BCL), Urbana Ill., in the sixties (he held a research position there) with the intention (among others) to establish a new logical basis for 2nd order cybernetics. Published material on PCL can be found, among others, in [13], [12], [14], [15].

Subsequently, we include some comments taken from PCL literature, for the convenience of reading. Basic principles are among others: Distribution of classical (2-valued) logics ("loci"). At least 3 loci are involved; the individual spaces are pairwise isomorphic ("locally"). "Transition" ("communication") between subsystems is of essential structural importance. The global system will always be a many-valued logical system. As we shall see later, it can be interpreted as a "fibered system" where a single fiber corresponds to the local system of an idividual "agent".

"The world has infinitely many logical places, and it is representable by a two-valued system in each of the places, when viewed isolately. However, a coexistence of such places can only be described in a many-valued system — if we intend to work with values in the first place." (cf. Günther, [13], Vol.2, p.199).

Previously, we mentioned 2nd order cybernetics and we would like to point here to the interesting article by Daniel Dubois, [6], where he discusses first order cybernetics, second order cybernetics, and a third order cybernetics proposed by himself in 1996.

Some Basic Notions from PCL follow. We use L or L_i (if an index is necessary) to denote a classical 2-valued logical space (a 1st order language or in the simplest case just a set of two truth values). Following the PCL literature, we introduce some basic notions; for more details and motivating comments we refer to the literature. The classical logical places (loci) within a PCL are denoted by L_i, $i = 1, 2, \ldots, n$, where $n = \binom{m}{2}$ and $1, 2, \ldots, m$ is an enumeration of the set of (global) truth values, thus leading to an m-valued PCL system with n classical subsystems L_i. The total (global) system is denoted by $\mathcal{L}^{(m)}$, The two (classical) truth values within L_i are denoted by T_i, F_i (with total order $T_i < F_i$). In addition to these basic constituents of a PCL the following so-called *mediation scheme - MS*, for short - represents basic information of a PCL system. In the following figure we present such a scheme for the case $m = 3$ (hence $n = 3$), we denote it by *MS3*.



It contains the following information: The arrow $T_i \to F_i$ expresses an ordering of the two values within the subsystem L_i, and $\supset\!\!\!-\!\!\!-\!\!\!\subset$ expresses that an F-value in one system (L_1) becomes a T-value in another (L_2) (a "change" of truth values when changing the corresponding subsystems)- i.e. a "semantical change". The vertical lines have to be interpreted as identifications. Thus, a *MS* describes the global relations between the local values and represents information about the passages from one subsystem to another. The *mediation scheme* describes how the *local* T-, F-values of the individual subsystems relate to each other *globally*, or how the collection of the $\Omega_i = \{T_i, F_i\}$ (local truth values of an individual L_i) form the global set of values $\{1, 2, 3\}$, respectively. In logical fibering notation (as introduced below) we shall express this by an equivalence relation on the union Ω of all the local value sets $\Omega_i = \{T_i, F_i\}$: $T_1 \equiv T_3$, $F_1 \equiv T_2$, $F_2 \equiv F_3$. As we will see, from the set of all local values one obtains the global values as set of equivalence classes. Exactly this information is encoded in a mediation scheme. Since every subsystem is a classical 2-valued system bivariate operations can be defined componentwise, for example in $\mathcal{L}^{(3)}$: $X \wedge \vee \wedge Y$ has to be interpreted as the operation where a conjunction is performed in subsystem L_1, a disjunction in L_2 and a conjunction in L_3. Analogously, the operation $X \wedge \vee \to Y$ has to be understood. In vector notation: $X \wedge \vee \wedge Y = \begin{pmatrix} x_1 \wedge y_1 \\ x_2 \vee y_2 \\ x_3 \wedge y_3 \end{pmatrix}$, $X \wedge \vee \to Y = \begin{pmatrix} x_1 \wedge y_1 \\ x_2 \vee y_2 \\ x_3 \to y_3 \end{pmatrix}$.

2.1. Fiber Bundles, Logical Fiberings

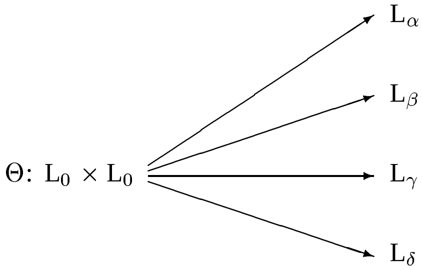
As previously mentioned, the development of logical fiberings originates in work on a case study in PCL. It has been strongly motivated and influenced by the classical theory of fiber bundles, a powerful modeling language from geometry and topology, where typical local-global interaction of different structures can be integrated in one concept. Thus, for example, in a vector bundle the fibers of such a fiber bundle are vector spaces of a fixed dimension. Roughly spoken, vertically one does algebra and horizontally (across the fibers) one is doing geometry/topology.

Concerning the (notion of) *logical fiberings*, as introduced in [20], the idea was to replace the fibers of a fiber bundle by a logical space (e.g. a 2-valued logic) and try to preserve all the expressive power of the classical fiber bundle notion as good as possible. A typical aspectarising here is that we are going to mix discrete with continuous structures in one integrated concept, but this is part of the challenge. More generally, our aim is to extend this logical fiberings approach to a generic modeling principle which allows to mix various logics in the sense that we take various different logical spaces as fibers, putting them together (as a bundle of fibers) over a base space manifold that serves as an index system with its own structure. The development of such logical fiberings in terms of a flexible logical operational modeling tool is a difficult task.

To safe space we use a short notation for the truth tables of logical connectives. For example, the truth table of a classical 2-valued conjunction will be abbreviated by the $2 \times 2-$matrix that is, we just list the images corresponding to the four input pairs of the conjunction.

$$\boxed{\begin{array}{cc} T & F \\ F & F \end{array}}\,.$$

Transjunctions A new nonclassical type of bivariate logical operation arises in PCL, and more general, in logical fiberings. We introduce the notion of a so-called *transjunction* (we adopt this name from PCL).. Apart from the possibility of forming (bivariate) 2-valued logical connectives in each subsystem L_i another, more general, non-classical operation arises naturally. Considering a "local" bivariate operation as a mapping $\Theta : L_i \times L_i \to \mathcal{L}^I$, we can distribute the images of different input pairs $(x_i, y_i) \in L_i \times L_i$ under Θ over different subsystems L_j, L_k, ... in the image space $\mathcal{L}^I$. The following picture shows the basic structure of such a bivariate operation:



More explicitly, semantically there can be up to four different subsystems for the images of the four possible local input pairs $\Omega_i \times \Omega_i$. For example, for the four image truth values we could obtain $\Theta(T_i, T_i) = T_\alpha$, $\Theta(T_i, F_i) = F_\beta$, $\Theta(F_i, T_i) = F_\gamma$, $\Theta(F_i, F_i) = F_\delta$, as displayed in the truth value matrix below. In other words, such bivariate operations can distribute images over several subsystems - this is a new situation. In [20], a classification of all such bivariate operations, called *transjunctions*, is given. For example, the truth value matrix of a *"conjunctional" transjunction* looks like: $\boxed{\begin{matrix} T_\alpha & F_\beta \\ F_\gamma & F_\delta \end{matrix}}$. First simple demonstrations how transjunctions can be applied for "logical control" of cooperating robots were presented in [29], [22]. This was the basis for a generalization of this logical modeling principle to more general cooperating agent systems (cf. [16], [17]).

2.2. Free and Derived Logical Fiberings

The simplest form of a fibering or bundle is the "trivial fibering" $\xi = (E, \pi, B, F)$ with $E = B \times F$, π the first projection; the fiber over $i \in B$ is: $\pi^{-1}(i) = \{i\} \times F$. In our considerations of logical fiberings such a trivial fibering is a *parallel system of (classical) logics* L_i over an index set I as base space B. We can think of reasoning processes running in parallel and independently within each fiber $L_i = \pi^{-1}(i)$. Transition ("communication") between fibers (loci) is described by suitable maps (cf. [20]). We call such a logical fibering a *"(free) parallel system"* denoted by $\mathcal{L}^I$. We shall make a difference between *local truth values* $\Omega_i = \{T_i, F_i\}$ in each 2-valued subsystem $L_i, i \in I$, and the set of *global values* Ω^I of the whole fibering. Parallel systems are characterized by the fact that there are no relations between different local values, i.e. the set of global values is just the mere coproduct (disjoint union) $\Omega^I = \coprod_{i \in I} \Omega_i$. We are using the notation $\mathcal{L}^n := \mathcal{L}^I$, for $I = \{1, \ldots, n\}$, n a natural number. In this case $\Omega^I = \Omega^n = \{T_1, F_1, T_2, \ldots, F_n\}$. In a free parallel system $\mathcal{L}^I$ many bivariate operations can be introduced without any problem combining various bivariate operations defined independently on each component (subsystem) L_i, $i \in I$. More generally, in a (free) parallel system logical formulas can be formed independently in each fiber in parallel. In this sense, we interpret a global deduction as a parallel process of local, fiberwise deductions running in parallel.

Starting with a free logical fibering, a **derived logical fibering** is obtained as a system where a nontrivial equivalence relation $\equiv$ is defined on Ω^I, we denote it by $\mathcal{L}^{(I)}$, with $\Omega^{(I)} := \Omega^I / \equiv$. In this sense the PCL system $\mathcal{L}^{(3)}$ can be derived from the free parallel system $\mathcal{L}^3$ by introducing the equivalence relation $\equiv$ (given by *MS3*) on the set of global values Ω^3 yielding $\Omega^{(3)} = \{1, 2, 3\}$.

All considerations are motivated by the main idea to use free parallel systems as basic objects from which we derive all other logical fiberings by introducing suitable equivalence relations on the global value set. Furthermore, our emphasis is on functional notation, i.e. to express logical operations by corresponding mappings and, accordingly, manipulate formulas in an operational way. We recall that in a derived logical fibering $\mathcal{L}^{(I)}$ all operations which we consider are induced by corresponding operations in the free parallel system $\mathcal{L}^{I}$ where we define the logical expressions and operations componentwise (in parallel, in the subsystems).

The technical process of passing from $\mathcal{L}^{I}$ to a new system $\mathcal{L}^{(I)}$, w.r.t. a certain equivalence relation on Ω^{I} is discussed in more detail in [20].

At this stage of our initial work in the development of logical fiberings we were able to state that a given PCL system is a special instance of a derived logical fibering determined by a corresponding $\equiv$-relation imposed on a suitable free logical fibering.

3. Decomposition, Representation Based on Logical Fiberings

The subsequent considerations concerning decompositions are taken from an experimental case study where many examples were examined (cf. [28]) suggesting a general algorithmic decomposition procedure and general representation of many-valued logics by logical fiberings (cf. also [24]). We use the same notation as in [20]. Recalling notation, in that work we briefly discussed an example of a bivariate logical function like $X \wedge \vee \wedge Y$. With that it shall be expressed that the whole bivariate logical function (operation) is formed by "putting together" local components defined in a $2-$valued classical logical space (subsystem) – considered as a "fiber" of the whole system which as a whole forms the underlying *logical fibering*.

Here we are going to interpret the fiberings model exactly in the opposite direction: we intend to use this approach to *decompose* a given $m-$valued bivariate logical function into a number of $2-$valued components based on a corresponding underlying logical fibering which has to be constructed. Such a decomposition procedure might be of interest at least with respect to the following aspects which arise naturally.

- Decomposing $m-$valued operations into components as mentioned above results in a *parallelization* of logical operations where one can work with classical operations in the components in parallel ("fiberwise"), respectively.

- A general representation theory of many valued logics by means of logical fiberings is expected where it should be possible to represent a many valued space by a fibering of $2-$valued spaces. More general, it should be possible also to "mix" logics and to model "carse" and "fine grain" decompositions, i.e. having also many valued logics as fibers (and not only $2-$valued) – conceptually this is possible, in principle, with our approach.

We start with the following motivating example of a 3-valued logic $\mathcal{L}$ taken from [31], p.169, as a first illustration of our method. The bivariate operations AND, OR, IMPLY of that example are given by the tables (cf. [31], loc.cit.)

AND	T	$*$	F
T	T	$*$	F
$*$	$*$	$*$	F
F	F	F	F

In the following considerations we shall use a technically shorter notation for displaying the truth table of such bivariate logical functions just using the $3 \times 3-$matrix consisting of the image values of such a function. Fortechnical reasons we rename the symbols for the truth values as follows, setting $a :=$ T, $b := *$, $c :=$ F. Symbolically, we then obtain the set of ("global") values $\Omega = \{a, b, c\}$.

a	b	c
b	b	c
c	c	c

Analogously, the OR and IMPLY operation is given by a corresponding matrix (as displayed in the considerations below).

Below we are briefly describing the principle how the given 3-valued logic $\mathcal{L}$ can be derived from the free parallel system $\mathcal{L}^3$, the logical fibering consisting of 3 classical 2-valued subsystems denoted L_1, L_2, L_3 (fibers) — cf.[20] for the details.

The *global values* of $\mathcal{L}^3$ are given by the 6 *local values* $\{\mathrm{T}_i, \mathrm{F}_i\}$, $i = 1, 2, 3$, of the three *subsystems*, forming the set $\Omega^3 = \{\mathrm{T}_1, \mathrm{F}_1, \mathrm{T}_2, \mathrm{F}_2, \mathrm{T}_3, \mathrm{F}_3\}$.

The procedure of deriving $\mathcal{L}$ from $\mathcal{L}^3$ can be briefly formulated as follows:

- Find a suitable equivalence relation $\equiv$ on Ω^3 such that the set of residue classes $\Omega := \Omega^3/\equiv$ yields the global value set $\{a, b, c\}$ of the given logic $\mathcal{L}$.
- Express each given logical connective AND, OR, IMPLY in terms of a family of local classical connectives (i.e. triples) defined in each of the three local subsystems L_1, L_2, L_3, respectively.
- These representations have to be compatible with the equivalence relation on Ω^3 ([20]).

In some respect this procedure can be considered to be a a construction following the *generators and relations principle*.

Method of decomposition:

¿From a 3-valued bivariate logical operation represented by a 3 × 3-value matrix consisting of the image values of a corresponding logical function $\{a, b, c\} \times \{a, b, c\} \rightarrow \{a, b, c\}$ we derive three 2-valued classical operations given by the three 2 × 2-submatrices along the diagonal of the 3 × 3-schema — interpreted in $\mathcal{L}^3$.

That means each of the 2 × 2-submatrices belongs to the possible 4 index pairs formed by a selected pair of indices i, j, namely $\{(i, i), (i, j), (j, i), (j, j)\}$, where $i < j$ and i, j is running through $1, 2, 3$.

Thus, for a general m-valued bivariate operation we obtain the amount of $n := \binom{m}{2}$ suboperations defined in a corresponding $2-$valued "local subsystem" (forming a fiber of the decomposition). For the logical values of each 2 × 2-submatrix we use the same total ordering which we define on the global values $\{a, b, c\}$, here we take $a < b < c$. For example, the AND operation leads to the first 2 × 2-submatrix ($\{1, 2\}$-submatrix)

a	b
b	b

which is interpreted in the first subsystem L_1 of $\mathcal{L}^3$, hence in terms of the local truth values $\{\mathrm{T}_1, \mathrm{F}_1\}$ of L_1. We obtain locally: $a = \mathrm{T}_1, b = \mathrm{F}_1$. Obviously this represents locally a classical conjunction – symbolically expressed by $x_1 \wedge y_1$.

Analogously, we derive the $\{2, 3\}$-submatrix and the $\{1, 3\}$-submatrix and obtain the local conjunctions $x_2 \wedge y_2$ and $x_3 \wedge y_3$, respectively.

These are all considered as local logical operations in the free parallel logical system (fibering) $\mathcal{L}^3$. Summarizing, we obtain the bivariate operation: $X \wedge\wedge\wedge Y = \begin{pmatrix} x_1 \wedge y_1 \\ x_2 \wedge y_2 \\ x_3 \wedge y_3 \end{pmatrix}$.

In order to find the representation of the originally given OR in $\mathcal{L}$ derived from that decomposition $X \wedge\wedge\wedge Y$ in $\mathcal{L}^3$ we have to define a suitable $\equiv$-relation on Ω^3, to identify the corresponding equivalence classes and to check whether this is compatible with all the three originally given bivariate operations. We demonstrate this with the example AND:

The second submatrix w.r.t. the indices $\{2, 3\}$ is given by:

b	c
c	c

This induces the local T,F-scheme

T_2	F_2
F_2	F_2

w.r.t. the correspondence $T_2 \leftrightarrow b$, $F_2 \leftrightarrow c$. This again yields a classical conjunction $x_2 \wedge y_2$ in the subsystem L_2 of $\mathcal{L}^3$.

The (third) $\{1,3\}$-submatrix which can be derived from the given 3×3-matrix is

a	c
c	c

leading to

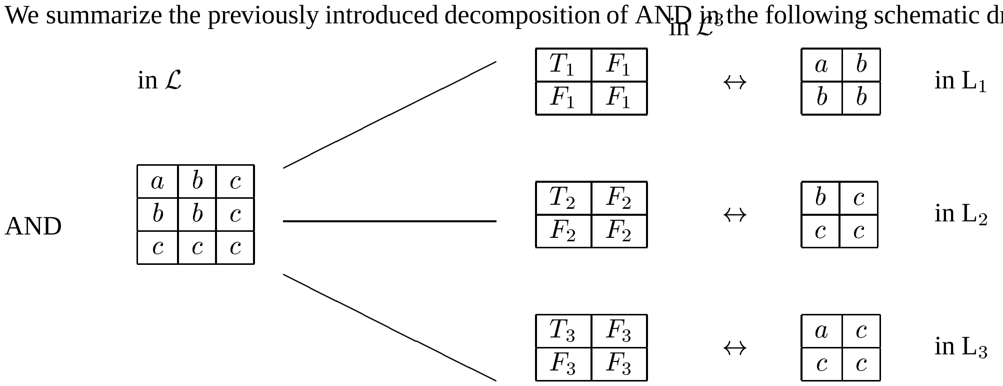
T_3	F_3
F_3	F_3

w.r.t. $T_3 \leftrightarrow b$, $F_3 \leftrightarrow c$ in subsystem L_3 of $\mathcal{L}^3$. This yields $x_3 \wedge y_3$. Altogether we obtain $X \wedge\wedge\wedge Y$ in $\mathcal{L}^3$.

Noticing that $\Omega = \{a,b,c\}$ is the set of global values which shall be obtained from Ω^3 as a set of equivalence classes $\Omega^3/\equiv$ we can read off the following equivalence relation on Ω^3 from the above identities:

$T_1 \equiv T_3$ leads to class a. Let $[T_1]$ denote the equivalence class of T_1 then $[T_1] = [T_3] = a$, analogously $F_1 \equiv T_2$, $F_2 \equiv F_3$ such that $[F_1] = [T_2] = b$, and $[F_2] = [F_3] = c$.

We summarize the previously introduced decomposition of AND in the following schematic drawing:



AND leads to

$$X \wedge\wedge\wedge Y = \begin{pmatrix} x_1 \wedge y_1 \\ x_2 \wedge y_2 \\ x_3 \wedge y_3 \end{pmatrix}$$

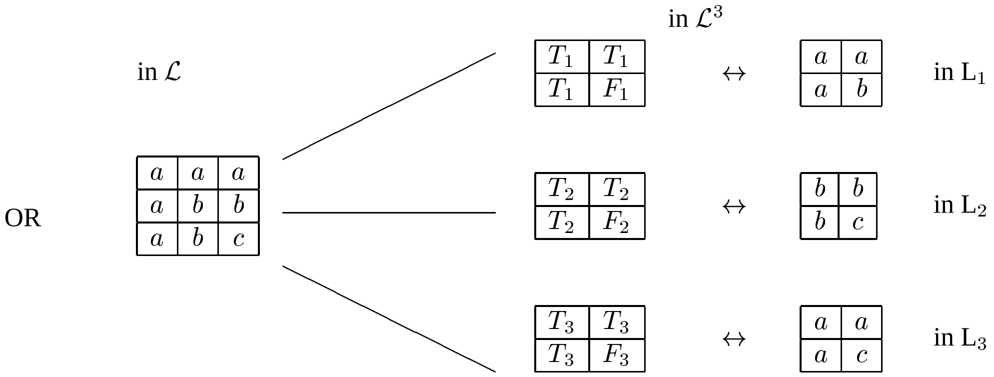
in $\mathcal{L}^3$ with $\equiv$ on Ω^3 given by

$$a = [T_1] = [T_3], b = [F_1] = [T_2], c = [F_2] = [F_3].$$

We can therefore represent AND by the operation $X \wedge\wedge\wedge Y$ interpreted in $\mathcal{L}$ w.r.t. $\Omega = (\Omega^3/\equiv) = \{a,b,c\}$ corresponding to the above identities on the truth values of Ω^3.

In this setting we have *parallelized* AND by the operation $X \wedge\wedge\wedge Y$ consisting of 3 conjunctions deduced from $\mathcal{L}^3$.

Analogously, we give the presentation of OR and IMPLY following the above method.



The corresponding identifications are:

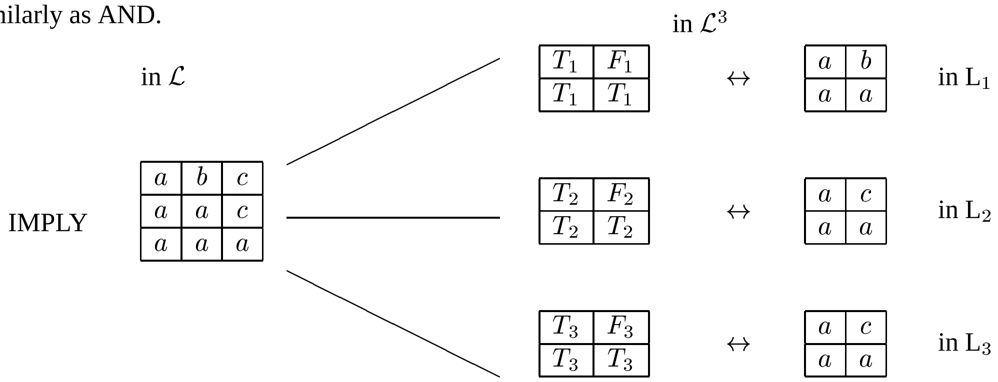
$$a = [\mathrm{T}_1] = [\mathrm{T}_3], b = [\mathrm{F}_1] = [\mathrm{T}_2], c = [\mathrm{F}_2] = [\mathrm{F}_3].$$

These are the same relations as obtained from AND — in this sense both representations are compatible with the given equivalence relation $\equiv$.

OR is thus representable as

$$X \vee\vee\vee Y = \begin{pmatrix} x_1 \vee y_1 \\ x_2 \vee y_2 \\ x_3 \vee y_3 \end{pmatrix},$$

similarly as AND.



In $\mathcal{L}^3$ we obtain:

$$X \to\to\to Y = \begin{pmatrix} x_1 \to y_1 \\ x_2 \to y_2 \\ x_3 \to y_3 \end{pmatrix}.$$

The induced identifications on the value set Ω^3 are given by

$$a = [\mathrm{T}_1] = [\mathrm{T}_2] = [\mathrm{T}_3], b = [\mathrm{F}_1], c = [\mathrm{F}_2] = [\mathrm{F}_3].$$

This differs from the above $\equiv$-relation in the class represented by a, namely for imply we have to identify $\mathrm{T}_2 \equiv \mathrm{T}_1$ in contrast to $\mathrm{T}_2 \equiv \mathrm{F}_1$ in AND and OR.

This yields an incompatibility: with respect to the previous $\equiv$-relation IMPLY cannot be represented as $X \to\to\to Y$ in the same way as we did this for OR, AND.

But it is possible to repair this, i.e. to make all 3 connectives compatible with the originally chosen $\equiv$-relation on Ω^3 if we apply the concept of transjunction (cf. [20] for the definition). Recalling, a transjunction in subsystem L_i is a local bivariate operation, defined on the values of L_i, which distributes its image values over different subsystems. In accordance with our notation it is defined by a value matrix like, e.g.

$$\boxed{\begin{array}{cc} T_i & F_i \\ F_i & F_i \end{array}} \rightarrow \boxed{\begin{array}{cc} T_\alpha & F_\beta \\ F_\gamma & F_\delta \end{array}}$$

The second submatrix $\begin{array}{|c|c|}\hline \mathrm{a} & \mathrm{c} \\ \hline \mathrm{a} & \mathrm{a} \\ \hline\end{array}$ corresponding to $\begin{array}{|c|c|}\hline \mathrm{T}_2 & \mathrm{F}_2 \\ \hline \mathrm{T}_2 & \mathrm{T}_2 \\ \hline\end{array}$ in subsystem L_2 leads to an incompatibility. But it would be compatible with the $\equiv$-relation obtained from the AND, OR representation if it were of the form $\begin{array}{|c|c|}\hline \mathrm{b} & \mathrm{c} \\ \hline \mathrm{b} & \mathrm{b} \\ \hline\end{array}$.

We can achieve a representation of IMPLY of the form $X \rightarrow \Rightarrow_t \rightarrow Y$ deducing it from the parallel system $\mathcal{L}^3$ where the second local operation $x_2 \Rightarrow_t y_2$ is a suitable transjunction defined by a bivariate operation defined as $\Omega_2 \times \Omega_2 \rightarrow \Omega_1 \cup \Omega_2$ given by $\begin{array}{|c|c|}\hline \mathrm{T}_1 & \mathrm{F}_2 \\ \hline \mathrm{T}_1 & \mathrm{T}_1 \\ \hline\end{array}$ and denoted by $\Rightarrow_t$. Note that this is a local bivariate operation defined in L_2 with values distributed over the two subsystems L_1 and L_2. As a T-F-pattern this is of the type of an implication table, we therefore also can describe this transjunction by: $\Rightarrow_t = \mathcal{D} \circ \rightarrow$, where $\rightarrow$: $\Omega_2 \times \Omega_2 \rightarrow \Omega_2$ is a classical implication and $\mathcal{D} : \Omega_2 \rightarrow \Omega_1 \cup \Omega_2$ distributes the values over 2 subsystems corresponding to $T_2 \mapsto T_1$, $F_2 \mapsto F_2$.

In this way we can express IMPLY by $X \rightarrow \Rightarrow_t \rightarrow Y$ and this is compatible with the original equivalence relation.

We verify this only for the local input $[T_2, F_2]$ to the second operation $\Rightarrow_t$ since this is the only critical situation:

Recall that IMPLY is evaluated by the local evaluations of $\rightarrow$ and $\Rightarrow_t$ and $\rightarrow$ corresponding to our representation, which can be performed in parallel. Note that we must take into account the given $\equiv$-relation.

Inputing the four possible pairs formed by $b = [T_2], c = [F_2]$ we obtain the correct second submatrix of IMPLY. We point out again that the whole 3×3-value matrix that defines IMPLY is represented by the evaluation procedures of the three bivariate operations $\rightarrow, \Rightarrow_t, \rightarrow$; this can be done in parallel.

Remark: The compatibility condition with respect to the three suboperations (submatrices) can be expressed as follows (cf. [20]): The three 2×2-matrices have to be merged to a 3×3-matrix scheme along the diagonal of the 3×3-matrix such that the corresponding diagonal elements match (i.e. the 2×2-matrices are the suitable submatrices). In this sense our decomposition method is the reverse process to this merge.

After these “experimental considerations”, in [28] more general cases are considered from Łukasiewicz Logic Ł$_n$, Bochvar’s system, Kleene’s system, providing further insight into the basic decomposition principle. Concluding, we give a brief summary of the algorithmic decomposition procedure which points to a theory of logical fibering representations of many-valued logics.

The decomposition procedure for L can be stated in meta-language as follows (cf. [28]).

Input: L, the set of truth values of the logical system $\mathcal{L}$, a total order $<$ on L, a set of (bivariate) logical connectives of $\mathcal{L}$, the truth tables for these logical connectives.

Problem: Find a decomposition of the connectives into a number of 2-valued components, based on a corresponding underlying logical fibering.

Begin

Step 1 Let $I = \{(i, j) \mid i, j \in L, i < j\}$ be the set of all possible pairs of distinct elements of L. Let $\Omega_{i,j} = \{T_{i,j}, F_{i,j}\}$ be the local truth values of the local subsystems corresponding to the elements of I. Let $\Omega = \coprod_{i,j} \Omega_{i,j}$ be the set of global truth values.

Step 2 Express each given connective on L in terms of a family of local classical connectives defined on each of these logical subsystems.

Step 3 Find a suitable equivalence relation $\equiv$ on Ω (compatible with the operations - cf. the demonstration examples above), such that the quotient set $\Omega/\equiv$ yields the set L of truth values of the given logical system $\mathcal{L}$.
End

For a more detailed version with a short proof we refer to [16].

4. Automated Deduction and Buchberger Algorithm (Gröbner Bases)

A canonical way to introduce and exploit the computational power of symbolic computation approaches. especially computer algebra, to treat problems in mathematical logic is the polynomial representation of logical operations. The Stone isomorphism uses the following polynomial representation of connectives in classical 2-valued logic.

$X \wedge Y \ \mapsto\ X.Y, \quad X \vee Y \ \mapsto\ XY + X + Y, \quad X \Rightarrow Y \ \mapsto\ XY + X + 1, \quad \neg X \ \mapsto\ X + 1.$ The polynomials are considered over the coefficient field with two elements $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

In [5] the authors applied a clever idea to generalize the Stone isomorphism to m-valued logical connectives. This deals with polynomials over finite fields (Galois fields). A crucial point is that the authors can translate a *problem of consequence* of a formula from a given set of formulas into a corresponding *problem of ideal membership*. This naturally leads to the application of *Buchberger's algorithm* for computing Gröbner bases and thus to the deployment of computer algebra systems allowing automated deduction. We mention here the interesting work [30] which is closely related to [5].

For the convenience of reading we give a brief summary of some notions and results of the article following the short presentation of the material by W.Meixl ([16]) - who worked on his thesis under my supervision. One objective of the thesis was to exploit the logical fibering approach to reduce complexity in the symbolic computation treatment of logical deduction problems.

4.1. Algebraic Representation of m-valued Operations

As already mentioned, we keep the following summary very short in order to save space. In [5] the authors consider finite coefficient domains for the multivariate polynomials. For a natural number m the finite ring $\mathbb{Z}/m\mathbb{Z}$ is abbreviated by $\mathbb{Z}_m$. For an $m-$valued logic the set of variables is denoted by $Var = \{x_1, \ldots, x_n\}$ and $v : Var \longrightarrow \mathbb{Z}_m$ is a valuation. Let p denote the smallest prime number greater than or equal to m ($m \leq p$), then the coefficient field $\mathbb{Z}_p$ and the polynomial ring $A = \mathbb{Z}_p[x_1, \ldots, x_n]$ are chosen for the continuation of development and work.

In a very clever way, the authors develop a method to associate to a logical connective φ a corresponding unique multivariate polynomial $T_\varphi \in A = \mathbb{Z}_p[x_1, \ldots, x_n]$. They exploit Lagrange interpolation unsing the logical values of the truth table of a given logical connective φ as points (interpolation nodes) for the corresponding interpolation polynomial. The approach leads to a mapping T on formulas with values in te polynomial ring A, where for the variables it holds $T(x_i) = x_i$.

Heading towards a polynomial ring criterion for a problem of consequence $\Phi \models \varphi$, the authors define a set of elementary polynomials U_i in A (of type $\prod_{k=0}^{m-1}(x_i - k)$) and the polynomial ideal $\mathcal{U}$, generated by the U_i, for $i = 1, ..., n$. With the help of this the following polynomial criterion for a consequence relation can be shown.

Let $\Phi = \{\varphi_1, ..., \varphi_r\}$ be a set of formulas, then the following equivalence holds:

$$\Phi \models \varphi \quad \Leftrightarrow \quad T(\varphi) - 1 \in (T(\varphi_1) - 1, ..., T(\varphi_r) - 1, U_1, ..., U_n).$$

This important theorem shows the formulation of a problem of consequence in terms of a corresponding ideal membership problem.

Coming back to the logical fiberings decomposition approach for m-valued logical connectives, in the diploma thesis [16] the following procedure has been established - we can only give a brief description here.

Let $fib(\varphi)$ denote the local connectives (local components) of the (logical fibering) decomposition of a given connective φ. Then, consequently, the polynomial T_φ will have an associated "fibered" polynomial $fib(T_\varphi)$ consisting of those polynomials which we obtain locally for each local subsystem of the decomposition. These are polynomials corresponding to classical 2-valued connectives (cf. Stone isomorphism) or special transjunctions. There is a natural unique correspondence between T_φ and $fib(T_\varphi)$. This leads to a fibered version of the Stone isomorphism corresponding to the generalized Stone isomorphism of [5]. Finally, all these considerations lead to a "fibered version" of the polynomial consequence criterion.

4.2. Parallelized Automated Deduction

After these technical considerations we conclude with a brief résumé of the basic ideas and objectives concerning the possibility to apply the decomposition method to results presented in [5]. In their work these authors develop a method to translate logical formulas of an m-valued logic into corresponding polynomials. This approach generalizes the *Stone isomorphism* in classical 2-valued logic. It is working with multivariate polynomials over finite fields (Galois fields). A crucial point in that article is that the authors can translate a *problem of consequence* – when a formula φ is a consequence of a given set Φ of formulas (i.e. $\Phi \models \varphi$)) – into a corresponding *problem of ideal membership*. In a natural way, this leads to the application of *Buchberger's Gröbner basis algorithm* and thus to the deployment of computer algebra systems. As a selection of references we cite here Bruno Buchberger's original work [2], and two survey papers [3], [4] and [1].

This is the point where we argue that our decomposition method can be usefully exploited in the following direction. Assuming that the decomposition of a given many valued space into a fibering can also be transformed canonically into the polynomial algebra case leads to the following aspects. A given decision problem then can be fully parallelized, i.e. manipulations can be done fiberwise in parallel, and even more, in each fiber (component) we have in many cases classical logical formulas to handle that means that the corresponding polynomials have degree not greater than two (!). If we have to deal with certain transjunctions (as discussed in examples above) we only have to consider a well known restricted class of operations and, again, the corresponding polynomials have bounded degree (maximally degree four). Having bounded small polynomial degrees might be a big advantage in Gröbner basis applications, because high polynomial degrees can cause heavy problems to the performance of computer algebra systems. Thus, the possibility to represent many-valued logics by logical fiberings provides a decomposition, parallelization approach for many-valued connectives yielding fiberwise simpler expressions. This, consequently, leads to an overall reduction of complexity, especially in the corresponding symbolic computation applications.

To save space, we conclude these considerations with a simple illustration of the basic idea and principle, presenting the following short example. We take the 3-valued disjunction OR given by the truth table

0	1	2
1	1	1
2	1	2

, its decomposition consists of three local 2-valued disjunctions, displayed in vector notation as

$$(x_1 \vee y_1, x_2 \vee y_2, x_3 \vee y_3)^T.$$

Following [5], the corresponding polynomial of the 3-valued OR is

$$T_{OR} = 2X_1^2X_2^2 + X_1^2X_2 + X_1X_2^2 + X_1X_2 + X_1 + X_2.$$

Since a classical OR corresponds to the polynomial $XY + X + Y$, the polynomial T_{OR} corresponds to the decomposition vector of polynomials $(X_1Y_1 + X_1 + Y_1, X_2Y_2 + X_2 + Y_2, X_3Y_3 + X_3 + Y_3)^T$.

This representation clearly demonstrates the decomposition (parallelization) of the original polynomial into smaller components with lower degree.

With these considerations, we conclude this section. A more detailed discussion with further examples will be subject of future work.

5. Conclusions and Prospects

The concept of logical fiberings offers a natural approach to assign a system of distributed logics to a multiagent system (MAS), where the basic modeling principle is the idea to attach an individual logical fiber to every agent which models the local logical state space of an agent. The entire logical fiber bundle forms the global logical state space of the whole MAS. We point to existing work as published (among others) in the following articles where basic ideas and prospects are presented, cf. [22], [29], [27], [17], [23], [26], [16], [7]. These publications always provides the basis for further investigations and developments in the wide area of multiagent systems and, especially, cooperating robots scenarios. First steps in this directions were part of our work in the MEDLAR poroject mentioned above. Furthermore, these aspects form the basis of a fruitful cooperation contact with my colleague Bernhard Mitterauer, director of the institute of forensic neuropsychiatry at the university of Salzburg. He is an expert in brain research and cybernetics. Our cooperation work focuses on developing mathematical models for some notions from his brain research being of basic relevance for agent models and multiagent system behaviour and for modeling connectionist networks and learning systems (cf. our work [25]). B.Mitterauer has a profound knowledge of G.Günther's theory of polycontexturality. Based on this, he proposed a new brain theory with the aim of technical implementations in robotics, cf. his work [18]. This brain model may also be explanatory for so-called mental disorders, like schizophrenia (cf. [19]). Currently, we are working on a mathematical formalization of the notions "intention" and "rejection" that are relevant to agent and MAS modeling.

Concerning the topics previously discussed in this contribution, we aim at exploiting it in the direction to devise a "general automated reasoning machine for an agent" in a MAS, using computational power to provide an agent and a MAS with automated deduction techniques. This point of view would be in the spririt of the old idea of MEDLAR practical reasoners (cf. the Esprit projects MEDLAR I,II).

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