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A note about a priori estimates for indefinite problems in unbounded domains

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Abstract. In this paper, we are dealing with the following superlinear elliptic problem :

$$(P_\Omega)\begin{cases} -\Delta u = \lambda\, u + h(x)u^p \ \text{ in } \Omega \\ u = 0 \text{ in } \ \partial\Omega,,\ u \geq 0 \end{cases}$$

where Ω a smooth domain not necessary bounded, h is a C^2 function from $\mathbb{R}^N$ to $\mathbb{R}$ changing sign such that $h(x) \ \to \ -\infty$ when $\|x\| \ \to +\infty$ and $1 \ < \ p \ < \ \frac{N+2}{N-2}$. We give existence and uniform a priori estimates for solutions to (P_Ω).

Una nota sobre estimaciones a priori para problemas indefinidos en dominios no acotados

Resumen. Consideramos el siguiente problema elíptico superlineal

$$(P_\Omega)\begin{cases} -\Delta u = \lambda\, u + h(x)u^p \ \text{ en } \Omega \\ u = 0 \text{ en } \ \partial\Omega,,\ u \geq 0 \end{cases}$$

donde Ω es un dominio regular no necesariamente acotado, h es una función C^2 de $\mathbb{R}^N$ en $\mathbb{R}$ cambiando de signo tal que $h(x) \ \to \ -\infty$ cuando $\|x\| \ \to +\infty$ y $1 \ < \ p \ < \ \frac{N+2}{N-2}$. Obtenemos la existencia y estimaciones a priori de las soluciones de (P_Ω).

1. Introduction

In this paper, we consider the following superlinear elliptic problem :

$$(P_\Omega)\begin{cases} -\Delta u = \lambda\, u + h(x)u^p \ \text{ in } \Omega \\ u \geq 0 \ u\,|_{\partial\Omega} = 0 \end{cases}$$

Our goal is to extend the results in [8] where (P_Ω) is also investigated in the case $\Omega = \mathbb{R}^N$. Precisely, in [8] assuming that $\Omega^+ \ = \{x \ \in \mathbb{R}^N \,/\, h(x) \ > \ 0\}$ is a bounded domain, that $\Gamma \ := \{x \ \in \mathbb{R}^N \,/\, h(x) = 0\}$ satisfies a nondegeneracy condition :

$$\forall\, x \ \in \ \Gamma\ ,\, \nabla h(x) \neq 0,$$

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the authors prove that there exist a global branch bifurcating from the essential spectrum, in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. For this, they prove that for λ bounded, the solutions to $(P_{\mathbb{R}^N})$ obtained by a local approach are uniformly bounded in $L^\infty(\mathbb{R}^N)$. The method they use involves studying a “local problem”, (P_{Ω_R}), in a bounded domain $\Omega_R \supset B_R$ where B_R is the ball centered at 0 and with radius R

$$(P_{\Omega_R}) \begin{cases} -\Delta u = \lambda\, u + h(x)u^p & \text{in } \Omega_R, \\ u \in H_0^1(\Omega_R), & u \geq 0, \end{cases}$$

and then they pass to the limit when R goes to $+\infty$.

The crucial step in this procedure is to get a priori estimates for solutions to (P_{Ω_R}) independent of R.

Here we prove that on some conditions, we can remove the nondegeneracy assumption : h vanishes in a non zero measure set and get the sames results as in [8] for any large domain Ω with smooth boundary. Furthermore, the a priori estimate we obtain concern all solutions to (P_Ω). Note that a large class of unbounded domains are considered.

Here, we suppose that h satisfies the following assumptions :

(H1) $h \in C^2(\mathbb{R}^N\, \mathbb{R})$, $\Omega^+ := \left\{x \in \mathbb{R}^N,\ h(x) > 0\right\}$ is bounded domain with non zero measure and smooth boundary.

Supposing that there exists $\Omega_0 := \{x \in \mathbb{R}^N \,/\, h(x) = 0\}/\partial\Omega^- \cup \partial\Omega^+$, we assume in addition

(H2) Ω_0 is bounded with smooth boundary and $\partial\Omega_0 \cap \partial\Omega^- \cap \partial\Omega^+ = \emptyset$,

(H3) For x close to $\partial\Omega_0 \cap \partial\Omega^+$, $h(x) \equiv C$ dist$(x, \Omega_0 \cap \partial\Omega^+)^\gamma$, $\gamma > 0$, $C > 0$.

Let $\Gamma := \partial\Omega^+ \cap \partial\Omega^-$, then if Γ is non empty, Γ satisfies either

(H4) for $x \in \Omega^+$ close to Γ, $h(x) \equiv C$ dist$(x, \Gamma)^{\gamma'}$, $\gamma' > 0$, $C > 0$

or

(H4bis) for any $x \in \Gamma$, $\nabla h(x) \neq 0$.

Remark 1

1. Clearly (H1) and (H2) imply that $\Gamma = \overline{\Omega^+} \cap \overline{\Omega^-}$ and it is bounded.
2. (H2) implies that Ω_0 is far from Γ.
3. (H3) and (H4) give some flatness condition on h near $\partial\Omega_0$ and Γ. We use (H3) and (H4) in a blow up technique as in [6]. A similar argument is also used in [3].
4. (H4bis) is the nondegeneracy condition as in [8]. ■

Our purpose is to prove the existence of solutions and to give the structure of solutions set with respect to the bifurcation parameter λ.

When $h(x)$ changes sign, the proof of existence of a priori estimates is more difficult to obtain. Let us mention some previous works in this direction :

In [6], the authors use a blow up technique combined with some Liouville theorems in cones to obtain uniform a priori bounds and some existence results for equation (P_Ω) with Ω a bounded domain for $1 < p < \frac{n+2}{n-1}$ and $\lambda = 0$. (H4bis) is used to get $\gamma' = 1$ in this paper. The question which follows then is : is it true for any p less the critical exponent?

In [12] Chen and Li answer positively to that question $i.e.$ they obtain some a priori bounds for positive solutions when p is subcritical ($i.e.\ p < \frac{N+2}{N-2}$). Precisely they consider the following problem

$$\begin{cases} -\Delta u = h(x)u^p & \text{in } \Omega, \\ u \in H_0^1(\Omega)\ u \geq 0, \end{cases}$$

where h satisfies (H1),(H4bis), $\Omega_0 = \emptyset$ and $\Gamma \subset \Omega$. They prove that every solution is uniformly bounded and that the a priori bound depends only on the geometry of Ω, p and h.

The proof of this result is carried out dividing the domain in three regions and then solving the following steps :

1. boundedness of solutions in the region where $h(x) \leq -\delta$, for a fixed $\delta > 0$,
2. boundedness of solutions in the region where $|h(x)|$ is small,
3. boundedness of solutions in the region where $h(x) \geq \delta$.

Each step involves different techniques :

1. In the region where $h(x)$ is strictly negative, the uniform estimate is obtained by an Harnack inequality and an integral estimate.
2. In the region where $|h(x)|$ is small, the a priori bound results from the moving plane technique (here (H4bis) plays a crucial role) and from the above estimate.
3. In the last region, a classical blow up analysis (see [16]) is used.

In [3], the authors remove the nondegeneracy condition using the same blow-up technique as in [6] but they keep some restrictions on p due to the restriction in Liouville's theorem they apply. In [19], the authors prove that if we restrict to some type of solutions (with finite Morse index solutions precisely) the restriction of p can be removed also in the degenerate case.

In the present work, combining different techniques, some of them booked from [6], [12] and [3], we get uniform a priori estimates in inbounded domains case or (if Ω is bounded) independent of the measure of the domain considered and independent of λ bounded. Precisely, we prove the following main results :

Theorem 1 . *Suppose that (H1), (H2), (H3) are satisfied, that $1 < p < \frac{N+2}{N-2}$ and that Ω is large enough that $\Gamma \cup \partial\Omega_0 \subset \Omega$ and $\partial\Omega \subset supp\, h^-$. Let $\lambda_1(\Omega^+)$ (resp. $\lambda_1(\Omega_0)$) be the first eigenvalue to $-\Delta$ in Ω^+ (resp. in Ω_0). We also assume that Ω_0 is nonzero measure set. Then,*

(i) If $\lambda \geq \inf\,(\lambda_1(\Omega^+), \lambda_1(\Omega_0))$, there are no non trivial solutions of (P_Ω).

(ii) Assume in addition that Γ is nonempty and (H4). Let p such that $1 < p < \frac{N+1+\inf\,(\gamma,\gamma')}{N-1}$. For any $\lambda_0 < \lambda_2 < inf(\lambda_1(\Omega^+), \lambda_1(\Omega_0))$, there is a constant C ($= C(\lambda_0, \lambda_2)$) such that if (λ, u) is a solution of (P_Ω) and $\lambda_0 \leq \lambda \leq \lambda_2$ then

$$\|u\|_{L^\infty} \leq C \tag{1}$$

and C depends only on λ_0, λ_2, Ω^+, Ω_0, p and h.

(iii) If $\partial\Omega_0 \cap \partial\Omega^+ = \emptyset$ and (H4bis) holds, then (1) is also true for any p subcritical.

(iv) If Γ is empty or if (H4bis) holds instead of (H4), then (1) is true for $1 < p < \frac{N+1+\gamma}{N-1}$ and C depends also on Γ.

Remark 2

1) Theorem 1 concern the case where Ω_0 is non zero measure set. If Ω_0 is empty and if (H4bis) holds, then we can apply results in [8].

2) Theorem 1 handle unbounded domains Ω as bounded domains. For unbounded domains, the dirichlet conditions are replaced by a limit condition at infinity.

Next, we show that if h has radial symetry properties and $\partial\Omega^+$ has only one piece component then no restriction on p subcritical and no nondegeneracy condition are necessary. A similar observation was previously made in [3] for bounded domains. ■

Proposition 1 . *Assume that h is radial symmetric continuous function and that Ω^+ is a ball. Then, (1)is also true for radial symmetric solutions.*

Remark 3 If $\partial\Omega^+$ has two pieces component and if (H4bis) holds then Proposition 1 is also true. ■

Finally, in the next result we show that the asymptotic behaviour of h is relevant to determine the behaviour of solutions to (P).

Proposition 2 . *Assume that h is continuous function on $\mathbb{R}^N$ such that $\lim_{|x|\to+\infty} h(x) = a < 0$ and finite. Let $\lambda > 0$. Then, for any nontrivial solution u to (P), we have*

$$\liminf_{|x|\to+\infty} u(x) > 0. \tag{2}$$

Remark 4

1. Proposition 2 is also valid in the case where h satisfies $h(x) < 0$ for $|x|$ large and $\lim_{|x|\to+\infty} h(x) = 0$.
2. We can extend easily Proposition 2 in the case where u^p is replaced by $g(u)$, g $C^2(\mathbb{R}^+)$ satisfying $s \to \frac{g(s)}{s}$ nonincreasing, $\lim_{s\to 0^+} \frac{g(s)}{s} = 0$ and $\lim_{s\to\infty^+} \frac{g(s)}{s} = +\infty$. ■

Using the above a priori estimates and the global bifurcation Rabinowitz (see [18]) Theorem, we get existence of solutions to (P) and the behaviour of solutions with respect to the bifurcation parameter λ. Consider $\phi_\Omega > 0$ the eigenfunction associated to the first eigenvalue $\lambda_1(\Omega)$ which satisfies :

$$\begin{cases} -\Delta\phi_\Omega = \lambda_1(\Omega)\phi_\Omega & \text{in } \Omega \\ \phi \geq 0. \end{cases}$$

and $\int_\Omega \phi_\Omega = 1$. Let $\Pi_{\mathbb{R}}$ denote the projection onto $\mathbb{R}$. We will prove as application of a priori estimates the following results :

Theorem 2 .
Assume that the assumptions of Theorem 1 are satisfied. We have the following :
If $\lambda_1(\Omega^+) < \lambda_1(\Omega_0)$, then there is a global branch of nontrivial solutions of (P), $\mathcal{C}$, connected in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$, bifurcating from $(0,0)$ such that

(i) $\Pi_{\mathbb{R}}\mathcal{C} =]-\infty\,, \lambda_0[$ *where* $0 < \lambda_0 < \lambda_1(\Omega^+)$.

(ii) Let $(\lambda_n, u_n) \in \mathcal{C}$ such that $\lambda_n \to -\infty$ as $n \to +\infty$. Then, up to subsequences, $\|u_n\|_{H^1, L^\infty} \to +\infty$.

Finally, if Ω^+ is empty, then we have :

Theorem 3 .
Assume that $\Omega^+ = \emptyset$ and Ω_0 bounded.

(i) if Ω_0 is a nonzero measure set, then there exists a global branch of nontrivial solutions of (P), $\mathcal{C}$, connected in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$, bifurcating from $(0,0)$ such that

   a) $\Pi_{\mathbb{R}}\mathcal{C} =]0\,, \lambda_1(\Omega_0)[$, *b) Let $(\lambda_n, u_n) \in \mathcal{C}$ such that $\lambda_n \to \lambda_1(\Omega_0)$ as $n \to +\infty$. Then, up to subsequences, $\|u_n\|_{L^\infty} \to +\infty$.*

(ii) If Ω_0 is empty, then (i) is also valid replacing $\lambda_1(\Omega_0)$ by $+\infty$.

The outline of the paper is as follows :
In Section 2, We prove the results concerning a priori estimates. Theorem 1, Proposition 1, Propositon 2.
In Section 3, we prove Theorems 2 and 3.

2. A Priori Estimates.

First, we prove Theorem 1 :
PROOF OF THEOREM 1.
Let us prove (i). We use a standard argument for superlinear elliptic problems. Multiply (P_Ω) by ϕ_{Ω^+} and integrate by parts in Ω^+, we obtain :

$$\lambda_1(\Omega^+)\int_{\Omega^+} u\phi_{\Omega^+} + \int_{\partial\Omega^+} \frac{\partial \phi_{\Omega^+}}{\partial n} u \;=\; \int_{\Omega^+} h(x)u^p\phi_{\Omega^+} \;+\; \lambda\int_{\Omega^+} u\phi_{\Omega^+}. \tag{3}$$

From (3), and Hopf lemma :

$$(\lambda_1(\Omega^+) - \lambda)\int_{\Omega^+} u\phi_{\Omega^+} \;\geq\; \int_{\Omega^+} h(x)u^p\phi_{\Omega^+} > 0,$$

which implies that $\lambda < \lambda_1(\Omega^+)$. Repeating the argument with ϕ_{Ω_0}, (i) is proved. Let us prove (ii). For this, we divide the proof in several parts :

1. Local Estimates in $\Omega_\delta^- := \{x \in \overline{\Omega} \cap \text{supp}h^-/d(x, \Gamma \cup \Omega_0) \geq \delta\}$. The estimate is obtained as in [8] by a uniform local L^p-estimate + the Harnack inequality (see proof of Proposition 1.1 step 1 in [8]).

2. Estimates in Ω^+. We use a blow up technique as in [6] and [3].

3. Estimates in Ω_0. We use a super and sub solutions argument.

4. L^∞-bound for $|x|$ large, obtained by the construction of a supersolution.

Step 1: A priori estimates in Ω_δ^-. See [8] or [14] (Proposition 5.1 and step 1 in the proof of Proposition 5.2). Note that since $\partial\Omega \subset \text{supp}h^-$ and by maximum principle we get a priori bounds near $\partial\Omega$.
Step 2: A priori estimates in Ω^+. For this, suppose by contradiction that there exists a sequence (λ_k, u_k) solution to (P) with $\lambda_0 \leq \lambda_k \leq \lambda_2$ and $\sup\limits_{\overline{\Omega^+}} u_k \to +\infty$. Let x_k such that $u_k(x_k) = \sup\limits_{\overline{\Omega^+}} u_k$. Up to a subsequence, we can suppose that $x_k \to x_0 \in \overline{\Omega^+}$.

We now deal with two cases :
First, suppose that $x_0 \in \Omega^+$ then we can conclude using [16] and get the contradiction. Suppose now that $x_0 \in \partial\Omega^+$. Then, either $x_0 \in \partial\Omega^+ \cap \partial\Omega_0$ or $x_0 \in \partial\Omega^+ \cap \partial\Omega^-$. Now following the same blow up analysis in [3] (see theorem 4.3) and using (H3) in the case $x_0 \in \partial\Omega^+ \cap \partial\Omega_0$ (resp. (H4) in the case $x_0 \in \Gamma$) we get a contradiction by a Liouville theorem in cones (see Theorem 2.1 in [6]).
Step 3: A priori estimates in Ω_0. Let Ω_0^i one of the connected component of Ω_0. By (H2), $\partial\Omega_0^i$ belongs to $\partial\Omega^-$ or $\partial\Omega^+$. Suppose that $\partial\Omega_0^i$ belongs to $\partial\Omega^-$, then we construct a supersolution in a ϵ-neighborhood of Ω_0^i, denoted by Ω_ϵ : Let λ_ϵ such that $\lambda_2 < \lambda_\epsilon < \lambda_1(\Omega_0)$ and ξ_ϵ the solution to

$$\begin{cases} -\Delta\phi = \lambda_\epsilon\phi \text{ in } \Omega_\epsilon \\ \phi = M \text{ in } \partial\Omega_\epsilon. \end{cases}$$

The existence and uniqueness of ξ_ϵ is provided by the fact that $\lambda_\epsilon < \lambda_1(\Omega_0)$ and ϵ small enough such that $\lambda_\epsilon < \lambda_1(\Omega_\epsilon)$.

Next, we choose $M = \sup_{\partial\Omega_\epsilon} u$ which do not depend on u since in Step 1 we have proved uniform local a priori estimates in Ω_δ^- (note that $\partial\Omega_\epsilon$ belongs to Ω_δ^- for an appropriated δ). Then, by maximum principle, we have $u \le \xi_\epsilon$.

Now, consider the case $\partial\Omega_0^i \subset \partial\Omega^+$. Therefore, by step 2, we have that u is uniformly bounded on $\partial\Omega_0^i$. Let M be the uniform bound of u in $\partial\Omega_0^i$. Now, consider ξ the unique solution to

$$\begin{cases} -\Delta\xi = \lambda_2\xi \ \text{ in } \ \Omega_0^i \\ \phi = M \text{ in } \partial\Omega_0^i. \end{cases}$$

Therefore, by the maximum principle, $u \le \xi$ in Ω_0^i. Finally, we get an uniform bound in Ω_0 since Ω_0 is bounded and have only finetely many components.

Step 4: A priori estimates for $|x|$ large. This part concerns the case where Ω is unbounded and we can suppose here that $\Omega = \mathbb{R}^N$. Let R_0 be such that $\{\Gamma \cup \partial\Omega_0\} \subset B_{R_0}$ and ϕ :

$$(P*)\begin{cases} -\Delta\phi = \lambda_2\phi + h^*(x)\phi^p \ \text{ in } \mathbb{R}^N/B_{R_0} \\ \phi = M \text{ in } \partial B_{R_0}\text{ , } \phi \to 0 \text{ when } |x| \to +\infty. \end{cases}$$

From a priori estimates in step 1, we choose M such that for any solution u, $\sup_{\partial B_{R_0}} u(x) \le M$ and h^* a continuous function such that $h^* \ge h$ we fix later. Then, by the maximum principle, $u \le \phi$. Next, thanks to $\lim_{|x|\to\infty} h(x) = -\infty$ we prove that $\phi(x)$ tends to 0 when $|x| \to +\infty$. For this, we choose h^* negative, radial symmetric, decreasing for large $r = |x|$ and $h^*(r) \to -\infty$ when $r \to +\infty$.

To prove the existence of ϕ, we consider the following problem :

$$(P_R*)\begin{cases} -\Delta\phi = \lambda_2\phi + h^*(x)\phi^p \ \text{ in } \ B_R/B_{R_0} \\ \phi = M \text{ in } \partial B_{R_0} \ \text{ and } \phi = 0 \text{ in} \partial B_R. \end{cases}$$

For R large, we claim that there exists a unique solution to (P_R*). For this, consider ψ_M a smooth continuation of M with compact support and the following minimization problem :

$$I_R = \min_{v\in H_0^1(R_0<|x|<R)} \mathcal{E}(v) := \frac{1}{2}\int_{R_0<|x|<R} \left(|\nabla(v+\psi_M)|^2 - \lambda_2(v+\psi_M)^2\right) + \frac{1}{p+1}\int_{R_0<|x|<R} |h*|(v+\psi_M)^{p+1}$$

By Sobolev imbeddings, we get easily $I_R > -\infty$ then, a global minimizer solution ϕ_R to (P_R*) exists. The uniqueness is a standard argument using the concavity of the nonlinearity (see appendix II in [9]). By uniqueness of ϕ_R and doing $R \to +\infty$, we get a minimal solution ϕ to $(P*)$ and ϕ is radial. Note that ϕ is bounded and if x_0 is a local maximum, then $\phi(x_0) \le (\frac{\lambda_2}{|h*(|x_0|)|})^{p-1}$. Therefore, since $h*(r) \to -\infty$ when $r \to +\infty$, either ϕ is decreasing for $|x|$ large either $\phi(x) \to 0$ when $|x| \to +\infty$. Assume that the first possibility holds then $\phi(x) \to l$ when $|x| \to +\infty$. If $l \ne 0$ then the O.D.E satisfied by ϕ shows that $\phi''(r) \to +\infty$ when $r \to +\infty$ which is impossible since ϕ is bounded. Note that we need here the local approach. Indeed, using that ψ_M has a compact support, for R large we claim that $I_R \to -\infty$ when $R \to +\infty$. For this, define v_R as follows :

$$v_R(x) = \begin{cases} 0 \ \text{ if } |x| \le \frac{R}{2} - 1 \ \text{ or } |x| \ge R+1 \\ (\frac{\lambda_2}{R^\alpha})^{\frac{1}{p-1}} \ \text{ if } \frac{R}{2} \le |x| \le R \\ (\frac{\lambda_2}{R^\alpha})^{\frac{1}{p-1}}(|x| - \frac{R}{2} + 1) \ \text{ if } \frac{R}{2} - 1 \le |x| \le \frac{R}{2} \\ (\frac{\lambda_2}{R^\alpha})^{\frac{1}{p-1}}(|x| - R) \ \text{ if } R \le |x| \le R+1 \end{cases}$$

Now, observing that by a simple computation

$$\mathcal{E}(v_R - \psi_M) \le C_1(\frac{\lambda_2}{R^\alpha})^{\frac{1}{p-1}} R^{N-1} - C_2(\frac{\lambda_2}{R^\alpha})^{\frac{2}{p-1}} R^N$$

which implies that $\mathcal{E}(v_R - \psi_M) \to -\infty$ when $R \to +\infty$, we have for R large enough

$$\mathcal{E}(v_R) = \mathcal{E}(v_R - \psi_M) + \mathcal{E}(0) \to -\infty \text{ when } R \to +\infty.$$

This completes the proof of assertion (ii).

(iii) and (iv) are a direct application of step 2 in [8] to get an a priori bound in a neighborhood of Γ instead of the blow up analysis when $x_0 \in \Gamma$. The proof of Theorem 1 is now complete. ■

Now, let us deal with the case where h is radial symmetric.

PROOF OF PROPOSITION 1 :

Note first that if Ω^+ is a ball B_{R_1}, then by the maximum principle, for any solution u to (P_Ω), we have for $r \le R_1$, $\min_{B_r} u(x)$ is attained on ∂B_r. Furthermore, using the L^p-estimate in [8] (see p 23-24 or Proposition 5.1 in [14]), we have :

$$\int_{B_{\frac{r}{2}}} u^p \le C := C(r, \inf_{B_r} h, \phi_1(B_{R_1}), \lambda_1, \lambda_2, p). \tag{4}$$

which implies that

$$\min_{B_{R_1}} u(x) \le \min_{B_{\frac{r}{2}}} u(x) \le C := C(r, \inf_{B_r} h, \phi_1(B_{R_1}), \lambda_1, \lambda_2, p).$$

Since u is radial symmetric, then u is uniformly bounded on $\partial\Omega^+$. Therefore doing again the blow up analysis as in step 2 of the proof of Theorem 1, we have that $x_0 \in \Omega^+$ and results in [16] are sufficient to get the contradiction. This completes the proof of Proposition 1. ■

Finally, let us prove Proposition 2

PROOF OF PROPOSITION 2 :

Let (λ, u) a non trivial solution to (P) and R_0 such that $\overline{\Omega^+ \cup \Omega_0} \subset B_{R_0}$. By Harnack inequality (see p. 199 in [17]), $\min\limits_{B_{R_0}} u(x) = m_u > 0$. For $R >> R_0$, consider the following problem :

$$(P_R) \begin{cases} -\Delta\phi_R = \lambda\phi_R - b\phi_R^p \text{ in } B_R \\ \phi = 0 \text{ in} \partial B_R, \end{cases}$$

where b satisfies $b > \sup_{|x| \ge R_0} |h(x)|$ and $(\dfrac{\lambda}{b})^{\frac{1}{p-1}} < m_u$. Then, by maximum principle, for all R large enough, it is easy to prove that $\phi_R < u$ in B_R / B_{R_0} (since $\sup \phi_R < (\frac{\lambda}{b})^{\frac{1}{p-1}}$). Now, we will show that $\phi_R \to (\frac{\lambda}{b})^{\frac{1}{p-1}}$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ when $R \to +\infty$ which completes the proof of Proposition 2. For this, note that ϕ_R is the unique nontrivial solution to (P_R) (see [5]) for R large. From [15] results, ϕ_R is radial symmetric and decreasing. Then, ϕ_R is also the global minimizer to

$$I_R = \min_{v \in H_0^1(B_R)} \mathcal{E}(v) := \frac{1}{2}\int_{B_R} |\nabla v|^2 - \lambda v^2 + \frac{1}{p+1}\int_{B_R} b\, v^{p+1}$$

and $\mathcal{E}(\phi_R) \to -\infty$ when $R \to +\infty$. For this, consider the following testing function :

$$k_R(x) = \begin{cases} 0 \text{ if } |x| \ge R-1 \\ (\frac{\lambda}{b})^{\frac{1}{p-1}} \text{ if } R-1 \le |x| \\ (\frac{\lambda}{b})^{\frac{1}{p-1}}(|x| - R + 1) \text{ if } R-1 \le |x| \le R. \end{cases}$$

Then,

$$\mathcal{E}(\phi_R) \le \mathcal{E}(k_R) \le C_1(\frac{\lambda}{b})^{\frac{2}{p-1}} R^{N-1} - C_2(\frac{\lambda}{b})^{\frac{2}{p-1}} R^N \to -\infty \text{ when } R \to +\infty. \quad (5)$$

If $R < R'$, then $\phi_R < \phi_{R'} < (\frac{\lambda}{b})^{\frac{1}{p-1}}$. Consequently, $\phi_R \to v$ when $R \to +\infty$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$. $v > 0$ is also radial symmetric and decreasing and satisfies :

$$-v_{rr} - (N-1)\frac{v_r}{r} = \lambda v - bv^p \text{ in } (0, +\infty)$$

Suppose that $v \neq (\frac{\lambda}{b})^{\frac{1}{p-1}}$, then using the above equation, it is easy to show that $v(+\infty) = 0$ and since $\phi_R \le v$, ϕ_R tends uniformly to 0 when $|x| \to +\infty$. Then for all ϵ, there exists R_ϵ such that $\phi_R(x) < \epsilon$ if $|x| \ge R_\epsilon$. Therefore, since ϕ_R is a solution to (P_R)

$$\begin{aligned} \mathcal{E}(\phi_R) &= -(\frac{1}{2} - \frac{1}{p+1}) \int_{B_R} b|\phi_R|^{p+1} \\ &\ge -C(R_\epsilon) - \epsilon^{p-1} \int_{B_R} b|\phi_R|^2 \\ &\ge -C(R_\epsilon) - \epsilon^{p-1} (\frac{\lambda}{b})^{\frac{2}{p-1}} R^N. \end{aligned}$$

But, from (5) and from ϵ small enough, we get the contradiction for large R. This completes the proof of Proposition 2. ■

Remark 5 Note that the result is not valid if $\lambda \le 0$. Indeed, for K large, the function $K|x|^{\frac{2}{p-1}}$ is a supersolution to $(P*)$ (see proof of Theorem 1 step 4). In [13], the authors investigate the decay of weak solutions of such problems. ■

3. Applications

In this section, we prove Theorems 2 and 3.
PROOF OF THEOREM 2 :
Since $\inf \lambda_1(\Omega_0), \lambda_1(\Omega^+) = \lambda_1(\Omega^+)$, we get from Theorem 1 that any nontrivial solution to (P) (resp.(P_Ω)) is uniformly bounded in $L^\infty(\mathbb{R}^N)$ (resp. in $L^\infty(\Omega)$). Then, we can proceed exactly as in [8] and get the same results. Note that the compactness of solutions in $L^\infty(\mathbb{R}^N)$ is provided by the uniform decay of solutions at infinity (see step 4 in the proof of Theorem 1). ■

Now, let us consider the case where $\Omega^+ = \emptyset$.
PROOF OF THEOREM 3 :
Let us prove (i). The existence of a global unbounded branch of solutions to (P), $\mathcal{C}$, is obtained exactly as in Theorem 2. So we don't repeat the arguments (see [8] for details). To prove the global behaviour of $\mathcal{C}$, note that if $(\lambda, u) \in \mathcal{C}$, then $0 \le \lambda < \lambda_1(\Omega_0)$. Indeed, suppose by contradiction that there exists $(\lambda, u) \in \mathcal{C}$ with $0 > \lambda$, then since $u \to 0$ when $|x| \to +\infty$, by the maximum principle, $u \equiv 0$.

Therefore, $\mathcal{C}$ has asymptotic bifurcation points. Using the a priori estimates in Theorem 1, we have that for $\lambda \neq \lambda_1(\Omega_0)$, the solutions u are uniformely bounded. Consequently, there is one and only one bifurcation point $\lambda = \lambda_1(\Omega_0)$. This completes the proof of assertion (i).

Finally, to prove (ii), we should remark that if $\Omega_0 = \emptyset$, then $\lambda_1(\Omega_0) = +\infty$. Furthermore, when $\lambda \to +\infty$, $u(\lambda) \to +\infty$ in any compact $K \subset \mathbb{R}^N$. For this, note that if $\lambda_1 < \lambda_2$ then $u(\lambda_1) < u(\lambda_2)$. This completes the proof of theorem 3. ■

Remark 6 In Assertion (ii) of Theorem 3, the branch $\mathcal{C}$ is a smooth curve. Indeed, it is easy to prove that for λ fixed, there is a unique non trivial solution to (P) (the proof is the same as in bounded domain, see [5]). Then using results in [10], we can prove that $\mathcal{C}$ is C^1. ■

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