RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **97** (3), 2003, pp. 461–466
Matemática Aplicada / Applied Mathematics
Comunicación Preliminar / Preliminary Communication

On the existence of multiple principal eigenvalues for some indefinite linear eigenvalue problems

J. Fleckinger, J. Hernández and F. de Thélin

Abstract. We study the existence of principal eigenvalues for differential operators of second order which are not necessarily in divergence form. We obtain results concerning multiplicity of principal eigenvalues in both the variational and the general case. Our approach uses systematically the Krein-Rutman theorem and fixed point arguments for the inverse of the spectral radius of some associated problems. We also use a variational characterization for both the self-adjoint and the general case.

Sobre la existencia de valores propios principales múltiples para algunos problemas lineales indefinidos de valores propios

Resumen. Estudiamos la existencia de valores propios principales para operadores diferenciales de segundo orden que no están necesariamente en forma de divergencia. Obtenemos resultados sobre la multiplicidad de valores propios principales tanto en el caso variacional como en el general. Usamos el teorema de Krein-Rutman y argumentos de punto fijo para el inverso del radio espectral de algunos problemas asociados. Utilizamos también la caracterización variacional, tanto en el caso autoadjunto como en el general.

1. Introduction

If Ω is a regular bounded domain in $\mathbb{R}^N$, with boundary $\partial\Omega$, and if a_0 and m are positive on $\overline{\Omega}$ and smooth enough, it is well known that the eigenvalue problem:

$$\begin{cases} -\Delta u + a_0(x)u = \lambda m(x)u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

possesses an infinite sequence of positive eigenvalues :

$$0 < \lambda_1 < \lambda_2 \leq \ldots \lambda_k \leq \ldots; \quad \lambda_k \to \infty, \text{ as } k \to \infty$$

with finite multiplicity. Moreover λ_1 is simple and its associate eigenfunction φ_1 is positive in Ω and $\partial\varphi_1/\partial n < 0$ on the boundary. In the following we also write λ_1, the first eigenvalue of the above problem, as $\lambda_1(-\Delta + a_0, m, \Omega)$.

Presentado por Jesús Ildefonso Díaz.
Recibido: 8 de Mayo de 2002. *Aceptado*: 4 de Diciembre de 2002.
Palabras clave / Keywords: Principal Eigenvalue, Indefinite Problems, Spectral Radius, Variational Characterization, Krein-Rutman Theorem.
Mathematics Subject Classifications: 35P15, 35P20, 35P05, 47A10, 47A75

Here we consider the case of (1) when the coefficients a_0 and m change sign.
We use the following notations:

$$\Omega^+ = \{x \in \Omega \mid m(x) > 0\},\ \Omega^- = \{x \in \Omega \mid m(x) < 0\},\ \Omega_0 = \{x \in \Omega \mid m(x) = 0\}. \qquad (2)$$

We assume that Ω^+ and Ω^- (which are actually defined up to a set of measure zero) are smooth enough subdomains of Ω such that both have positive measure. $|A|$ is the Lebesgue measure of the set A.
The classical result in [6] for continuous coefficients and self-adjoint operators was extended by Manes and Micheletti [14] (see also [9], [5]) in the sense of Theorem 1 below. A similar result was obtained by Hess and Kato [11] for operators in general form by using the Krein-Rutman Theorem ([12], [1]). All these results correspond to the case $\lambda_1(-\Delta + a_0, 1, \Omega) > 0$ in our notation.
The situation is more involved if $\lambda_1(-\Delta + a_0, 1, \Omega) < 0$ and interesting results were given in [13] for operators in general form. Here we give a much more general and unified view of the problem, providing a description of the number of principal eigenvalues depending on a parameter introduced in the weight m. Our approach uses systematically a reformulation in terms of fixed points for (the inverse of) the spectral radius of an associated eigenvalue problem. We use a more general version of the Krein-Rutman Theorem in [8] (see also [15]) and rely heavily on the variational characterization of the first eigenvalue and on a result by Dancer ([7], see also [2]) which is only available for operators in divergence form in this context. Results not using Dancer's theorem are still valid for general operators and this allows us to extend results in [11] and [3]. Detailed statements and proofs are given in [10].

2. The variational case for unbounded indefinite coefficients.

We assume that

$$a_0, m \in L^r(\Omega),\ r > \frac{N}{2}. \qquad (3)$$

We can rewrite equation (1) as

$$-\Delta u \,+\, a_0^+(x)u \,+\, \lambda m^-(x)u \,=\, (\lambda m^+(x) + a_0^-(x))u,\ x \in \Omega,$$

and we are led to study the following eigenvalue problem

$$\begin{cases} -\Delta u \,+\, (a_0^+(x)+1)u \,+\, \lambda m^-(x)u \,=\, r(m^+(x) + \frac{a_0^-(x)+1}{\lambda})u, & x \in \Omega, \\ u(x) \,=\, 0, & x \in \partial\Omega, \end{cases} \qquad (4)$$

By using Krein-Rutman Theorem [8], we can show the existence of a positive eigenvalue $r(\lambda) > 0$ to problem (4); moreover, since the coefficients in (4) are positive, we have also the variational characterization:

$$r(\lambda) \,=\, \inf_{\phi \in H_0^1(\Omega);\phi\not\equiv 0} \frac{\int_\Omega \mid \nabla\phi \mid^2 + \int_\Omega (a_0^+(x)+1)\phi^2 + \lambda \int_\Omega m^-(x)\phi^2}{\int_\Omega m^+(x)\phi^2 + \frac{1}{\lambda}\int_\Omega (a_0^-(x)+1)\phi^2}. \qquad (5)$$

It can be seen easily that $r(\lambda)$ is increasing in λ, depends monotonically on the domain and is continuous ([14], [9]). Moreover by continuity, $r(0) = 0$, and

$$r'(0) \,=\, \lim_{\lambda\to 0^+} \frac{r(\lambda)}{\lambda} \,=\, \lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega).$$

With these notations we have that:

$$\begin{cases} \lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega) > 1 \,\Leftrightarrow\, \lambda_1(-\Delta + a_0, 1, \Omega) > 0, \\ \lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega) = 1 \,\Leftrightarrow\, \lambda_1(-\Delta + a_0, 1, \Omega) = 0, \\ \lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega) < 1 \,\Leftrightarrow\, \lambda_1(-\Delta + a_0, 1, \Omega) < 0. \end{cases} \qquad (6)$$

Using the monotone dependence with respect to both the coefficients and the domain we obtain the first estimate:

$$r(\lambda) \ < \ \lambda_1(-\Delta + a_0^+ + 1, m^+, \Omega^+), \tag{7}$$

and we deduce

Theorem 1 *Assume that a_0 and m satisfy* (3)*, and that $|\Omega^+| > 0$, $|\Omega^-| > 0$.*

If $\lambda_1(-\Delta + a_0, 1, \Omega) \ > 0$, then problem (1) *possesses a unique positive (resp. negative) eigenvalue λ_1^+ ($resp.$ λ_1^-); this eigenvalue is such that*

$$0 < \lambda_1^+ < \lambda_1(-\Delta + a_0, m^+, \Omega^+), \quad (\text{resp. } 0 > \lambda_1^- > -\lambda_1(-\Delta + a_0, m^-, \Omega^-));$$

moreover λ_1^+ (resp. λ_1^-) is algebraically simple and is the unique positive (resp. negative) eigenvalue associated with a positive eigenfunction.

If $\lambda_1(-\Delta + a_0, 1, \Omega) \ = 0$, then problem (1) *has a positive (*resp. *negative)principal eigenvalue if and only if*

$$\int_\Omega m\phi_0^2 < 0 \quad (\text{resp.} > 0),$$

where ϕ_0 is the (positive) principal eigenfunction associated with $\lambda_1(-\Delta + a_0, 1, \Omega)$. In this case it is unique. ■

Remark 1 In the second case, if $\int_\Omega m\phi_0^2 \ = \ 0$, then 0 is the only possible principal eigenvalue to problem (1). ■

We consider now the case

$$\lambda_1(-\Delta + a_0, 1, \Omega) \ < 0. \tag{8}$$

From (8), (6), we have $\lim_{\lambda\to 0} \frac{r(\lambda)}{\lambda} \ < \ 1$ and we need further estimates. More relevant results can be obtained if we have an estimate of the slope at infinity of $r(\lambda)$ which is a consequence of the following proposition due to Dancer [7]. (See also [2]).

Proposition 1 *Let D be a regular bounded domain in $\mathbb{R}^N$. Let b, q, g in $L^r(D)$, $r > N/2$, be such that $b \geq 0, q \geq 0, g > 0$ on Ω. We define $D_0 \ := \ \{x \in D \ / \ q(x) = 0\}$. Assume that*

(H) $$D_0 = \overline{int(D_0)}, \quad |int(D_0)| > 0 \quad \textit{and}$$

$int(D_0)$ satisfies the cone property except may be for a set of capacity zero.
Then we have

$$\lim_{\alpha\to+\infty} \lambda_1(-\Delta + b + \alpha q, g, D) = \lambda_1(-\Delta + b, g, D_0). \quad ■$$

We study first the particular case $m^+ \equiv 0$ and rewrite the problem as

$$\begin{cases} -\Delta u \ + \ (a_0^+(x) + 1)u \ + \ \lambda m^-(x)u \ = \ \rho(\frac{a_0^-(x)+1}{\lambda})u, & x \in \Omega, \\ u(x) \ = \ 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

The first eigenvalue is given by the expression

$$\bar{r}(\lambda) \ = \ \lambda \inf_{\phi \in H_0^1(\Omega); \phi \not\equiv 0} \frac{\int_\Omega | \nabla\phi |^2 + \int_\Omega (a_0^+(x) + 1)\phi^2 + \lambda \int_\Omega m^-(x)\phi^2}{\int_\Omega (a_0^-(x) + 1)\phi^2}. \tag{10}$$

Thus $\bar{r}(\lambda)$ is convex and $\bar{r}(0) \ = \ 0$. If (H) is satisfied, it follows from Proposition 1, that its "slope at infinity" is given by

$$\lim_{\lambda\to+\infty} \frac{\bar{r}(\lambda)}{\lambda} \ = \ \lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega_0).$$

It turns out that, if $\lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega_0) \le 1$, or what is equivalent $\lambda_1(-\Delta + a_0, 1, \Omega_0) \le 0$, $\bar{r}(\lambda)$ will never intersect the diagonal and the problem has no solution. On the opposite side, if $\lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega_0) > 1$, or equivalently

$$\lambda_1(-\Delta + a_0, 1, \Omega_0) > 0, \tag{11}$$

then, since $\bar{r}(\lambda)$ is strictly convex, it will intersect exactly once the diagonal. Thus we have proved

Theorem 2 *Assume that a_0 and m satisfy* (3)*, $m^+ \equiv 0$, $|\Omega_0| > 0$, and (H) and* (8) *are satisfied. Then there is a unique positive principal eigenvalue to* (1) *if and only if* (11) *holds.* ■

Finally, we will consider the problem for a weight function which changes sign on Ω. The second estimate is derived from (5):

$$0 < \; r(\lambda) \; < \; r_2(\lambda), \tag{12}$$

where

$$r_2(\lambda) \; = \; \lambda \inf_{\phi \in H_0^1(\Omega); \phi \not\equiv 0} \frac{\int_\Omega \mid \nabla \phi \mid^2 + \int_\Omega (a_0^+(x) + 1)\phi^2 + \lambda \int_\Omega m^-(x)\phi^2}{\int_\Omega (a_0^-(x) + 1)\phi^2}. \tag{13}$$

We also have that $r_2(\lambda)$ is convex, since $\lambda \to \frac{r_2(\lambda)}{\lambda}$ is strictly increasing. Hence $r_2(\lambda)$ will intersect the straight line $r = \lambda_1(-\Delta + a_0^+, m^+, \Omega^+)$ in a unique point which will be denoted by λ^*.

Proposition 2 *If we have*

$$\lambda_1(-\Delta + a_0^+, m^+, \Omega^+) \; \le \lambda^*,$$

then there is no positive principal eigenvalue to (1). ■

Hence we have obtained the necessary condition

$$\lambda_1(-\Delta + a_0^+, m^+, \Omega^+) \; > \lambda^*, \tag{14}$$

for the existence of a positive eigenvalue.
Now we assume that (14) holds and consider a family of eigenvalue problems

$$\begin{cases} -\Delta u \; + \; a_0(x)u \; = \; \lambda(tm^+(x) \; - \; m^-(x))u, & x \in \Omega, \\ u(x) \; = \; 0, & x \in \partial\Omega, \end{cases} \tag{15}$$

where $t \ge 0$ plays the role of a parameter and we rewrite the equation as

$$-\Delta u \; + \; (a_0^+(x) + 1)u \; + \; \lambda m^-(x)u \; = \; \rho(tm^+(x) \; + \; \frac{a_0^-(x) + 1}{\lambda})u, \quad x \in \Omega. \tag{16}$$

Using again Proposition 1 and perturbation and continuity arguments , exploiting the associated variational characterization, we derive

Theorem 3 *Suppose that Ω is a regular bounded domain in $\mathbb{R}^N$ such that $|\Omega^+| > 0$, $|\Omega^-| > 0$ and moreover $\Omega^+ \cup \Omega_0$ satisfies condition (H) in Proposition 1. Assume also that conditions* (3), (14), *$\lambda_1(-\Delta + a_0, 1, \Omega) < 0$, $\lambda_1(-\Delta + a_0, 1, \Omega^+ \cup \Omega_0) > 0$ are satisfied . Then there exists a $\bar{t} > 0$ such that the eigenvalue problem* (15) *has two positive principal eigenvalues for any $t \in (0, \bar{t})$, exactly one for $t = \bar{t}$, and none if $t > \bar{t}$.* ■

Proposition 3 *Assume that the hypotheses of Theorem 3 are satisfied and let $\phi_0 > 0$ be the eigenfunction associated to $\lambda_1(-\Delta + a_0, 1, \Omega) < 0$. Then, for any $t > 0$ such that* (15) *has a positive principal eigenvalue, we have*

$$t \; < \; \frac{\int_\Omega m^- \phi_0^2}{\int_\Omega m^+ \phi_0^2}. \quad ■$$

Corollary 1 *If $\int_\Omega m\phi_0^2 \ge 0$, then there is no positive principal eigenvalue to problem* (1). ■

3. The case of a general operator

3.1. Smooth domains

We consider here the case of a differential operator in general form on a bounded domain Ω in $\mathbb{R}^N$. The corresponding eigenvalue problem can be written as

$$\begin{cases} Lu = \lambda m(x)u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{17}$$

where

$$Lu \;=\; -\sum_{i,j} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} \;+\; \sum_i b_i(x)\frac{\partial u}{\partial x_i} \;+\; a_0(x)u \tag{18}$$

is a second order uniformly elliptic differential operator. We assume that both Ω and the coefficients in L are such that the L^p-regularity theory applies; in particular we have

$$a_0 \in L^\infty(\Omega). \tag{19}$$

Moreover we assume that

$$m^- \in L^\infty(\Omega), \tag{20}$$

$$m^+ \in L^r(\Omega),\; r > \frac{N}{2}, \tag{21}$$

are satisfied.
As above (17) can be written equivalently as

$$\begin{cases} Lu + \lambda(m^-(x) + \chi_{\Omega^- \cup \Omega_0})u \;=\; \lambda(m^+(x) + \chi_{\Omega^- \cup \Omega_0})u, & x \in \Omega, \\ \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

and the associated eigenvalue problem is now

$$\begin{cases} Lu + \lambda(m^-(x) + \chi_{\Omega^- \cup \Omega_0})u \;=\; \rho(m^+(x) + \chi_{\Omega^- \cup \Omega_0})u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

By using again the Krein-Rutman Theorem in [8] we prove

Theorem 4 *Suppose that the above assumptions, as well as* (19) *to* (21) *are satisfied. Then there exists a unique positive (resp. negative) eigenvalue* $\lambda_1^+(L, m, \Omega)$ *(*$resp.\ \lambda_1^-(L, m, \Omega)$*) to* (17)*; moreover*

$$0 < \lambda_1^+(L, m, \Omega) < \lambda_1(L, m^+, \Omega^+) \quad \textit{(resp. } -\lambda_1(L, m^-, \Omega^-) < \lambda_1^-(L, m, \Omega) < 0\textit{).}$$

Moreover, $\lambda_1^+(L, m, \Omega)$ *(resp.* $\lambda_1^-(L, m, \Omega)$*) is algebraically simple and it is the only positive (*$resp.$ *negative) eigenvalue having a positive eigenfunction.* ■

Remark 2 Again similar arguments allow to extend some of the results in [3] and [4] for non-smooth domains in the same direction. ■

Acknowledgement. The second author is supported by REN 2000-0766, Spain and European Network IHP-RTN-002.

References

[1] Amann, H. (1976). Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Review*, **18**, 620–709.

[2] Belaud, Y. (2004). Asymptotic estimates for a variational problem involving a quasilnear operator in the semi-classical limit, *Ann. Global Anal. Geom.*, to appear.

[3] Berestycki, H., Nirenberg, L. and Varadhan, S. R. S. (1994). The principal eigenvalue and Maximum Principle for second-order elliptic operators in general domains, *Comm. Pure Appl. Math.*, **47**, 47–92.

[4] Birindelli, I. (1995). Hopf's Lemma and Anti-Maximum Principle in general domains, *J. Differential Equations*, **119**, 450–472.

[5] Brown, K. J. and Lin, C. C. (1980). On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight-function, *J. Math. Anal. Appl.*, **75**, 112–120.

[6] Courant, R. and Hilbert, D. (1962). *Methods of Mathematical Physics*, Vol. **2**, Wiley-Interscience, New York.

[7] Dancer, E. N. (1996). Some remarks on classical problems and fine properties of Sobolev spaces, *Differential Integral Equations*, **9**, 437–446.

[8] Daners, D. and Koch-Medina, P. (1992). *Abstract Evolution Equations, Periodic Problems and Applications*, Longman Research Notes, **279**.

[9] Figueiredo, D. G. de (1982). Positive solutions of semilinear equations, *Lecture Notes in Math.*, **957**, Springer, 34–87.

[10] Fleckinger, J., Hernández, J. and Thélin, F. de (2004). Existence of multiple principal eigenvalues for some indefinite linear eigenvalue problems, *Boll. Un. Mat. Ital.*, **8** 7-B, 159–188.

[11] Hess, P. and Kato, T. (1980). On some nonlinear eigenvalue problems with an indefinite weight-function, *Comm. Partial Differential Equations*, **5**, 999–1030.

[12] Krein, M. G. and Rutman, M. A. (1950). Linear operators leaving invariant a cone in a Banach space, *A.M.S. Transl.*, **26**, 1–128.

[13] López-Gómez, J. (1996). The Maximum Principle and the existence of principal eigenvalues for some linear weighted boundary value problems, *J. Differential Equations*, **127**, 263–294.

[14] Manes, A. and Micheletti, A. M. (1973). Un estensione della teoria variazionale classica degli autovalori per operatori ellitici del secondo ordine, *Boll. Un. Matem. Ital.*, **7**, 285–301.

[15] Pagter, B. de (1986). Irreducible compact operators, *Math. Zeit.*, **192**, 149–153.

J. Fleckinger
MIP-UT1,
Univ. Toulouse1
Pl. A. France
31042 Toulouse Cedex
France
`jfleck@univ-tlse1.fr`

J. Hernández
Departamento de Matemática
Univ. Autónoma
28049 Madrid
Spain
`Jesus.Hernandez@uam.es`

F. de Thélin
MIP
Univ. Paul Sabatier
118 rte de Narbonne
31062 Toulouse Cedex 04
France
`dethelin@mip.ups-tlse.fr`