**RAC**SAM
*Rev. R. Acad. Cien. Serie A. Mat.*
VOL. **97 (**3), 2003, pp. 423–445
Matemática Aplicada / Applied Mathematics

# Nonlinear nonlocal evolution problems

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**Abstract.** We consider a class of nonlinear parabolic problems where the coefficients are depending on a weighted integral of the solution. We address the issues of existence, uniqueness, stationary solutions and in some cases asymptotic behaviour.

**Problemas no locales y no lineales de evolución**

**Resumen.** Se considera una clase de ecuaciones parabólicas no lineales en las que algunos de los coeficientes dependen de una integral, con un cierto peso, de la solución. Se estudia la existencia y unicidad de soluciones, así como para el problema estacionario asociado, y, en ciertos casos, se analiza el comportamiento asintótico.

## 1. Introduction

In this note we would like to present some of the new techniques introduced recently to study nonlocal time dependent problems. We will restrict ourselves to a special class of problems hoping raising the interest of the reader to develop further tools. Our main effort will be devoted to the dynamical behaviour of such problems. As we will stress out, one of the main difficulty there is the absence of obvious Lyapunov functions. Let us first introduce our notation.

We will denote by $\Omega$ a bounded open subset of $\mathbb{R}^n$, $n \geq 1$. We suppose the boundary $\Gamma$ of $\Omega$ divided into two measurable subsets $\Gamma_D$ and $\Gamma_N = \Gamma \setminus \Gamma_D$. We denote by $a_{ij} = a_{ij}(\zeta)$, $i, j = 1, \ldots, n$ and $a_0 = a_0(\zeta)$ functions satisfying:

$$a_{ij}, a_0 \text{ are bounded, continuous from } \mathbb{R} \text{ into } \mathbb{R}, \tag{1}$$

there exist positive constants $\lambda, \Lambda$ such that

$$\lambda|\xi|_2^2 \leq \sum_{i,j=1}^{n} a_{ij}(\zeta)\xi_i\xi_j \leq \Lambda|\xi|_2^2 \quad \forall\, \xi \in \mathbb{R}^n,\ \forall\, \zeta \in \mathbb{R}, \tag{2}$$

$$0 \leq a_0(\zeta) \leq \Lambda \quad \forall\, \zeta \in \mathbb{R}. \tag{3}$$

In other words the operator that we will use below will be uniformly elliptic. If $\partial_{x_i}$ denotes the partial derivative in the direction $x_i$ we introduce the operator defined for any $\zeta \in \mathbb{R}$ by

$$A = A(\zeta) = \sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(\zeta)\partial_{x_i}). \tag{4}$$

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Presentado por Jesús Ildefonso Díaz.
*Recibido*: 26 de Noviembre de 2002. *Aceptado*: 5 de Noviembre de 2003.
*Palabras clave / Keywords*: Nonlocal problems, Elliptic equations, Parabolic equations, Asymptotic behaviour, Stationary problems
*Mathematics Subject Classifications*: 35A05, 35B40, 35Jxx, 35Kxx

If $\nu$ denotes the outward unit normal to $\Gamma$ we define the conormal derivative of a function $u$ by

$$\partial_{\nu_A} u = \partial_{\nu_{A(\zeta)}} u = \sum_{i,j=1}^{n} a_{ij}(\zeta)\partial_{x_i} u \nu_j \tag{5}$$

where $\nu = (\nu_1, \dots, \nu_n)$. Then we would like to consider the problem

$$\begin{cases} u_t - A(\ell(u(t)))u + a_0(\ell(u(t)))u = f \text{ in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 \text{ on } \Gamma_D \times \mathbb{R}^+, \\ \partial_{\nu_{A(\ell(u(t)))}} u = 0 \text{ on } \Gamma_N \times \mathbb{R}^+, \\ u(\cdot, 0) = u_0 \text{ in } \Omega. \end{cases} \tag{6}$$

In the above system $\ell$ is defined by

$$\ell(u(t)) = \int_\Omega g(x)u(x,t)\,dx. \tag{7}$$

The functions $f$, $g$, $u_0$ are such that

$$f, g, u_0 \in L^2(\Omega). \tag{8}$$

Of course for these kinds of problems many variants of $\ell$ are possible. For instance in [11] $\ell$ is no more a linear form on $L^2(\Omega)$ but represents some elastic energy given by

$$\ell(u) = \int_\Omega |\nabla u(x,t)|^2\,dx. \tag{9}$$

It is also possible – depending on the application that we have in mind – to have different $\ell$'s in the coefficients and to have coefficients depending on several of them, see [6]. However for simplicity we will restrict ourselves to the problem (6). Note that it would be also interesting to address the case of nonhomogeneous boundary conditions. Let us give few examples of problems (6). In what follows $a$ is a positive continuous function.

**Example 1.**

$$(a_{ij}) = a(\zeta)\,\mathrm{Id}, \qquad a_0 \equiv 0, \qquad \Gamma_D = \Gamma, \tag{10}$$

where $\mathrm{Id}$ is the identity matrix. The problem (6) becomes

$$\begin{cases} u_t - a(\ell(u(t)))\Delta u = f \text{ in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 \text{ on } \Gamma \times \mathbb{R}^+, \\ u(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{11}$$

where $\Delta$ is the usual Laplace operator. This problem has been investigated in [6], [7], [8], [10]. From a physical point of view, it describes the evolution of a population whose diffusion velocity depends on a nonlocal quantity. The rate of supply in this population is $f$. Note that we will choose it here most of the time independent of $t$ even so some variants of our results could be obtained in the time dependent case. Note that (10) does not take death into account (see [4] for more details on the modelisation).

For $\ell$ – especially in the case of population dynamics – several obvious candidates come in mind. For instance – for $g \equiv 1$ –

$$\ell(u(t)) = \int_\Omega u(x,t)\,dx \tag{12}$$

is the total population in our system. If $\Omega'$ denotes a subdomain of $\Omega$ and $g = \mathcal{X}_{\Omega'}$ then

$$\ell(u(t)) = \int_{\Omega'} u(x,t)\,dx \tag{13}$$

takes only into account the population of $\Omega'$. Now some parts of the population could play a crucial rôle which could lead to introduce a "weight" $g$ as in the formula (7). Note that our analysis with population could apply also to a model of heat propagation – i.e. $u$ could be a temperature – for some special class of bodies which is left to the imagination of the reader.

**Example 2.**

$$(a_{ij}) = a(\zeta)\,\mathrm{Id}, \qquad a_0 \equiv 1. \tag{14}$$

The problem is then:

$$\begin{cases} u_t - a(\ell(u(t)))\Delta u + u = f \text{ in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 \text{ on } \Gamma_D \times \mathbb{R}^+, \quad \partial_\nu u(x,t) = 0 \text{ on } \Gamma_N \times \mathbb{R}^+, \\ u(\cdot,0) = u_0 \text{ in } \Omega. \end{cases} \tag{15}$$

$\partial_\nu$ is the usual normal derivative. This problem is studied in [2]. Note that with respect to the preceding example the variant consists only in introducing a constant death rate. However, from a mathematical point of view the analysis has to be more involved. In particular the research of the stationary points requires to solve an equation which is not so explicit as in Example 1.

**Example 3.**

$$(a_{ij}) = \mathrm{Id}, \qquad a_0 = a(\zeta). \tag{16}$$

Then the problem becomes nonlocal with respect to the lower order term i.e. we have to solve

$$\begin{cases} u_t - \Delta u + a(\ell(u(t)))u = f \text{ in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 \text{ on } \Gamma_D \times \mathbb{R}^+, \quad \partial_\nu u(x,t) = 0 \text{ on } \Gamma_N \times \mathbb{R}^+, \\ u(\cdot,0) = u_0 \text{ in } \Omega. \end{cases} \tag{17}$$

It corresponds to a constant diffusion rate and a nonlocal death rate. For various results in this case we refer the reader to [3].

Perhaps some comments on nonlocality are in order. By opposition to the nonlocal problem (6), one calls local the variant of (6) given by

$$\begin{cases} u_t - \partial_{x_j}\{a_{ij}(u(x,t))\partial_{x_i} u\} + a_0(u(x,t))u = f \text{ in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 \text{ on } \Gamma_D \times \mathbb{R}^+, \quad \partial_{\nu_{A(u(x,t))}} u = 0 \text{ on } \Gamma_N \times \mathbb{R}^+, \\ u(\cdot,0) = u_0. \end{cases} \tag{18}$$

At each step in time the system is driven by the knowledge of $u(x,t)$ at every point. In the so called nonlocal case (6) the information known is only of integral type. Thus a lot of information is lost. In particular one can very well have

$$\ell(u(t)) = \int_\Omega g(x)u(x,t)\,dx = \int_\Omega g(x)v(x,t)\,dx = \ell(v(t)) \tag{19}$$

but

$$u \neq v. \tag{20}$$

As a consequence, the comparison principle

$$\underline{u}_0 \leq \overline{u}_0 \quad \Rightarrow \quad \underline{u} \leq \overline{u}$$

which holds for (18) fails for (6) (see [4], [8], $\underline{u}$ is the solution corresponding to the initial data $\underline{u}_0$, $\overline{u}$ the one corresponding to $\overline{u}_0$). Moreover, if the stationary problem associated to (18) admits a unique solution (under some mild assumptions – see [4]) this fails for the stationary problem associated to (6) as we will see below. This, together with the difficulty of exhibiting Lyapunov functions makes the asymptotic behaviour of (6) very challenging.

We will divide our note in three further sections. In Section 2**.** we will address the issue of existence and uniqueness of a solution to (6). In Section 3**.** we will study the stationary problem corresponding to (6). Finally in the case of an example we will consider some asymptotic behaviour.

## 2. Existence and uniqueness

There are various techniques to address the problem (see [15]). We will rely here on a very simple fixed point argument. Let us fix some positive time $T$. Then define the space $V$ as

$$V = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \}. \tag{21}$$

(We refer the reader to [1], [12], [13], [14] for the different spaces introduced here). We will suppose $V$ equipped by the topology of $H^1(\Omega)$ defined by the norm

$$\|v\|_{1,2}^2 = \int_\Omega \{ |\nabla v(x)|^2 + v(x)^2 \} \, dx. \tag{22}$$

($\nabla$ is the usual gradient, $|\cdot|$ the euclidean norm in $\mathbb{R}^n$). Let us denote by $V'$ the strong dual of $V$. Since it does not complicate the problem we will suppose in the theorem below that $f$ depends also on $t$ and assume

$$f \in L^2(0,T;L^2(\Omega)), \qquad g, u_0 \in L^2(\Omega). \tag{23}$$

Then the existence of a weak solution to (6) is given by the result below.

**Theorem 1** *Under the assumptions* (1)–(3), (23) *there exists a function* $u$ *such that*

$$\begin{cases} u \in L^2(0,T;V) \cap C([0,T], L^2(\Omega)), \; u_t \in L^2(0,T;V'), \\ \dfrac{d}{dt}(u,v) + \displaystyle\sum_{i,j=1}^n a_{ij}(\ell(u(t))) \int_\Omega \partial_{x_i} u \partial_{x_j} v \, dx + a_0(\ell(u(t)))(u,v) \\ \qquad = (f,v) \quad \forall\, v \in V, \; \textit{in } \mathcal{D}'(0,T), \\ u(\cdot,0) = u_0 \; \textit{in } \Omega. \end{cases} \tag{24}$$

*Moreover if the* $a_{ij}$*'s,* $a_0$ *are locally Lipschitz continuous the solution is unique. (In the above,* $(u,v)$ *denotes the usual scalar product in* $L^2(\Omega)$ *– we refer to [1], [4], [12] for the definition of the different spaces introduced).*

PROOF. Let us set – if $|\cdot|_{L^2(0,T;L^2(\Omega))}$ denotes the norm in this space –

$$B = \{ v \in L^2(0,T;L^2(\Omega)) \mid |v|_{L^2(0,T;L^2(\Omega))} \leq C_0 \} \tag{25}$$

where $C_0$ is a constant that we will fix later on. Recall that $L^2(0,T;L^2(\Omega))$ can be identified with $L^2((0,T)\times\Omega)$ – see also [1], [4] for the definition of the different norms introduced below. We are going to apply the Schauder fixed point theorem for the convex ball $B$ of $L^2(0,T;L^2(\Omega))$. If $w \in B$ we introduce

$$u = F(w) \tag{26}$$

the solution to

$$\begin{cases} u \in L^2(0,T;V) \cap C([0,T],L^2(\Omega)), \quad u_t \in L^2(0,T;V'), \\ \dfrac{d}{dt}(u,v) + \displaystyle\sum_{i,j=1}^{n} a_{ij}(\ell(w(t))) \int_\Omega \partial_{x_i} u \partial_{x_j} v \, dx + a_0(\ell(w(t)))(u,v) \\ \qquad = (f,v) \quad \forall\, v \in V, \text{ in } \mathcal{D}'(0,T), \\ u(\cdot,0) = u_0. \end{cases} \tag{27}$$

(27) is a linear problem and the existence and uniqueness of $u$ results from a well known result of J.L. Lions (see [12]). Taking $v = u$ in (27) we get by (2) and the Cauchy–Schwarz inequality

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \lambda \big||\nabla u|\big|_2^2 \le |f|_2|u|_2 \tag{28}$$

($|\cdot|_2$ is the usual $L^2(\Omega)$-norm). Let us set

$$\|u\|_2 = |u|_{L^2(0,T;L^2(\Omega))} = \left\{ \int_0^T \int_\Omega u^2(x,t)\, dx\, dt \right\}^{1/2}. \tag{29}$$

Integrating (28) on $(0,t)$ for $t \le T$ we obtain

$$\begin{aligned} \frac{1}{2}|u(t)|_2^2 + \lambda \int_0^t \big||\nabla u|\big|_2^2\, dt &\le \frac{1}{2}|u_0|_2^2 + \int_0^T |f(t)|_2 |u(t)|_2\, dt \\ &\le \frac{1}{2}|u_0|_2^2 + \|f\|_2\|u\|_2. \end{aligned} \tag{30}$$

With a further integration in $t$ we get easily – dropping the gradient term – and using the Young inequality -

$$\begin{aligned} \|u\|_2^2 &\le T|u_0|_2^2 + 2T\|f\|_2\|u\|_2 \\ &\le T|u_0|_2^2 + \frac{1}{2}\|u\|_2^2 + 2T^2\|f\|_2^2. \end{aligned} \tag{31}$$

From this we deduce that

$$\|u\|_2^2 \le 2T|u_0|_2^2 + 4T^2\|f\|_2^2 = C_0^2 \tag{32}$$

where we have set $C_0 = \{2T|u_0|_2^2 + 4T^2\|f\|_2^2\}^{1/2}$. This shows that the map $F$ defined by (26) applies $B$ into itself. Moreover, combining (30), (32) we derive easily that it holds that

$$|u|_{L^2(0,T;V)} \le C' \tag{33}$$

and by using the equation that we have

$$|u_t|_{L^2(0,T;V')} \le C' \tag{34}$$

where $C'$ is some constant independent of $w$. It is not difficult to show that $F$ is continuous from $B$ into $B$ (see [4]) – since from (33), (34), $F(B)$ is relatively compact in $B$ – this completes the existence part by the Schauder fixed point theorem.

To show uniqueness let us assume – without loss of generality since $u \in C([0,T],L^2(\Omega))$ – that the $a_{ij}$'s and $a_0$ are Lipschitz continuous i.e. that

$$|a_{ij}(\zeta) - a_{ij}(\zeta')|, |a_0(\zeta) - a_0(\zeta')| \le A|\zeta - \zeta'| \quad \forall\, \zeta, \zeta' \in \mathbb{R}. \tag{35}$$

Consider $u_1$, $u_2$ two solutions to (27). By difference we get in $\mathcal{D}'(0,T)$

$$\begin{aligned}\frac{d}{dt}(u_1-u_2,v) &+ \sum_{i,j=1}^{n} a_{ij}(\ell(u_1(t)))\int_\Omega \partial_{x_i}(u_1-u_2)\partial_{x_j}v\,dx \\ &+ a_0(\ell(u_1(t)))(u_1-u_2,v) \\ &= \sum_{i,j=1}^{n}\{a_{ij}(\ell(u_2(t)))-a_{ij}(\ell(u_1(t)))\}\int_\Omega \partial_{x_i}u_2\partial_{x_j}v\,dx \\ &+ \{a_0(\ell(u_2(t)))-a_0((\ell(u_1(t)))\}(u_2,v) \quad \forall\, v\in V.\end{aligned} \tag{36}$$

Taking $v=u_1-u_2$ we derive by (35)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 &+ \lambda\big||\nabla(u_1-u_2)|\big|_2^2 \\ &\le nA|\ell(u_2(t))-\ell(u_1(t))|\big||\nabla u_2|\big|_2\big||\nabla(u_1-u_2)|\big|_2 \\ &+ A|\ell(u_2(t))-\ell(u_1(t))||u_2|_2|u_1-u_2|_2.\end{aligned} \tag{37}$$

Recalling (7) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 &+ \lambda\big||\nabla(u_1-u_2)|\big|_2^2 \\ &\le nA|g|_2\big||\nabla u_2|\big|_2|u_1-u_2|_2\big||\nabla(u_1-u_2)|\big|_2 + A|g|_2|u_2|_2|u_1-u_2|_2^2.\end{aligned} \tag{38}$$

Applying in the first term of the right hand side of (38) the Young inequality

$$ab \le \frac{\lambda}{2}a^2 + \frac{1}{2\lambda}b^2$$

it comes

$$\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + \lambda\big||\nabla(u_1-u_2)|\big|_2^2 \le \frac{\lambda}{2}\big||\nabla(u_1-u_2)|\big|_2^2 + c(t)|u_1-u_2|_2^2 \tag{39}$$

where

$$c(t) = \frac{n^2A^2|g|_2^2\big||\nabla u_2|\big|_2^2}{2\lambda} + A|g|_2|u_2|_2 \in L^1(0,T). \tag{40}$$

Thus

$$\frac{d}{dt}|u_1-u_2|_2^2 \le 2c(t)|u_1-u_2|_2^2$$

and the uniqueness follows by the Gronwall inequality. This completes the proof of the theorem. ■

**Remark 1** In what follows we will assume that we are under the assumptions of Theorem 1. By a solution to (6) we will then mean the weak solution to (27) defined for every $T$. Note that $f=f(x)$ implies that $f$ belongs to $L^2(0,T,L^2(\Omega))$ for every $T$ so that Theorem 1 applies for every $T$. ■

# 3. Stationary solutions

A stationary solution to (6) – recall (4), (5) – is a solution to

$$\begin{cases} -A(\ell(u))u + a_0(\ell(u))u = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_D, \quad \partial_{\nu_{A(\ell(u))}}u = 0 \text{ on } \Gamma_N, \end{cases} \tag{41}$$

where

$$\ell(u) = \int_\Omega g(x)u(x)\,dx. \tag{42}$$

Introducing the operator $\mathcal{A}$ defined by

$$\mathcal{A}(\zeta)v = A(\zeta)v - a_0(\zeta)v = \sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(\zeta)\partial_{x_i} v) - a_0(\zeta)v, \tag{43}$$

(41) can be written as

$$\begin{cases} -\mathcal{A}(\ell(u))u = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_D, \quad \partial_{\nu_{\mathcal{A}(\ell(u))}} u = 0 \text{ on } \Gamma_N. \end{cases} \tag{44}$$

We will deal with the weak formulation of (44) – i.e.

$$\begin{cases} u \in V, \\ \sum_{i,j=1}^{n} a_{ij}(\ell(u))(\partial_{x_i} u, \partial_{x_j} v) + a_0(\ell(u))(u,v) = (f,v) \quad \forall\, v \in V. \end{cases} \tag{45}$$

(Recall that $(\cdot,\cdot)$ is the usual scalar product in $L^2(\Omega)$. To simplify our exposition, in all this section, we will suppose

$$|\Gamma_D| \neq 0 \quad \text{or} \quad |\Gamma_D| = 0 \quad \text{and} \quad a_0(\zeta) > 0 \quad \forall\, \zeta \in \mathbb{R}. \tag{46}$$

($|\Gamma_D|$ denotes the superficial measure of $\Gamma_D$).

Under the assumption (46), by the Lax–Milgram theorem, for any $\zeta \in \mathbb{R}$ there exists a unique

$$\varphi = \varphi_{\mathcal{A}(\zeta)} \tag{47}$$

solution to

$$\begin{cases} \varphi \in V, \\ \sum_{i,j=1}^{n} a_{ij}(\zeta)(\partial_{x_i} \varphi, \partial_{x_j} v) + a_0(\zeta)(\varphi,v) = (f,v) \quad \forall\, v \in V. \end{cases} \tag{48}$$

Then we have

**Theorem 2** *Under the assumptions* (1)–(3), (8), (46) *the mapping*

$$u \mapsto \ell(u) \tag{49}$$

*is a one-to-one mapping from the set of solutions of* (45) *onto the set of the solutions of the equation in* $\mathbb{R}$

$$\mu = \ell(\varphi_{\mathcal{A}(\mu)}) = \int_\Omega g\varphi_{\mathcal{A}(\mu)}\,dx. \tag{50}$$

PROOF. Suppose that $u$ is solution of (45) – then by (48) we have

$$u = \varphi_{\mathcal{A}(\ell(u))}. \tag{51}$$

It follows that

$$\ell(u) = \ell(\varphi_{\mathcal{A}(\ell(u))})$$

i.e. $\ell(u)$ is a solution of (50). This shows that $\ell$ maps the solutions of (45) into the set of solutions of (50).

Consider now $\mu$ a solution to (50). Set

$$u = \varphi_{\mathcal{A}(\mu)}. \tag{52}$$

Applying $\ell$ to both sides of the equality we get

$$\ell(u) = \ell(\varphi_{\mathcal{A}(\mu)}) = \mu \tag{53}$$

and by (52)

$$u = \varphi_{\mathcal{A}(\ell(u))} \tag{54}$$

i.e. $u$ is solution to (45). This shows that the map $\ell$ is onto. Clearly now, if $u_1$, $u_2$ are solutions to (45) with $\ell(u_1) = \ell(u_2)$ then $u_1 = u_2$. This completes the proof of the theorem. ■

**Remark 2** To solve the stationary problem (45) reduces to solve an equation in $\mathbb{R}$. Such a phenomenon for nonlocal problems was already observed in [9]. ■

Using Theorem 2 we can then solve the stationary problem (45). We have

**Theorem 3** *Suppose*

$$|\Gamma_D| \neq 0 \quad \textit{or} \quad |\Gamma_D| = 0 \quad \textit{and} \quad a_0 \geq \lambda > 0 \tag{55}$$

*for some positive constant $\lambda$ that without loss of generality we can take as before. Then the stationary problem* (45) *admits at least one solution.*

PROOF. Consider the bilinear form $a(u, v)$ defined by

$$a_\zeta(u, v) = a(u, v) = \sum_{i,j=1}^{n} a_{ij}(\zeta)(\partial_{x_i} u, \partial_{x_j} v) + a_0(\zeta)(u, v). \tag{56}$$

We claim that for some constant $c$ – independent of $\zeta$– it holds that

$$\lambda c \|u\|_{1,2}^2 \leq a(u, u) \quad \forall\, u \in V. \tag{57}$$

Indeed we have by (2)

$$a(u, u) \geq \lambda \big| |\nabla u| \big|_2^2 + a_0(\zeta)|u|_2^2.$$

If $|\Gamma_D| = 0$ the result is clear with $c = 1$. If $|\Gamma_D| \neq 0$ it follows from the fact that $\big||\nabla v|\big|_2$ and $\|v\|_{1,2}$ are two equivalent norms on $V$.

Let us now consider $\varphi$ the solution to (48). Taking $v = \varphi$ in (48) it follows from (57) that it holds that

$$\lambda c \|\varphi\|_{1,2}^2 \leq (f, \varphi) \leq |f|_2 |\varphi|_2$$

and thus

$$\|\varphi\|_{1,2} \leq \frac{|f|_2}{c\lambda}. \tag{58}$$

Let $\zeta, \zeta' \in \mathbb{R}$. We denote by $\varphi$, $\varphi'$ the solutions to (48) corresponding to $\zeta$, $\zeta'$ respectively. We have for $v \in V$

$$\begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(\zeta)(\partial_{x_i} \varphi, \partial_{x_j} v) + a_0(\zeta)(\varphi, v) \\ &\quad = \sum_{i,j=1}^{n} a_{ij}(\zeta')(\partial_{x_i} \varphi', \partial_{x_j} v) + a_0(\zeta')(\varphi, v) \end{aligned}$$

and thus

$$
\begin{aligned}
&\sum_{i,j=1}^{n} a_{ij}(\zeta)(\partial_{x_i}(\varphi-\varphi'),\partial_{x_j}v)+a_0(\zeta)(\varphi-\varphi',v)\\
&\quad=\sum_{i,j=1}^{n}\{a_{ij}(\zeta')-a_{ij}(\zeta)\}(\partial_{x_i}\varphi',\partial_{x_j}v)+\{a_0(\zeta')-a_0(\zeta)\}(\varphi',v).
\end{aligned}
\tag{59}
$$

Taking $v=\varphi-\varphi'$ we deduce easily that it holds that – see (57)

$$
\lambda c\|\varphi-\varphi'\|_{1,2}^2 \le \left\{\sum_{i,j=1}^{n}|a_{ij}(\zeta')-a_{ij}(\zeta)|+|a_0(\zeta')-a_0(\zeta)|\right\}\|\varphi'\|_{1,2}\|\varphi-\varphi'\|_{1,2}
$$

and thus, using (58), we get for some constant $c$

$$
\|\varphi-\varphi'\|_{1,2} \le \left\{\sum_{i,j=1}^{n}|a_{ij}(\zeta')-a_{ij}(\zeta)|+|a_0(\zeta')-a_0(\zeta)|\right\}\frac{|f|_2}{c^2\lambda^2}. \tag{60}
$$

By Theorem 1, to have a solution to (45) it is enough to show that (50) has a solution. From (60) it is clear that the mapping

$$
\zeta \mapsto \varphi_{\mathcal{A}(\zeta)} \tag{61}
$$

is continuous from $\mathbb{R}$ into $H^1(\Omega)$ and thus from $\mathbb{R}$ into $L^2(\Omega)$. Thus the mapping

$$
\mu \mapsto \mu - \int_\Omega g\varphi_{\mathcal{A}(\mu)}\,dx \tag{62}
$$

is continuous. Moreover, by (58), $\varphi_{\mathcal{A}(\mu)}$ is bounded independently of $\mu$ so that it holds that

$$
\lim_{\mu\to\pm\infty} \mu - \int_\Omega g\varphi_{\mathcal{A}(\mu)}\,dx = \pm\infty \tag{63}
$$

and there is – by the intermediate value theorem – a solution to (50) and thus to (45). This completes the proof of the theorem. ■

As we mentioned above the solution to (45) might fail to be unique. This was already observed in [7]. To see it consider for instance the case of example 1. Let us introduce $\tilde\varphi$ the solution to

$$
\begin{cases}
-\Delta\tilde\varphi = f \quad \text{in } \Omega,\\
\tilde\varphi = 0 \quad \text{on } \Gamma.
\end{cases}
\tag{64}
$$

Then, we have clearly in this case

$$
\varphi_{\mathcal{A}(\zeta)} = \frac{\tilde\varphi}{a(\zeta)}, \tag{65}
$$

and the equation (50) becomes

$$
\mu = \ell\Big(\frac{\tilde\varphi}{a(\mu)}\Big) \quad \Longleftrightarrow \quad a(\mu) = \frac{\ell(\tilde\varphi)}{\mu}. \tag{66}
$$

Then the set of solutions to (50) is the intersection of the curve defined by $a$ and a branch of hyperbola (if $\ell(\tilde\varphi)\neq 0$). Several cases can occur that are described in the pictures below.

The same phenomenon can occur in the case of the examples 2 and 3. However it is more difficult to show it since the equation (50) is, in these cases, not so simple as (66). We refer the reader to [2], [3] for details, see also below after (108).

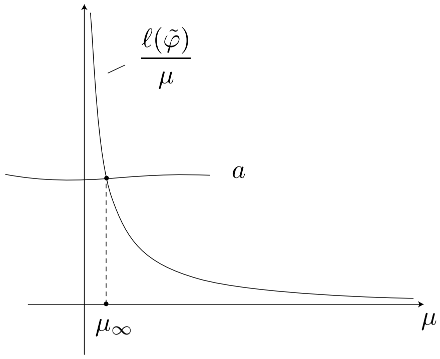

Fig. 1 The case $(^{-}) > 0$, a single solution

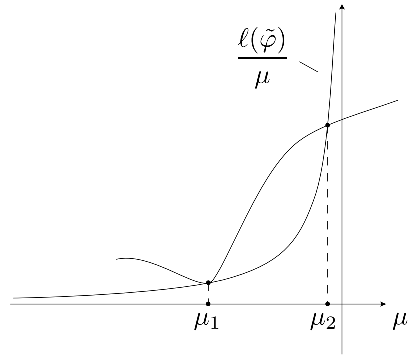

Fig. 2 The case $(^{-}) < 0$, two solutions

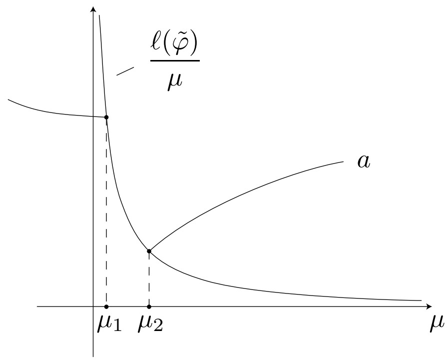

Fig. 3 The case of a continuum of solutions

# 4. Asymptotic behaviour

## 4.1. A linearized stability result

We give here a local stability result – however our assumptions are rather weak. For a matrix $A = (a_{ij})$ we denote by $|A|$ the euclidean norm defined as

$$|A| = \left\{ \sum_{i,j=1}^{n} a_{ij}^2 \right\}^{1/2}. \tag{67}$$

If $A = (a_{ij})$ where $a_{ij}$ are $C^1$-functions we denote by $A'$ the matrix of the derivatives of the $a_{ij}$'s – i.e.

$$A' = (a'_{ij}). \tag{68}$$

**Theorem 4** *Suppose that $a_{ij}$, $a_0$ are $C^1$-functions satisfying* (1)–(3)*. Suppose in addition that* (55) *holds. Let $u$ be the weak solution to* (6) *and let $u_\infty$ be a stationary point that is to say a solution to* (45)*. If $\mu_\infty = \ell(u_\infty)$ assume that*

$$|A'(\mu_\infty)| + |a'_0(\mu_\infty)| < \lambda^2 c^2 / |f|_2 |g|_2 \tag{69}$$

*where $c$ is the constant appearing in* (57), (58) *then $u_\infty$ is locally exponentially stable in the sense that there exist positive constants $\varepsilon$, $\delta$ such that*

$$|u_0 - u_\infty|_2 < \varepsilon \tag{70}$$

*implies*

$$|u(t)-u_\infty|_2 \le e^{-\delta t}|u_0-u_\infty|_2 \quad \forall\, t>0. \tag{71}$$

PROOF. From (24), (45) we derive for every $v \in V$ – see (56)

$$\frac{d}{dt}(u,v)+a_{\ell(u)}(u,v)=a_{\ell(u_\infty)}(u_\infty,v). \tag{72}$$

This can be written as

$$\frac{d}{dt}(u-u_\infty,v)+a_{\ell(u)}(u-u_\infty,v)=a_{\ell(u_\infty)}(u_\infty,v)-a_{\ell(u)}(u_\infty,v) \quad \forall\, v\in V. \tag{73}$$

We set

$$h=u-u_\infty. \tag{74}$$

Taking $v=h$ in (73) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|h|_2^2+a_{\ell(u)}(h,h)&=\sum_{i,j=1}^{n}\{a_{ij}(\ell(u_\infty))-a_{ij}(\ell(u))\}(\partial_{x_i}u_\infty,\partial_{x_j}h)\\&\quad+\{a_0(\ell(u_\infty))-a_0(\ell(u))\}(u_\infty,h).\end{aligned} \tag{75}$$

Noting that

$$\ell(u_\infty)=\mu_\infty, \qquad \ell(u)=\ell(u_\infty+h)=\mu_\infty+\ell(h)$$

it comes by the mean value theorem and (57)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|h|_2^2+\lambda c\|h\|_{1,2}^2 &\le -\sum_{i,j=1}^{n}a'_{ij}(\mu_\infty+\theta_{ij}\ell(h))(\partial_{x_i}u_\infty\partial_{x_j}h)\ell(h)\\&\quad-a'_0(\mu_\infty+\theta\ell(h))(u_\infty,h)\ell(h)\end{aligned} \tag{76}$$

for some numbers $\theta_{ij}$, $\theta\in(0,1)$. It follows then easily that it holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|h|_2^2+\lambda c\|h\|_{1,2}^2\\&\qquad\le\{|A'(\mu_\infty+\theta\ell(h))|+|a'_0(\mu_\infty+\theta\ell(h))|\}|g|_2\|u_\infty\|_{1,2}\|h\|_{1,2}^2\end{aligned} \tag{77}$$

where $A'(\mu_\infty+\theta\ell(h))$ denotes the matrix $(a'_{ij}(\mu_\infty+\theta_{ij}\ell(h)))$. Since $u_\infty$ is a special $\varphi$, from (58) we derive finally

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|h|_2^2\\&\quad+\left\{\lambda c-\{|A'(\mu_\infty+\theta\ell(h))|+|a'_0(\mu_\infty+\theta\ell(h)|\}\frac{|f|_2|g|_2}{\lambda c}\right\}\|h\|_{1,2}^2\le 0.\end{aligned} \tag{78}$$

We can select $\varepsilon$ so that

$$|h|_2<\varepsilon \;\Rightarrow\; \lambda c-\{|A'(\mu_\infty+\theta\ell(h))|+|a'(\mu_\infty+\theta\ell(h))|\}\frac{|f|_2|g|_2}{\lambda c}>\delta>0, \tag{79}$$

(see (69)). Then for

$$|h(0)|_2=|u_0-u_\infty|_2<\varepsilon$$

we see that

$$\frac{1}{2}\frac{d}{dt}|h|_2^2+\delta|h|_2^2\le 0$$

for any $t$ – ($|h|_2 \searrow$ and remains always less than $\varepsilon$). Thus

$$\frac{d}{dt}\{e^{2\delta t}|h|_2^2\} \le 0$$

and the result follows. This completes the proof of the theorem. ■

**Remark 3** Somehow Theorem 4 is a perturbation result from the constant coefficient case. ■

## 4.2. Some global asymptotic behaviour

We will need the following lemma.

**Lemma 1** *Let $u_0^n \in L^2(\Omega)$ be a sequence such that*

$$u_0^n \rightharpoonup u_0 \quad \textit{in} \quad L^2(\Omega) \tag{80}$$

*when $n \to +\infty$. Let $u^n$, $u$ be the solutions to* (24) *corresponding to the initial data $u_0^n$, $u_0$ respectively. Then it holds that*

$$u^n(t) \rightharpoonup u(t) \quad \forall\, t \ge 0 \quad \textit{in} \quad L^2(\Omega). \tag{81}$$

*($u^n(t) = u^n(\cdot,t), u(t) = u(\cdot,t)$).*

PROOF. The above result is a simple generalization of a result in [8]. We give the proof for the reader's convenience.

By (80) it is clear that $u_0^n$ is bounded in $L^2(\Omega)$ independently of $n$. It follows then from (30), (33), (34) that for some constant $C$ independent of $n$ it holds that

$$|u^n|_{L^2(0,T;V)}, |u^n|_{L^\infty(0,T;L^2(\Omega))}, |u_t^n|_{L^2(0,T;V')} \le C. \tag{82}$$

Thus, one can extract a subsequence from $n$ – that will still label $n$ – such that when $n \to +\infty$

$$\begin{aligned} &u^n \rightharpoonup u^\infty \quad \text{in} \quad L^2(0,T;V), \qquad u^n \to u^\infty \quad \text{in} \quad L^2(0,T;L^2(\Omega)),\\ &u^n \rightharpoonup u^\infty \quad \text{in} \quad L^\infty(0,T,L^2(\Omega))\text{*-weak}, \quad u_t^n \rightharpoonup u_t^\infty \quad \text{in} \quad L^2(0,T;V'). \end{aligned} \tag{83}$$

(We used the compactness of the canonical embedding from $H^1(0,T;V,V')$ into $L^2(0,T;L^2(\Omega))$ – see [12], [13], [4]). By definition $u^n$ satisfies

$$\begin{aligned} -\int_0^T (u^n,v)\varphi'\,dt + \int_0^T \{a_{ij}(\ell(u^n(t)))(\partial_{x_i}u^n,\partial_{x_j}v) + a_0(\ell(u^n(t)))(u^n,v)\}\varphi\,dt \\ = \int_0^T (f,v)\varphi\,dt \quad \forall\,\varphi \in \mathcal{D}(0,T), \quad \forall\, v \in V. \end{aligned} \tag{84}$$

We made above the summation convention. Clearly from (83) we have

$$\ell(u^n(t)) \to \ell(u^\infty(t)) \quad \text{in} \quad L^2(0,T). \tag{85}$$

Up to a subsequence we can assume that this convergence holds for a.e. $t$. By the Lebesgue convergence theorem we have then for every $i$

$$\begin{cases} a_{ij}(\ell(u^n(t)))\varphi\partial_{x_j}v & \to \quad a_{ij}(\ell(u^\infty(t)))\varphi\partial_{x_j}v,\\ [3pt]a_0(\ell(u^n(t)))\varphi v & \to \quad a_0(\ell(u^\infty(t)))\varphi v, \end{cases} \tag{86}$$

in $L^2(\Omega \times (0,T)) = L^2(0,T;L^2(\Omega))$. Passing to the limit in (84) we obtain then

$$\frac{d}{dt}(u^\infty, v) + a_{ij}(\ell(u^\infty(t)))(\partial_{x_i} u^\infty, \partial_{x_j} v) + a_0(\ell(u^\infty(t)))(u^\infty, v) \\ = (f, v) \quad \forall\, v \in V, \quad \text{in} \quad \mathcal{D}'(0,T). \tag{87}$$

Moreover, for every $v \in V$ we have

$$(u^n(t), v) - (u_0^n, v) = \int_0^t \langle u_t^n, v\rangle \, dt \quad \text{a.e. } t \tag{88}$$

(see [5]). Up to a subsequence we can assume that

$$u^n(t) \to u^\infty(t) \quad \text{in} \quad L^2(\Omega), \quad \text{a.e. } t.$$

Thus passing to the limit in (88) we have for $v \in V$

$$(u^\infty(t), v) - (u_0, v) = \int_0^t \langle u_t^\infty, v\rangle \, dt = (u^\infty(t), v) - (u^\infty(0), v). \tag{89}$$

Thus, $u^\infty(0) = u_0$ and by uniqueness of a solution to (24) it follows that $u^\infty = u$. By uniqueness of the possible limit we obtain that the whole sequence $u^n$ satisfies (83). Thus in particular we have

$$u^n \rightharpoonup u \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{*-weak.} \tag{90}$$

This implies that it holds

$$(u^n(t), v) \to (u(t), v) \quad \text{in} \quad L^\infty(0,T) \quad \text{*-weak.} \tag{91}$$

We have also for every $t^2 > t^1$, $t_i \in [0,T)$

$$(u^n(t_2), v) - (u^n(t_1), v) = \int_{t_1}^{t_2} \langle u_t^n, v\rangle \, dt \le \int_{t_1}^{t_2} |u_t^n|_{V'} |v|_V \\ \le (t_2 - t_1)^{1/2} |v|_V |u_t^n|_{L^2(0,T;V')} \le C(t_2 - t_1)^{1/2}. \tag{92}$$

($|\cdot|_{V'}$ denotes the strong dual norm in $V'$). It follows that the sequence of function $(u^n(t), v)$ is equicontinuous and thus relatively compact in $C[0,T]$ the space of continuous functions on $[0,T]$. By uniqueness of the possible limit it follows that for every $v \in V$

$$(u^n(t), v) \to (u(t), v) \quad \text{in} \quad C([0,T]). \tag{93}$$

Since $V$ is dense in $L^2(\Omega)$ it follows easily that (93) holds for every $v \in L^2(\Omega)$. This completes the proof of the Lemma. ∎

There are many asymptotic behaviour results available (cf. [2], [3], [4], [8], [10]). We are going to restrict ourselves to two of them. In these simple cases, as we will see, the situation is far from being complete.

Thus, consider the example 2 of Section 1**.**, with for simplicity $\Gamma_D = \Gamma$. Then the stationary problem (41) becomes

$$\begin{cases} -a(\ell(u))\Delta u + u = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma. \end{cases} \tag{94}$$

In its weak form it can be written

$$\begin{cases} u \in H_0^1(\Omega), \\ a(\ell(u)) \displaystyle\int_\Omega \nabla u \nabla v \, dx + \int_\Omega uv \, dx = \int_\Omega fv \, dx \quad \forall\, v \in H_0^1(\Omega). \end{cases} \tag{95}$$

On $a$ we will assume – see (2):

$$a \text{ continuous}, \quad 0 < \lambda \leq a(\zeta) \leq \Lambda \quad \forall\, \zeta \in \mathbb{R}. \tag{96}$$

For $a > 0$ we introduce $\varphi_a$ the solution to

$$\begin{cases} \varphi_a \in H_0^1(\Omega), \\ a \displaystyle\int_\Omega \nabla\varphi_a \nabla v\, dx + \int_\Omega \varphi_a v\, dx = \int_\Omega f v\, dx \quad \forall\, v \in H_0^1(\Omega). \end{cases} \tag{97}$$

it is clear – see (48) – that we have

$$\varphi_{\mathcal{A}(\zeta)} = \varphi_{a(\zeta)} \tag{98}$$

and the equation (50) becomes

$$\mu = \int_\Omega g\varphi_{a(\mu)}\, dx. \tag{99}$$

We know already by Theorem 3 that a solution to (99) and thus (95) does exist. In the spirit of what we have shown in Example 1 we are going to show that several solutions might also exist in the case of (95). To see that let us set for $a > 0$

$$K(a) = \int_\Omega g\varphi_a\, dx = \ell(\varphi_a). \tag{100}$$

Then we have:

**Lemma 2** *Suppose that $f$ satisfies*

$$f \in H^1(\Omega), \quad f \geq 0, \quad \Delta f \leq 0 \textit{ in } \Omega, \quad \Delta f \not\equiv 0 \textit{ in } \Omega \quad \textit{or} \quad f \not\equiv 0 \textit{ on } \Gamma, \tag{101}$$

*then the mapping $a \mapsto \varphi_a$ is decreasing – i.e.*

$$a_1 > a_2 \quad \Rightarrow \quad \varphi_{a_1} < \varphi_{a_2}. \tag{102}$$

PROOF. See [2], Theorem 3. In the above lemma $\Delta f \leq 0$ is meant for instance in the sense of distributions. The assumptions (101) hold for instance for $f = \text{cst}$. ■

Then, we prove:

**Lemma 3** *It holds that*

$$K(a) \textit{ is continuous on } (0, +\infty) \tag{103}$$

$$\lim_{a\to 0} K(a) = \int_\Omega fg\, dx, \qquad \lim_{a\to +\infty} K(a) = 0. \tag{104}$$

*Moreover if $f$ satisfies* (101)*, $g$ is such that*

$$g \geq 0, \qquad g \not\equiv 0, \tag{105}$$

*then it holds that*

$$K \textit{ is decreasing on } (0, +\infty). \tag{106}$$

PROOF. The continuity of $K$ is easy to establish – cf. the proof of Theorem 3. One can also show – see [2] – that

$$\lim_{a\to 0} \varphi_a = f, \qquad \lim_{a\to +\infty} \varphi_a = 0 \quad \text{in} \quad L^2(\Omega). \tag{107}$$

(104) follows then. (106) follows immediately from (102), (105). ■

**Remark 4** It is not possible to relax completely the assumption (101) to obtain the monotonicity of $K$, i.e. $f \geq 0$ is not enough – see [2]. ■

In what follows we will assume that (101), (105) hold in such a way that $K$ is decreasing. Then, equation (99) can also be written

$$a(\mu) = K^{-1}(\mu). \tag{108}$$

$K^{-1}$ is a function independent of the function $a$. Thus, it is clear that choosing $a$ can produce each of the situations that we encountered in the case of Example 1. We will restrict ourselves to the two cases of the figure below.

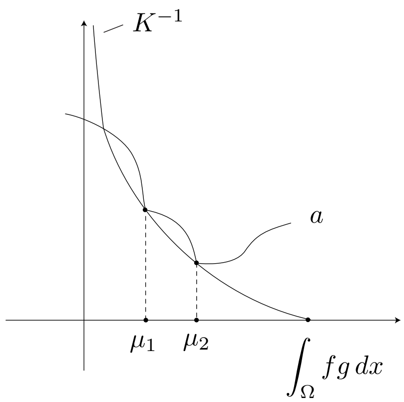


Fig. 4 A case of several equilibria

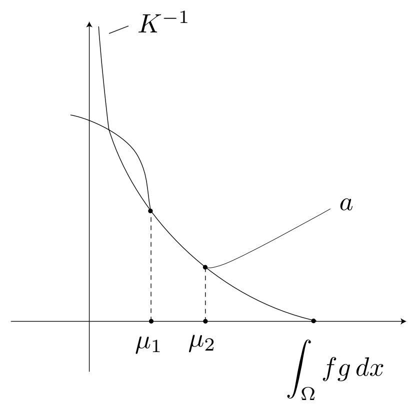


Fig. 5 A case of a continuum of equilibria

In particular we will suppose

$$a(\mu_i) = K^{-1}(\mu_i) \quad i = 1, 2, \tag{109}$$

$$a(\mu_2) \leq a(\mu) \leq a(\mu_1) \quad \forall\, \mu \in [\mu_1, \mu_2]. \tag{110}$$

We will denote by $u$ the weak solution of

$$\begin{cases} u_t - a(\ell(u(t)))\Delta u + u = f & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega, \end{cases} \tag{111}$$

and by $u_i$, $i = 1, 2$ the stationary points $u_i = \varphi_{a(\mu_i)}$ solution to

$$\begin{cases} u_i \in H_0^1(\Omega), \\ a(\ell(u_i)) \displaystyle\int_\Omega \nabla u_i \nabla v\,dx + \int_\Omega u_i v\,dx = \int_\Omega fv\,dx \quad \forall\, v \in H_0^1(\Omega). \end{cases} \tag{112}$$

Then we have:

**Lemma 4** *Suppose that* (101) *holds and that* $f > 0$ *in* $\Omega$. *Then it holds that*

$$0 < u_1 < u_2 \quad \textit{in} \quad \Omega, \tag{113}$$

$$\int_\Omega \nabla u_i \nabla v\,dx \geq 0 \quad \forall\, v \in H_0^1(\Omega), \quad v \geq 0, \quad i = 1, 2. \tag{114}$$

PROOF. We know that

$$u_i = \varphi_{a(\mu_i)}. \tag{115}$$

Then, the second inequality of (113) follows from (102). From (97) we have

$$a \int_\Omega \nabla(\varphi_a - f)\nabla v\, dx + \int_\Omega (\varphi_a - f)v\, dx = -a\int_\Omega \nabla f \nabla v\, dx \quad \forall\, v \in H_0^1(\Omega).$$

Taking $v = (\varphi_a - f)^+$, by (101) we get

$$a \int_\Omega |\nabla(\varphi_a - f)^+|^2\, dx + \int_\Omega (\varphi_a - f)^{+2}\, dx = -a\int_\Omega \nabla f \nabla(\varphi - f)^+\, dx \le 0$$

and thus

$$(\varphi_a - f)^+ = 0.$$

Going back to (97) we have

$$\begin{aligned} a\int_\Omega \nabla\varphi_a \nabla v\, dx &= -\int_\Omega (\varphi_a - f)v\, dx \\ &= \int_\Omega (\varphi_a - f)^- v\, dx \ge 0 \quad \forall\, v \ge 0, \quad v \in H_0^1(\Omega). \end{aligned} \tag{116}$$

Taking into account (115) this proves (114). Taking $v = \varphi_a^-$ in (116) we see easily that $\varphi_a \ge 0$ then the first inequality of (113) follows from (102). This completes the proof of the lemma. ■

From now on we will assume

$$f = g > 0 \quad \text{in} \quad \Omega \tag{117}$$

and we will choose $u_0$, the initial value to (111) such that

$$u_1 \le u_0 \le u_2. \tag{118}$$

Then, let us first establish

**Lemma 5** *Under the assumptions* (101), (109), (110), (117), (118)*and if $u$ is the weak solution to* (111) *it holds that*

$$u_1 \le u(\cdot, t) \le u_2 \quad \forall\, t. \tag{119}$$

PROOF. The proof is identical to the one in [7]. We reproduce it for the reader convenience. Denote by $E$ the set

$$E = \{\, t \mid \ell(u(s)) \in [\mu_1, \mu_2]\ \forall\, s \le t \,\}. \tag{120}$$

By (118), $E$ contains 0. (Recall that $g \ge 0$, $\ell(u_i) = \mu_i$). Set

$$t^* = \sup\{\, t \mid t \in E \,\}. \tag{121}$$

By continuity of the mapping $t \mapsto u(t)$ in $L^2(\Omega)$ (see (24)), $t \mapsto \ell(u(t))$ is continuous and

$$\ell(u(t^*)) \in [\mu_1, \mu_2] \tag{122}$$

so that $t^* \in E$.

We claim next that

$$u_1 \le u(t) \le u_2 \quad \forall\, t \in [0, t^*]. \tag{123}$$

Let us prove the left hand side inequality. Using the weak formulation of (111) and (112) we have in $\mathcal{D}'(0,t^*)$

$$\begin{aligned}\frac{d}{dt}(u-u_1,v)+a(\ell(u(t)))\int_\Omega \nabla(u-u_1)\nabla v\,dx+(u-u_1,v)\\ =\{a(\mu_1)-a(\ell(u(t)))\}\int_\Omega \nabla u_1\nabla v\,dx\quad \forall\, v\in H_0^1(\Omega).\end{aligned} \tag{124}$$

Due to (110), (122), (114) we have

$$a(\mu_1)-a(\ell(u(t)))\geq 0,\qquad \int_\Omega \nabla u_1\nabla v\,dx\geq 0\quad \forall\, v\in H_0^1(\Omega),\ v\geq 0.$$

Taking $v=-(u-u_1)^-$ we derive easily

$$\frac{1}{2}\frac{d}{dt}|(u-u_1)^-|_2^2+|(u-u_1)^-|_2^2\leq 0\quad\Longrightarrow\quad \frac{d}{dt}\big\{e^{2t}|(u-u_1)^-|_2^2\big\}\leq 0.$$

Since $(u-u_1)^-(0)=(u_0-u_1)^-=0$, it follows that $(u-u_1)^-(t)=0$ – i.e. $u(t)\geq u_1\ \forall\, t\in[0,t^*]$. This proves the left hand side inequality of (123). The right hand side inequality can be derived the same way. This proves (123). Next, by definition of $t^*$, if $t^*<+\infty$ we have

$$\ell(u(t^*))=\ell(u_1)\quad \text{or}\quad \ell(u_2).$$

Since $g$ is strictly positive by (123) this implies

$$u(t^*)=u_1\quad \text{or}\quad u_2$$

and by the uniqueness of the solution to (111) this equality remains valid for larger time which contradicts the definition of $t^*$. We thus have $t^*=+\infty$ and (123) gives (119). This completes the proof of the lemma. ■

**Remark 5** Here and subsequently the strict positivity (117) could be relaxed – see [8]. ■

Next assuming

$$a(\mu)\geq K^{-1}(\mu)\quad \forall\,\mu\in[\mu_1,\mu_2], \tag{125}$$

we have

**Lemma 6** *$|u(t)|_2^2$ is a Lyapunov function that is to say decreases with time. More precisely if $a=a(\ell(u(t)))$ we have*

$$\frac{1}{2}\frac{d}{dt}|u(t)|_2^2\leq -a\big\{\big|\big|\nabla(u-\varphi_a)\big|\big|_2^2+|u-\varphi_a|_2^2\big\}. \tag{126}$$

PROOF. We denote by $\langle\cdot,\cdot\rangle$ the duality bracket between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. In what follows $a$ denotes $a(\ell(u(t)))$. By (24) we have

$$\langle u_t,v\rangle+a\int_\Omega \nabla u\nabla v\,dx+(u,v)=(f,v)\quad \forall\, v\in H_0^1(\Omega).$$

Taking $v=\varphi_a$ it comes

$$\langle u_t,\varphi_a\rangle+a\int_\Omega \nabla u\nabla\varphi_a\,dx+(u,\varphi_a)=(f,\varphi_a)=(g,\varphi_a)=K(a(\ell(u(t)))).$$

From the definition of $\varphi_a$ and since $f = g$ we derive

$$\begin{aligned} \langle u_t, \varphi_a \rangle + (f, u) &= K(a(\ell(u(t)))) \\ \text{i.e.} \quad \langle u_t, \varphi_a \rangle &= K(a(\ell(u(t)))) - \ell(u(t)). \end{aligned} \tag{127}$$

Since $\ell(u(t)) \in [\mu_1, \mu_2]$, by (125) we obtain

$$\langle u_t, \varphi_a \rangle \le 0. \tag{128}$$

(Note that $\langle u_t, \varphi_a \rangle = 0$ when the equality holds in (125)). Next, combining (111) and (112) we get

$$\langle u_t, v \rangle + a \int_\Omega \nabla(u - \varphi_a)\nabla v \, dx + (u - \varphi_a, v) = 0 \quad \forall \, v \in H_0^1(\Omega).$$

Taking $v = u - \varphi_a$, by (128) we obtain

$$\langle u_t, u \rangle \le -\left\{ a \int_\Omega |\nabla(u - \varphi_a)|^2 \, dx + |u - \varphi_a|_2^2 \right\} \tag{129}$$

which is exactly (126). This completes the proof of the lemma. ■

We consider now the case of Figure 4. In particular we assume that (109), (110) hold with in addition – compare to (125)

$$a(\mu) > K^{-1}(\mu) \quad \forall \, \mu \in (\mu_1, \mu_2). \tag{130}$$

Then we have:

**Theorem 5** *Under the above assumptions, let* $u$ *be the solution to* (111) *with* $u_0$ *satisfying* (118) *and* $u_0 \neq u_2$. *Then it holds that*

$$\lim_{t \to +\infty} u(t) = u_1 \quad \textit{in} \quad L^2(\Omega). \tag{131}$$

PROOF. From (126) we derive by integration in $t$

$$\int_0^t a\big||\nabla(u - \varphi_a)|\big|_2^2 + |u - \varphi_a|_2^2 \, dt \le \frac{1}{2}|u_0|_2^2. \tag{132}$$

it follows that the above integral converges in $t$ and thus it holds that

$$\liminf_{t \to +\infty} a\big||\nabla(u - \varphi_a)|\big|_2^2 + |u - \varphi_a|_2^2 = 0. \tag{133}$$

It follows that we have for some sequence $t_n$, $t_n \to +\infty$

$$u(t_n) - \varphi_{a(\ell(u(t_n)))} \to 0 \quad \text{in} \quad H^1(\Omega). \tag{134}$$

(Recall that $a \ge a(\mu_2) > 0$). Since $u(t_n)$ is uniformly bounded in $L^2(\Omega)$ – see (119) – we can extract from $t_n$ a subsequence that for simplicity we still label $t_n$ such that for some $u_\infty$ we have

$$u(t_n) \rightharpoonup u_\infty \quad \text{in} \quad L^2(\Omega). \tag{135}$$

The set

$$C = \{ \, v \in L^2(\Omega) \mid u_1(x) \le v(x) \le u_2(x) \text{ a.e. } x \in \Omega \, \} \tag{136}$$

is closed and convex in $L^2(\Omega)$. It is also weakly closed and by lemma 5 and (135) we obtain

$$u_1 \le u_\infty \le u_2 \quad \text{in} \quad \Omega. \tag{137}$$

Moreover, from (134) we get

$$u_\infty = \varphi_{a(\ell(u_\infty))} \tag{138}$$

that is to say $u_\infty$ is a stationary point and by (137)

$$u_\infty = u_1 \quad \text{or} \quad u_2. \tag{139}$$

Since $|u(t)|_2^2$ is decreasing and $u_0 \neq u_2$, we can only have

$$u_\infty = u_1.$$

Thus we have found a sequence $t_n$, $t_n \to +\infty$ such that

$$u(t_n) \rightharpoonup u_1 \quad \text{in} \quad L^2(\Omega). \tag{140}$$

Next, consider another sequence $t'_n$, $t'_n \to +\infty$ such that

$$u(t'_n) \rightharpoonup v_\infty \quad \text{in} \quad L^2(\Omega). \tag{141}$$

Since $u(t'_n) \in C$ we have also $v_\infty \in C$ and in particular

$$v_\infty \geq u_1. \tag{142}$$

From (134), (140) we have in fact

$$u(t_n) \to u_1 \quad \text{in} \quad H^1(\Omega). \tag{143}$$

Since $|u(t)|_2^2$ is nonincreasing, it admits a limit when $t \to +\infty$ and by (143) this limit can only be $|u_1|_2^2$. Thus by passing to the limit in the inequality

$$|u(t'_n)|_2^2 - (u(t'_n), u_1) = (u(t'_n), u(t'_n) - u_1) \geq 0.$$

we get

$$|u_1|_2^2 - (v_\infty, u_1) = (u_1, u_1 - v_\infty) \geq 0.$$

Since $u_1 > 0$, $v_\infty \geq u_1$ this clearly imposes

$$v_\infty = u_1. \tag{144}$$

Thus, every sequence converging towards $u_1$, we have as $t \to +\infty$,

$$u(t) \rightharpoonup u_1 \quad \text{in} \quad L^2(\Omega). \tag{145}$$

The strong convergence follows from the fact that

$$|u(t)|_2 \to |u_1|_2.$$

This completes the proof of the theorem. ■

**Remark 6** In the case where (125) holds we have shown roughly speaking that $u_1$ is stable and $u_2$ unstable. ■

We consider now the case of a continuum of equilibria – i.e. the case of Figure 5. In particular we assume now that

$$a(\mu) = K^{-1}(\mu) \quad \forall \mu \in (\mu_1, \mu_2). \tag{146}$$

Then we have:

**Theorem 6** *Suppose that $u_0 \in C$. Then, under the above assumptions, and in the case of the figure 5 there exists a stationary point $u_\infty \in C$ solution to* (97) *with $a = a(\ell(u_\infty))$ such that*

$$u(t) \to u_\infty \quad \textit{in} \quad L^2(\Omega). \tag{147}$$

*($u$ is the solution to* (111) *corresponding to the initial value $u_0$ – recall that $C$ is defined in* (136)*).*

PROOF. We use a dynamical system technique. First we set for any $u_0 \in C$

$$u(t) = S(t)u_0. \tag{148}$$

Then we have

**Lemma 7** *$S(t)$ is a dynamical system on $C$ equipped with the weak topology of $L^2(\Omega)$.*

PROOF OF THE LEMMA. We refer to [1], [8], [4] for the definition of a dynamical system. The only difficult property to establish is to show that

$$S(t) : C \to C \quad \text{is continuous.}$$

This follows from the fact that if

$$u_0^n \rightharpoonup u_0 \quad \text{in} \quad L^2(\Omega) \quad \text{then} \quad S(t)u_0^n \rightharpoonup S(t)u_0 \quad \text{in} \quad L^2(\Omega).$$

See Lemma 4.1. ■

We then defined the $\omega$-limit set of $u_0$ as

$$\omega(u_0) = \{\, v_\infty \in C \mid \exists\, t_n,\ t_n \to +\infty \text{ such that } u(t_n) \rightharpoonup v_\infty \,\}. \tag{149}$$

Proceding exactly as above (138) one can show that there exists an equilibrium $u_\infty \in C$ and a sequence $t_n$, $t_n \to +\infty$ such that

$$u(t_n) \to u_\infty \quad \text{in} \quad H^1(\Omega). \tag{150}$$

It follows also due to Lemma 6 that

$$|u(t)|_2 \to |u_\infty|_2. \tag{151}$$

We would like to show that $u(t)$ converges toward $u_\infty$ in $L^2(\Omega)$. For that we will need the following lemma:

**Lemma 8** *Let $u$ be the solution to* (111)*. Then under the assumptions of Theorem 6 there exists a constant $K$ independent of $t > t_1$ such that*

$$\|u(t)\|_{1,2} \le K \quad \forall\, t > t_1. \tag{152}$$

PROOF OF THE LEMMA. Since $u(t) \in C\ \forall\, t$ we have of course for some constant $K_0$ independent of $t$

$$|u(t)|_2 \le K_0 \quad \forall\, t > 0. \tag{153}$$

Next, due to the smoothing effect for parabolic problem, for some $t_1 > 0$, it holds that

$$\int_\Omega |\nabla u(x, t_1)|^2 \, dx < +\infty. \tag{154}$$

Then we consider (111) for $t > t_1$. We have

$$u_t - a(\ell(u(t)))\Delta u + u = f \text{ in } \Omega \times \mathbb{R}^+.$$

Let us set

$$\sigma(t) = \int_0^t a(\ell(u(s)))ds,$$

$$v(x, \sigma(t)) = u(x, t).$$

Smoothing eventually $a$ and $f$ we can assume everything smooth. Then $v$ satisfies

$$v_t - \Delta v = \frac{f - v}{a(\ell(v(t)))} \text{ in } \Omega \times \mathbb{R}^+. \tag{155}$$

Squaring both sides of the equality and integrating over $\Omega$ we get easily

$$|v_t|_2^2 - 2(\Delta v, v_t) + |\Delta v|_2^2 \le K_1 \tag{156}$$

($K_1$ is a constant independent of $t$ and of the smoothing).

Moreover

$$(-\Delta v, v_t) = -\int_\Omega \nabla \cdot (\nabla v v_t)\, dx + \int_\Omega \nabla v \nabla v_t = \frac{1}{2}\frac{d}{dt}\big||\nabla v|\big|_2^2. \tag{157}$$

(This is due to our boundary conditions). From (156) we derive – recall (153) –

$$\frac{d}{dt}\big||\nabla v|\big|_2^2 + |\Delta v|_2^2 + |v|_2^2 \le K_2 \tag{158}$$

where $K_2$ is independent of $t$. Since on $H^2(\Omega)$ the norm

$$\{|\Delta v|_2^2 + |v|_2^2\}^{1/2}$$

is equivalent to the usual one – for some constant $c$ it holds that

$$\frac{d}{dt}\big||\nabla v|\big|_2^2 + c\big||\nabla v|\big|_2^2 \le K_2 \tag{159}$$

i.e.

$$\frac{d}{dt}\big\{\big||\nabla v|\big|_2^2 e^{2ct}\big\} \le e^{ct} K_2.$$

Integrating between $\sigma(t_1)$ and $\sigma(t)$ – it comes

$$\begin{aligned} &\big||\nabla v(\sigma(t))|\big|_2^2 e^{2ct} - \big||\nabla v(\sigma(t_1))|\big|_2^2 e^{2ct_1} \le \frac{1}{c} e^{ct} K_2 \\ \Longrightarrow \quad &\big||\nabla u(t)|\big|_2^2 \le \big||\nabla u(t_1)|\big|_2^2 + \frac{K_2}{c} \quad \forall\, t > t_1. \end{aligned} \tag{160}$$

Combined with (153) this completes the proof of the lemma. ■

END OF THE PROOF OF THEOREM 6. We claim that

$$|v_\infty|_2 = |u_\infty|_2 \qquad \forall\, v_\infty \in \omega(u_0). \tag{161}$$

Indeed, let $v_\infty \in \omega(u_0)$. By definition of $\omega(u_0)$ there exists a sequence $t_n', t_n' \to \infty$ such that

$$u(t_n') \rightharpoonup v_\infty.$$

Due to Lemma 8 – up to a subsequence – we have by the compactness of the canonical imbedding of $H^1(\Omega)$ into $L^2(\Omega)$

$$u(t_n') \to v_\infty \quad \text{in} \quad L^2(\Omega)$$

and thus

$$|u(t'_n)|_2 \to |v_\infty|_2.$$

By (151) this implies (161). Due to well known results regarding dynamical systems we have (see [1], [4])

$$S(t)\omega(u_0) = \omega(u_0). \tag{162}$$

Let $v_\infty \in \omega(u_0)$. Due to (161), (127)–(129) we have for $u(t) = S(t)v_\infty$,

$$\langle u_t, \varphi_a \rangle = 0$$

and

$$0 = \frac{1}{2}\frac{d}{dt}|u|_2^2 = -\left\{a(\ell(u(t)))\int_\Omega |\nabla(u - \varphi_{a(\ell(a(t)))})|^2 + |u - \varphi_{a(\ell(u(t)))}|^2\,dx\right\}.$$

Thus for any $t$, $u(t)$ is a stationary point. But there is only a stationary point of a given norm ($a \mapsto \varphi_a$ is decreasing). We thus have

$$u(t) = S(t)v_\infty = u_\infty$$

and thus $\omega(u_0) = \{u_\infty\}$. This means that when $t \to +\infty$

$$u(t) \rightharpoonup u_\infty.$$

The strong convergence follows from (151). This completes the proof of the theorem. ■

**Remark 7** We do not know how $u_\infty$ is selected depending on the initial data. It would be of course very interesting to remove the assumption $f = g$. ■

**Acknowledgement.** Part of these notes were presented at the Universidad Complutense de Madrid during the inaugural conference of the seminar of applied mathematics on October 2002. We thank this institution for its invitation. The second author would like also to acknowledge the support of the Swiss National Science foundation under the contract # 20-67618.02.

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