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The summability of solutions to variational problems since Guido Stampacchia

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Abstract. Inequalities concerning the integral of $|\nabla u|^2$ on the subsets where $|u(x)|$ is greater than k can be used in order to prove regularity properties of the function u. This method was introduced by Ennio De Giorgi e Guido Stampacchia for the study of the regularity of the solutions of Dirichlet problems.

Integrabilidad de soluciones de problemas variacionales desde Guido Stampacchia

Resumen. Adecuadas desigualdades sobre la integral de $|\nabla u|^2$ extendida a los subconjuntos donde $|u(x)|$ es mayor que k pueden ser usadas para obtener propiedades de regularidad de la función u. Este método fue introducido por Ennio De Giorgi y Guido Stampacchia para el estudio de la regularidad de las soluciones de problemas de Dirichlet.

1. The Stampacchia method

I recall the following regularity results by Guido Stampacchia, concerning solutions of linear Dirichlet problems.

Let Ω be a bounded subset of $\mathbb{R}^N$ $(N > 1)$ and $L(v) = -\mathrm{div}(M(x)\nabla v)$ be a differential operator, where M is a bounded elliptic matrix. Consider the Dirichlet problem

$$u \in W_0^{1,2}(\Omega) \; : L(u) = f(x) \in L^{\frac{2N}{N+2}}(\Omega) \tag{1}$$

The use of

$$G_k(u) = \begin{cases} u(x) + k, & \text{if } x : u(x) < -k; \\ 0, & \text{if } x : |u(x)| \le k; \\ u(x) - k, & \text{if } x : u(x) > k; \end{cases}$$

as test function implies

$$\alpha \int_\Omega |\nabla G_k(u)|^2 \le \int_\Omega f G_k(u) \tag{2}$$

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$$\leq \left(\int_{\{x\in\Omega:|u(x)|>k\}} |f|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}} \left(\int_\Omega |G_k(u)|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}}$$

If

$$f \in M^m(\Omega),\ m > \frac{2N}{N+2} \tag{3}$$

by Sobolev inequality we have

$$\left(\int_\Omega |G_k(u)|^{2^*}\right)^{\frac{1}{2^*}} \leq c_f |A_k|^{\frac{N+2}{2N}-\frac{1}{m}}$$

where $|E|$ denotes the measure of the subset E and

$$A_k = \{x \in \Omega : |u(x)| > k\}$$

Then, if $h > k > 0$, we have

$$(h-k)|A_h|^{\frac{1}{2^*}} \leq c_f |A_k|^{\frac{N+2}{2N}-\frac{1}{m}}$$

$$|A_h| \leq c_f \frac{|A_k|^{2^*(\frac{N+2}{2N}-\frac{1}{m})}}{(h-k)^{2^*}} \tag{4}$$

Here we use the following lemma in order to prove that, in (4),

• if $2^*\left(\frac{N+2}{2N}-\frac{1}{m}\right) > 1$ $\left(\text{that is } m > \frac{N}{2}\right)$, there exists $M > 0$ such that $|A_M| = 0$: u is bounded ($|u| \leq M$);

• if $2^*l\left(\frac{N+2}{2N}-\frac{1}{m}\right) < 1$ $\left(\text{that is } \frac{2N}{N+2} \leq m < \frac{N}{2}\right)$, then there esists $c_0 > 0$ such that

$$|A_k| \leq c_0 \frac{c_f}{k^{m^{**}}} :$$

u belongs to the Marcinkiewicz space $M^{m^{**}}(\Omega)$.

Lemma 1 (Stampacchia's Lemma) *Let $\phi(t)$ be a positive, decreasing real function such that*

$$h > k \Rightarrow \phi(h) \leq C \frac{\phi(k)^\theta}{(h-k)^a} \quad 0 < \theta < 1,\ a > 0 \tag{5}$$

then there exist c_0 and k_0 such that

$$\phi(k) \leq c_0 \frac{C^{\frac{1}{1-\theta}}}{k^{\frac{a}{1-\theta}}},\ \ k > k_0$$

if

$$h > k \Rightarrow \phi(h) \leq C \frac{\phi(k)^\lambda}{(h-k)^a},\ \lambda > 1 \tag{6}$$

then there exist M such that

$$\phi(M) = 0. \quad \blacksquare$$

We repeat the results proved in the previous page:

Theorem 1 (Stampacchia's regularity) *The solution u of the Dirichlet problem* (1) *is bounded, if $f \in M^m(\Omega)$, with $m > \frac{N}{2}$ and u belongs to $M^{m^{**}}(\Omega)$, if $f \in M^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$.* $\blacksquare$

Thanks to interpolation, the following theorem follows from the linearity of the differential operator.

Theorem 2 (Stampacchia's summability) *If $f \in M^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then u belongs to $L^{m^{**}}(\Omega)$.* ■

Developments of this method can be found in in [6], [8], [10], [17], [21], [22], [25], [26], [27], [18], [24].

1.1. Nonlinear operators

Consider, now, the nonlinear differential operator $A(v) = -\mathrm{div}(a(x, v, \nabla v))$ in $W_0^{1,p}(\Omega)$ ($p > 1$), with the usual Leray-Lions assumptions (see [23]), and the Dirichlet problem

$$u \in W_0^{1,p}(\Omega) \, : A(u) = f(x)$$

For sake of simplicity, we still take $p = 2$:

$$u \in W_0^{1,2}(\Omega) \, : A(u) = f(x) \in L^{\frac{2N}{N+2}}(\Omega) \tag{7}$$

The proofs of Theorem 1 still hold.

Theorem 3 (Stampacchia's regularity) *The solution u of the Dirichlet problem (7) is bounded, if $f \in M^m(\Omega)$, with $m > \frac{N}{2}$ and u belongs to $M^{m^{**}}(\Omega)$, if $f \in M^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$.* ■

Theorem 2 still holds, with a different proof (powers of as u test functions): see [14], [15] for the proof and applications (developments in [9]).

Theorem 4 (Summability) *If $f \in L^m(\Omega)$, with $m > \frac{N}{2}$, then u is bounded; if $f \in M^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then u belongs to $L^{m^{**}}(\Omega)$.*

PROOF. Use

$$v = \frac{|T_k(u)|^{2\lambda} T_k(u)}{2\lambda + 1}, \ \lambda = \frac{-mN + 2N - 2m}{4m - 2N}; \quad k > 0$$

as test function in the weak formulation of (7) and Sobolev inequality:

$$\frac{1}{2\lambda+1} \int_\Omega a(x, u, \nabla u) \nabla(|T_k(u)|^{2\lambda} T_k(u)) \geq c_1(\lambda) \alpha \left(\int_\Omega |T_k(u)|^{(\lambda+1)2^*} \right)^{\frac{2}{2^*}}.$$

Then the Hölder inequality implies that

$$\left(\int_\Omega |T_k(u)|^{(\lambda+1)2^*} \right)^{\frac{2}{2^*}} \leq c_2(\alpha, \lambda) \left(\int_\Omega |T_k(u)|^{(2\lambda+1)m'} \right)^{\frac{1}{m'}} \|f\|_{L^m(\Omega)}.$$

The definition of λ gets $(\lambda + 1)2^* = m'(2\lambda + 1)$, and

$$\|T_k(u)\|_{L^{m^{**}}(\Omega)} \leq c_3(\alpha, m) \|f\|_{L^m(\Omega)}.$$

Now, if $k \to \infty$, the Fatou Lemma implies

$$\|u\|_{L^{m^{**}}(\Omega)} \leq c_3(\alpha, m) \|f\|_{L^m(\Omega)}. \quad ■ \tag{8}$$

1.2. Minima of functionals

Consider, now, the following functional of the Calculus of Variations

$$J(v) = \int_\Omega j(x, v, \nabla v) - \int_\Omega f\, v$$

in $W_0^{1,p}(\Omega)$ ($p > 1$), with the usual assumptions on j (see [20], [19]), and the minimization problem

$$u \in W_0^{1,p}(\Omega) \; : J(u) \leq J(v), \;\; \forall v \in W_0^{1,p}(\Omega)$$

For sake of simplicity, we still take $p = 2$:

$$u \in W_0^{1,2}(\Omega) \; : J(u) \leq J(v), \;\; \forall v \in W_0^{1,2}(\Omega) \tag{9}$$

Let

$$T_k(u) = \begin{cases} -k, & \text{if } x : u(x) < -k; \\ u, & \text{if } x : |u(x)| \leq k; \\ +k, & \text{if } x : u(x) > k; \end{cases}$$

If we take $v = u - G_k(u) = T_k(u)$, we get again the inequality (2), so that the proof of Theorem 1 still holds.

Theorem 5 (Stampacchia's regularity) *The minima u of the minimization problem (9) is bounded, if $f \in M^m(\Omega)$, with $m > \frac{N}{2}$ and u belongs to $M^{m^{**}}(\Omega)$, if $f \in M^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$.* ■

If we assume that $f \in M^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$ the coice

$$v = u - \frac{|T_k(u)|^{2\lambda} T_k(u)}{2\lambda + 1}, \; \lambda = \frac{-mN + 2N - 2m}{4m - 2N}$$

it is not useful.

Theorem 6 (Summability [16]) *The minimum u of the minimization problem (9) belongs to $L^{m^{**}}(\Omega)$, if $f \in L^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$.*

PROOF. We take again $v = u - G_k(u) = T_k(u)$ and we get the inequality (2). Then

$$\int_{\{x\in\Omega:|u(x)|\geq k\}} |\nabla u|^2 \leq \left(\int_{\{x\in\Omega:|u(x)|\geq k\}} |f|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{N}}.$$

The previous inequality implies that, for every $k > 0$, and $\lambda = \frac{-mN+2N-2m}{4m-2N}$ as in Theorem 4

$$k^{2\lambda-1} \int_{\{x\in\Omega:|u(x)|\geq k\}} |\nabla u|^2 \leq k^{2\lambda-1} \left(\int_{\{x\in\Omega:|u(x)|\geq k\}} |f|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{N}}.$$

Then, as starting point of the proof, we write the previous inequality as

$$k^{2\lambda-1} \sum_{j=k}^{\infty} \int_{\{x\in\Omega:j\leq|u(x)|<j+1\}} |\nabla u|^2 \leq k^{2\lambda-1} \left(\int_{\{x\in\Omega:|u(x)|\geq k\}} |f|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{N}}.$$

which gets (**formally**)

$$\sum_{k=0}^{\infty} k^{2\lambda-1} \sum_{j=k}^{\infty} \int\limits_{\{x\in\Omega: j\le |u(x)|<j+1\}} |\nabla u|^2 \le \sum_{k=0}^{\infty} k^{2\lambda-1} \left(\int\limits_{\{x\in\Omega:|u(x)|\ge k\}} |f|^{\frac{2N}{N+2}}] \right)^{\frac{N+2}{N}}.$$

But it is possible (but not easy) to prove that

$$\sum_{k=0}^{\infty} k^{2\lambda-1} \sum_{j=k}^{\infty} \int\limits_{\{x\in\Omega: j\le |u(x)|<j+1\}} |\nabla u|^2 \approx \int\limits_{\Omega} |u|^{2\lambda}|\nabla u|^2$$

and

$$\sum_{k=0}^{\infty} k^{2\lambda-1} [\int\limits_{\{x\in\Omega:|u(x)|\ge k\}} |f|^{\frac{2N}{N+2}}]^{\frac{N+2}{N}} \approx \left(\int\limits_{\Omega} |u|^{(2\lambda+1)m'} \right)^{\frac{1}{m'}} \|f\|_{L^m(\Omega)}$$

in order to get again the inequality (8) and show the summability of the minimum u. ■

Developments of above method method ([16]) can be found in [10] (regularity of minimizing sequences) and in [7] (parabolic equations).

2. Singular data ([4])

2.1. Dirichlet problems in large Sobolev spaces

In this subsection, I report some results concerning Marcikiewicz estimates on the solutions of Dirichlet problems with irregular data. Aim of Theorem 7 is to give an easier and shorter proof of some results of [1].

Consider again the nonlinear differential operator $A(v) = -\mathrm{div}(a(x, v, \nabla v))$ and the boundary value problem

$$\begin{cases} -\mathrm{div}(a(x, u, \nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{10}$$

where on the right hand side we assume only that $f \in L^1(\Omega)$.

The existence and properties of solutions is proved in [11], [12], [2], [13], [3]. Moreover in [12] is proved that u belongs to $W_0^{1,m^*}(\Omega)$, if f belongs to $L^m(\Omega)$, if $1 < m < \frac{2N}{N+2}$.

Now we shall discuss the regularity of u if

$$f \in M^m(\Omega), \;\; 1 < m < \frac{2N}{N+2} \tag{11}$$

Theorem 7 *If f belongs to $M^m(\Omega)$, $1 < m < \frac{2N}{N+2}$, the weak solutions u of (10) belong to $M^{m^{**}}(\Omega)$ and $\nabla u \in M^{m^*}(\Omega)$.*

PROOF. We cannot use the approach of Stampacchia, since it is not possible to use u (and $G_k(u)$) as test function in the Dirichlet problem, because $|\nabla u|^2$ does not belong to $L^1(\Omega)$.

Use (**formally**) as test function $T_{h-k}[G_k(u)]$. Thus

$$\alpha \int\limits_{\Omega} |\nabla T_{h-k}[G_k(u)]|^2 \le \int\limits_{\Omega} f T_{h-k}[G_k(u)] \tag{12}$$

$$\alpha \mathcal{S}^2 \, (h-k)^2 |A_h|^{\frac{2}{2^*}} \le c_f (h-k)|A_k|^{1-\frac{1}{m}}$$

$$|A_h| \leq \left(\frac{c_f}{\alpha \mathcal{S}^2}\right)^{\frac{2^*}{2}} \frac{|A_k|^{(1-\frac{1}{m})\frac{2^*}{2}}}{(h-k)^{\frac{2^*}{2}}}$$

Here we use the Lemma 1 with $\theta = \left(1 - \frac{1}{m}\right)\frac{2^*}{2}$, so that $\frac{2^*}{2(1-\theta)} = m^{**}$: u belongs to $M^{m^{**}}(\Omega)$.

Moreover, if in (9) we take $h = k+1$ we have

$$\alpha \int_{B_k} |\nabla u|^2 \leq \int_{A_k} |f| \leq c_f |A_k|^{1-\frac{1}{m}} \leq c_0 \frac{c_f}{k^{m^{**}(1-\frac{1}{m})}}$$

where

$$B_k = \{x \in \Omega : k \leq |u(x)| < k+1\}$$

and $A_0 = \Omega$. Thus

$$\alpha \int\limits_{\Omega} |\nabla T_k(u)|^2 \leq |\Omega| + \sum_{i=1}^{i=k-1} c_0 \frac{c_f}{i^{m^{**}(1-\frac{1}{m})}}$$

Remark that

$$0 \leq m^{**}(1 - \frac{1}{m}) < 1 \iff 1 \leq m < \frac{2N}{N+2}$$

and $(0 < \theta < 1)$

$$\frac{(k-1)^{1-\theta}}{1-\theta} > 1 + \frac{(k-1)^{1-\theta} - 1}{1-\theta}$$

$$= 1 + \int_1^{k-1} \frac{1}{t^\theta} = 1 + \sum_{j=1}^{k-2} \int_j^{j+1} \frac{1}{t^\theta} > 1 + \sum_{j=1}^{k-2} \frac{1}{(j+1)^\theta} = \sum_{i=1}^{i=k-1} \frac{1}{i^\theta}$$

So, for $k \geq 1$,

$$\alpha \int\limits_{\Omega} |\nabla T_k(u)|^2 \leq |\Omega| + c_\theta c_f (k-1)^{1-m^{**}(1-\frac{1}{m})} \leq C_f k^{1-m^{**}(1-\frac{1}{m})} \tag{13}$$

Here we follow a technique of [2]. Estimate (15) implies also

$$t^2 \,\text{meas}(A_k \cap \{|\nabla u| > t\}) \leq \int_{A_k} |\nabla u|^2 \leq c_2 c_f k^{1-m^{**}(1-\frac{1}{m})}$$

On the other hand

$$\text{meas}\{|\nabla u| > t\} \leq \text{meas}\{|\nabla u| > t,\, |u| \leq k\} + \text{meas}\{|u| > k\}$$

$$\leq c_1 \frac{k^{1-m^{**}(1-\frac{1}{m})}}{t^2} + c_2 \frac{1}{k^{m^{**}}}$$

Note that

$$m^{**}(1 - \frac{1}{m}) = \frac{(m-1)N}{N-2m}, \quad 1 - m^{**}(1 - \frac{1}{m}) = \frac{2N - m(N+2)}{N-2m} \in (0,1]$$

The minimization with respect to k gives ($k = t^{\frac{N-2m}{N-m}}$)

$$\text{meas}\{|\nabla u| > t\} \leq \frac{\tilde{C}_f}{t^{m^*}}$$

as desired. ■

2.2. Functionals with nonregular data

Consider the set $\mathcal{T}_0^{1,2}(\Omega)$ $(p > 1)$ of all functions u which are almost everywhere finite and such that $T_k(u) \in W_0^{1,2}(\Omega)$ for every $k > 0$. For every $u \in \mathcal{T}_0^{1,2}(\Omega)$ there exists a measurable function $\Phi : \Omega \mapsto R^N$ such that $\nabla T_k(u) = \Phi\chi_{\{|u|\leq k\}}$ a.e. in Ω. This function Φ, which is unique up to almost everywhere equivalence, will be denoted by ∇u. Note that ∇u coincides with the distributional gradient of u whenever $u \in \mathcal{T}_0^{1,2}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ and $\nabla u \in L^1_{\mathrm{loc}}(\Omega, R^N)$.

Definition 1 Let $f \in L^1(\Omega)$. A function $u \in \mathcal{T}_0^{1,2}(\Omega)$ is a T-minimum for the functional

$$J(v) = \int_\Omega j(x, \nabla v) - \int_\Omega fv$$

if, for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and every $k > 0$,

$$\int_\Omega j(x, \nabla\varphi + \nabla T_k[u - \varphi]) \leq \int_\Omega j(x, \nabla\varphi) + \int_\Omega fT_k[u - \varphi], \tag{14}$$

Theorem 8 ([5]) *There exists a T–minimum u of $J(v)$ such that*

$$\int_\Omega |\nabla T_k(u)|^2 \leq k\left(\frac{\|f\|_{L^1(\Omega)}}{\alpha}\right) \quad (k > 0), \tag{15}$$

$$\int_{B_{h,k}} |\nabla u|^2 \leq \frac{1}{\alpha}\int_{A_h} |f| \quad (h,\ k > 0),$$

where

$$B_{h,k} = \{x \in \Omega : h \leq |u(x)| < h + k\},$$

$$A_h = \{x \in \Omega : h \leq |u(x)|\}.$$

and $u \in W_0^{1,q}(\Omega), q < \frac{N}{N-1}$.

Moreover (Marcikiewicz framework) u *belongs to* $M^{\frac{N}{N-2}}(\Omega)$ *and* ∇ *belongs to* $M^{\frac{N}{N-1}}(\Omega)$. ■

If, in (14), we write $h - k$ instead of k and take $\varphi = T_k(u)$, then

$$\int_\Omega j(x, \nabla T_k(u) + \nabla T_{h-k}[u - T_k(u)]) \leq \int_\Omega j(x, \nabla T_k(u)) + \int_\Omega fT_{h-k}[u - T_k(u)],$$

which implies the inequality (12), so that we have we can prove the following results.

Theorem 9 *If f belongs to $M^m(\Omega)$, $1 < m < \frac{2N}{N+2}$, then the T–minimum u belongs to $M^{m^{**}}(\Omega)$ and* $\nabla u \in M^{m^*}(\Omega)$. ■

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References

[1] Alvino, A., Ferone, V. and Trombetti G. (2000). Estimates for the gradient of solutions of nonlinear elliptic equations with L^1 data, *Ann. Mat. Pura Appl.*, **(4)** 178, 129–142.

[2] Bénilan, P., Boccardo, L., Gallouet, T., Gariepy, R., Pierre, M. and Vázquez, J. L. (1995). An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Annali Sc. Norm. Sup. Pisa*, **22**, 241–273.

[3] Boccardo, L. (1996). Some nonlinear Dirichlet problems in L^1 involving lower order terms in divergence form. Progress in elliptic and parabolic partial differential equations (Capri, 1994), 43–57, *Pitman Res. Notes Math. Ser.*, **350**, Longman, Harlow.

[4] Boccardo, L. Marcikiewicz estimates on the solutions of some elliptic problems with irregular data, *in ellaboration*.

[5] Boccardo, L. (2000). T–minima: An approach to minimization problems in L^1. Contributi dedicati alla memoria di Ennio De Giorgi, *Ricerche di Matematica*, **49**, 135–154.

[6] Boccardo, L. and Brezis, H. (2003). Some remarks on a class of elliptic equations with degenerate coercivity, *Boll. Unione Mat. Ital.*, **6**, 521–530.

[7] Boccardo, L. Dall'Aglio, A., Gallouet, T. and Orsina, L. (1999). Existence and regularity results for nonlinear parabolic equations, *Adv. Math. Sci. Appl.*, **9**, 1017–1031.

[8] Boccardo, L. Dall'Aglio, A. and Orsina, L. (1998). Existence and regularity results for some elliptic equations with degenerate coercivity. Dedicated to Prof. C. Vinti (Italian) (Perugia, 1996). *Atti Sem. Mat. Fis. Univ. Modena*, **46** suppl., 51–81.

[9] Boccardo, L., Díaz, J. I., Giachetti, D. and Murat, F. (1993). Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms, *J. Diff. Eq.*, **106**, 215–237.

[10] Boccardo, L., Ferone, E., Fusco, N. and Orsina, L. (1999). Regularity of minimizing sequences for functionals of the Calculus of Variations via the Ekeland principle, *Differential Integral Eq.*, **12**, 119–135.

[11] Boccardo, L. and Gallouet, T. (1989). Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.*, **87**, 149–169.

[12] Boccardo, L. and Gallouet, T. (1992). Nonlinear elliptic equations with right hand side measures, *Comm. Partial Differential Equations*, **17**, 641–655.

[13] Boccardo, L., Gallouet, T. and Orsina, L. (1997). Existence and nonexistence of solutions for some nonlinear elliptic equations., *J. Anal. Math.*, **73**, 203–223.

[14] Boccardo, L. and Giachetti, D. (1985). Some remarks on the regularity of solutions of strongly nonlinear problems and applications. (Italian), *Ricerche Mat.*, **34**, 309–323.

[15] Boccardo, L. and Giachetti, D. (1989). Existence results via regularity for some nonlinear elliptic problems, *Comm. Partial Differential Equations*, **14**, 663–680.

[16] Boccardo, L. and Giachetti, D. L. L^s- regularity of solutions of some nonlinear elliptic problems, *preprint*.

[17] Boccardo, L., Marcellini, P. and Sbordone, C. (1990). L^∞-regularity for variational problems with sharp nonstandard growth conditions, *Boll. UMI*, **4A**, 219–225.

[18] Boccardo, L., Murat, F. and Puel, J. P. (1992). L^∞-estimate for nonlinear elliptic partial differential equations and application to an existence result, *SIAM J. Math. Anal.*, **23**, 326–333.

[19] Dacorogna, B. (1989). *Direct methods in the calculus of variations*, Applied Mathematical Sciences, **78**, Springer-Verlag, Berlin-New York.

[20] De Giorgi, E. *Teoremi di semicontinuità nel Calcolo delle Variazioni*, Istituto Nazionale di Alta Matematica (1968–1969).

[21] Giachetti, D. and Porzio, M. M. (2000). Local regularity results for minima of functionals of Calculus of Variations, *Nonlinear Anal.*, **39**, 463–482.

[22] Giachetti, D. and Porzio, M. M. (2001). D. Existence results for some nonuniformly elliptic equations with irregular data, *J. Math. Anal. Appl.*, **257**, 100–130.

[23] Leray, J. and Lions, J.-L. (1965). Quelques résultats de Višik sur les problèmes elliptiques semi-linéaires par les méthodes de Minty et Browder, *Bull. Soc. Math. France*, **93**, 97–107.

[24] Orsina, L. and Porzio, M. M. (1992). $L^\infty(Q)$-estimate and existence of solutions for some nonlinear parabolic equations, *Boll. Un. Mat. Ital.* B (7) 6 (1992), no. 3, 631–647.

[25] Porzio, M. M. (1996). L^∞_{loc} estimates for a class of doubly nonlinear parabolic equations with sources, *Rend. Mat. Appl.*, **(7)** 16, 433–456.

[26] Porzio, M. M. (1997). L^∞-regularity for degenerate and singular anisotropic parabolic equations, *Boll. Un. Mat. Ital. A*, **(7)** 11, 697–707.

[27] Porzio, M. M. (1999). Local regularity results for some parabolic equations, *Houston J. Math.*, **25**, 769–792.

[28] Stampacchia, G. (1965). Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier (Grenoble)*, **15**, 189–258.

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