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Atomic decomposition of a weighted inductive limit

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Dedicated to the memory of Klaus Floret

Abstract. We study some structural questions concerning the locally convex space H_V^∞, which consists of analytic functions on the open unit disc. We construct an atomic decomposition in this space, using a lattice of points of the unit disc which is more dense than a usual one. The coefficient space is a Köthe sequence space. We also prove that H_V^∞ is not nuclear.

Descomposición atómica de un límite inductivo ponderado

Resumen. Estudiamos algunas cuestiones estructurales acerca del espacio localmente convexo H_V^∞, que está formado por funciones analíticas en el disco unidad abierto. Construimos una descomposición atómica de este espacio, usando un retículo de puntos del disco unidad que es más denso que el usual. También demostramos que H_V^∞ no es nuclear.

1. Introduction

In the paper [10] we introduced the space $H_V^\infty := H_V^\infty(\mathbf{D})$ consisting of analytic functions on the open unit disc. This space is slightly larger than the classical Banach space H^∞ of bounded analytic functions. The motivation of H_V^∞ is the fact that it behaves much better than H^∞ with respect to Bergman and Szegö projections and the harmonic conjugation operator. The space H_V^∞ is aimed to be a substitute for H^∞ in situations where the continuity of the above mentioned mappings is critical. In [10] we also proved that this space is a some sense the smallest possible substitute.

In this work we continue the study of H_V^∞ by showing that it admits an atomic decomposition: every analytic function in this space can be presented as a linear combination of "atoms" defined using the standard Bergman kernel. The coefficients form sequences belonging to a natural Köthe sequence space. For details, see Theorem 1. While atomic decompositions are interesting in itself in harmonic analysis, our results also give a representation of H_V^∞ as a complemented subspace of a Köthe–Schwartz coechelon space.

The atom functions are defined using a lattice of points of the unit disc. Our choice of the lattice is not a typical one with essentially constant hyperbolic distances between the lattice points. In the present work the lattice is much more dense. This leads to many simplifications in proofs.

Atomic decompositions were first studied by Coifman and Rochberg in the setting of Bergman spaces on the disc, see [3] and especially the book [12], Chapter 4, for a simple presentation. It is intuitively quite

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clear that our method could be used also in the classical case of Bergman spaces to make many proofs shorter or even trivial.

Before proceeding to the atomic decomposition let us recall from [10] the definition of H_V^∞. By dA we denote the normalized 2–dimensional Lebesgue measure on $\mathbf{D}$; moreover, $m(B) := \int_B dA$ for a measurable subset B of $\mathbf{D}$.

Definition 1 *By V we denote the set of radial weight functions $v : \mathbf{D} \to \mathbf{R}^+$ i.e. positive bounded continuous functions, which satisfy for every $n \in \mathbf{N}$*

$$\sup_{z\in\mathbf{D}} v(z)|\log(1-|z|)|^n \leq C_n \tag{1}$$

and moreover

$$\sqrt{1-|z|} \leq v(z) \leq 1 \quad \text{for all } z \in \mathbf{D}. \tag{2}$$

Notice that (2) is not explicitely required in [10]; nevertheless the spaces in the next definition are the same as those in [10].

Definition 2 *We define the locally convex space*

$$H_V^\infty := \Big\{ f : \mathbf{D} \to \mathbf{C} \text{ analytic} \,\Big|\, \|f\|_v := \sup_{z\in\mathbf{D}} |f(z)|v(z) < \infty \text{ for all } v \in V \Big\}. \tag{3}$$

The topology of H_V^∞ is determined by the uncountable family $\{\|\cdot\|_v | v \in V\}$ of seminorms. It is a complete countable inductive limit of Banach spaces: $H_V^\infty = \operatorname{ind}_{k\to\infty} H_{v_k}^\infty$, where $v_k(z) := \min\{1, |\log(1-|z|)|^{-k}\}$ and

$$H_{v_k}^\infty := \Big\{ f : \mathbf{D} \to \mathbf{C} \text{ analytic} \,\Big|\, \|f\|_{v_k} < \infty \Big\}. \tag{4}$$

It is a Schwartz space; in particular, the bounded and precompact subsets coincide. It is not so surprising that the weight system is not "steep" enough to make the space a nuclear one. We consider this question in Section 4.

The dual of H_V^∞ (with respect to the topology of uniform convergence on bounded sets) is the space

$$\begin{aligned} H_W^1 \quad := \quad & \Big\{ f : \mathbf{D} \to \mathbf{C} \text{ analytic} \,\Big|\, \|f\|_k := \\ & \int\limits_{z\in\mathbf{D}} |f(z)||\log(1-|z|)|^k dA(z) < \infty \text{ for all } k \in \mathbf{N} \Big\}. \end{aligned} \tag{5}$$

The dual pairing is the usual $\langle f, g\rangle := \int_{\mathbf{D}} f\bar{g}dA$. The spaces H_V^∞ and H_W^1 are reflexive, hence, the duality $(H_W^1)'_b = H_V^\infty$ also holds. The dual H_W^1 is a Fréchet–Schwartz space, whose topology is determined by the sequence of seminorms $(\|\cdot\|_k)_{k=1}^\infty$. For more details, especially for a proof of the duality, see [10].

We define $K(z,w) := 1/(1-z\bar{w})^2$, where z, $w \in \mathbf{D}$, and we denote by R the Bergman projection

$$Rf(z) = \int\limits_{\mathbf{D}} K(z,w)f(w)dA(w). \tag{6}$$

According to [10], R is a continuous projection from L_V^∞ onto H_V^∞, where L_V^∞ is defined as in (3) but "measurable" replacing "analytic" and "ess sup" replacing "sup".

To fix a constant later we remark that R also maps the space L^∞ continuously into $L^\infty_{v_1}$ (definition analogous to (4)); see [10]. Moreover, R is self–dual with respect to the dual pairing $\langle \cdot, \cdot \rangle$. Let us thus show that R is also a bounded operator from

$$L^1_1 := \Big\{ f : \mathbf{D} \to \mathbf{C} \text{ measurable} \Big| \, \|f\|_1 := \int\limits_{z\in\mathbf{D}} |f(z)|(1 + |\log(1 - |z|)|) dA(z) < \infty \Big\} \tag{7}$$

into $L^1 := L^1(\mathbf{D}, dA)$: for $f \in L^1_1$ and $g \in L^\infty$,

$$\begin{aligned} |\langle Rf, g\rangle| &= \Big| \int\limits_{\mathbf{D}} \int\limits_{\mathbf{D}} \frac{f(\zeta)\bar{g}(z)}{(1 - z\bar{\xi})^2} dA(\zeta) dA(z) \Big| \\ &\le \int\limits_{\mathbf{D}} |f(\zeta)| \Big| \overline{\int\limits_{\mathbf{D}} \frac{g(z)}{(1 - \bar{z}\xi)^2} dA(z)} \Big| dA(\zeta) \\ &= \int\limits_{\mathbf{D}} |f(\zeta)\overline{Rg}(\zeta)| dA(\zeta) \\ &\le C \int\limits_{\mathbf{D}} |f(\zeta)|(1 + |\log(1 - |\zeta|)|) dA(\zeta) \le C'. \end{aligned}$$

So, let us denote by C_0 the positive constant such that (denoting the norm of L^1 by $\| \cdot \|_0$)

$$\|Rg\|_0 \le C_0 \|g\|_1 \tag{8}$$

for all $g \in L^1_1$.

Concerning other notations and terminology, for analytic function spaces we refer to the book of Zhu, [12], and for locally convex spaces we mention [6], [7] and [8] and also the paper [10].

2. Preliminaries for the atomic decomposition.

We next turn to the construction of the atomic decomposition. Atomic decompositions for analytic function spaces on the unit disc were first introduced by Coifman and Rochberg, see [3]. A simple presentation of the classical result is given in [12], Chapter 4. By an "atomic decomposition" of some analytic function space we roughly mean a way to write an analytic function as an infinite linear combination of simple building blocks, "atoms", in such a way that the complex coefficients for a sequence belong to a relevant sequence space.

For our function space H^∞_V the "atom" functions, i.e. the building blocks, will be defined using the Bergman kernel, as usual. Another essential ingredient is a suitable decomposition of the unit disc $\mathbf{D}$. Our decomposition is finer than the usual one, which consists of sets with essentially constant hyperbolic area; see [3] or [12]. Our choice actually makes many preliminary lemmas nearly trivial to prove. It is very probable that this could be used to simplify the proofs also in the classical case of Bergman spaces $A^p(\mathbf{D})$.

We are aware of the fact that on the other hand our choice of the lattice is not an optimal, i.e. small, one. Hence we also do not particularly try to achieve optimal constants in various inequalities; the estimates may be quite crude in some cases.

Definition 3 *Let us fix the constant $M = (C_0 + 1)10000$, where C_0 is as in (8). For all $m \in \mathbf{N}$, define first $r_m = 1 - m^{-1/3}$; then define for every m, for every $m' := 0, 1, 2, \ldots, M$*

$$r_{m,m'} := r_m + \frac{m'}{M}(r_{m+1} - r_m) \tag{9}$$

and $\theta(m,k) := 2\pi k/[Mm^{4/3}]$ *for* $k = 0,1,\ldots,[Mm^{4/3}]$*. (Here* $[a]$ *stands for the largest integer strictly smaller than* $a+1$*.) Define a bijection* ρ *from* $\mathbf{N}$ *onto the set* $\{(m,m',k) \mid m \in \mathbf{N},\ m' = 0,1,2,\ldots,M-1,\ k = 0,1,2,\ldots,[Mm^{4/3}]-1\}$*. For every* $n \in \mathbf{N}$*, pick up* m*,* m' *and* k *such that* $n = \rho^{-1}(m,m',k)$ *and define*

$$D_n := \{z = re^{i\theta} \in \mathbf{D} \mid r_{m,m'} \le r \le r_{m,m'+1} \text{ and } \theta(m,k) \le \theta \le \theta(m,k+1)\}. \tag{10}$$

We clearly have

$$\bigcup_n D_n = \mathbf{D}, \tag{11}$$

and moreover, the set formed by the numbers $z \in \mathbf{D}$ belonging to more than one of the sets D_n has Lebesgue measure 0.

Lemma 1 *For every* n *the Euclidean diameter of* D_n *satisfies*

$$\mathrm{diam} D_n \le \inf_{z\in D_n} \frac{20(1-|z|)^4}{M} \tag{12}$$

PROOF. If n is given and m, m' and k are such that $n = \rho^{-1}(m,m',k)$, then

$$\inf_{z\in D_n}(1-|z|) = 1 - r_{m,m'+1} = 1 - \Big(r_m + \frac{m'+1}{M}(r_{m+1}-r_m)\Big) \ge 1 - r_{m+1} = (m+1)^{-1/3}. \tag{13}$$

On the other hand, the arguments of the points of D_n may vary at most from $\theta(m,k)$ to $\theta(m,k+1)$, hence at most $2\pi/(Mm^{4/3})$. The radii may vary from $r_{m,m'}$ to $r_{m,m'+1}$, which amounts to a change of (see (9))

$$M^{-1}(r_{m+1}-r_m) = \frac{1}{M}\Big(1 - \frac{1}{(m+1)^{1/3}} - 1 + \frac{1}{m^{1/3}}\Big) \le \frac{m^{-4/3}}{M}. \tag{14}$$

This implies the claim. ■

Corollary 1 *For every analytic function* f *on* $\mathbf{D}$*, for all* n*, we have*

$$|f(z) - f(w)| \le \frac{20}{M}(1-|\lambda|)^4 \sup_{\zeta\in D_n} |f'(\zeta)| \tag{15}$$

for all $z, w, \lambda \in D_n$*.* ■

The proof is a direct application of the previous lemma and the mean value theorem.

The crucial step for our main result is to strengthen the classical approximation lemma (e.g. [12], Lemma 4.3.4) using our new decomposition of $\mathbf{D}$. Notice that the (relatively easy) proof is based on the density of our decomposition.

Lemma 2 *Let* $v \in V$*, in particular, assume* $\sqrt{1-|z|} \le v(z) \le 1$*. For all* $n \in \mathbf{N}$*, let* λ_n *be an interior point of* D_n*.*
a) For all $g \in H^1_W$ *with* $\int_{\mathbf{D}} |g|dA \le 1$ *we have*

$$\sum_{n=1}^{\infty} \int_{D_n} |g(z) - g(\lambda_n)| \frac{1}{v(z)} dA(z) \le 200M^{-1} \tag{16}$$

b) For all $f \in H^\infty_V$ *with* $|f(z)| \le 1/v(z)$ *on* $\mathbf{D}$*, and for all* $n \in \mathbf{N}$*, we have*

$$\sup_{z\in D_n} |f(z) - f(\lambda_n)| \le 200M^{-1} \sup_{z\in D_n} (1-|z|)^{1/4}. \tag{17}$$

PROOF. In the case a) we estimate using Corollary 1 (choose $\lambda := z$ there) and (2)

$$\sum_{n=1}^{\infty}\int_{D_n}\frac{1}{v(z)}|g(z)-g(\lambda_n)|dA(z)$$

$$\leq \frac{20}{M}\sum_{n=1}^{\infty}\int_{D_n}(1-|z|)^{-1/2}(1-|z|)^4\sup_{\zeta\in D_n}|g'(\zeta)|dA(z) \tag{18}$$

Since $\sup\limits_{\zeta\in D_n}(1-|\zeta|)\leq(1+1/10)(1-|z|)$ for all $z\in D_n$ (e.g.by Lemma 1), we can bound (18) by

$$\frac{40}{M}\sum_{n=1}^{\infty}m(D_n)\sup_{z\in D_n}(1-|z|)^{7/2}|g'(z)|\leq\frac{40}{M}\sup_{z\in\mathbf{D}}(1-|z|)^{7/2}|g'(z)| \tag{19}$$

Here we differentiate the reproducing formula $g(z)=\int\limits_{\mathbf{D}}\frac{g(\zeta)}{(1-z\bar\zeta)^2}dA(\zeta)$ once under the integral sign to bound (19) by

$$\frac{80}{M}\sup_{z\in\mathbf{D}}(1-|z|)^{7/2}\Big|\int\limits_{\mathbf{D}}\frac{g(\zeta)\bar\zeta}{(1-z\bar\zeta)^3}dA(\zeta)\Big| \tag{20}$$

$$\leq\frac{80}{M}\int\limits_{\mathbf{D}}|g(\zeta)|dA(\zeta)\leq\frac{80}{M}. \tag{21}$$

As for b), let f be as indicated. Again by Corollary 1

$$\sup_{z\in D_n}|f(z)-f(\lambda_n)|\leq\frac{20}{M}\sup_{z\in D_n}(1-|z|)^4|f'(z)|$$

$$\leq \frac{20}{M}\Big(\sup_{z\in D_n}(1-|z|)^{7/2}|f'(z)|\Big)\Big(\sup_{z\in D_n}(1-|z|)^{1/4}\Big),$$

and as in the previous case we see that the first factor is bounded by

$$\frac{40}{M}\int\limits_{\mathbf{D}}|f(\zeta)|dA(\zeta)\leq\frac{40}{M}\int\limits_{\mathbf{D}}1/v(\zeta)dA(\zeta)\leq\frac{200}{M}. \tag{22}$$

The part b) of the above result immediately implies. ■

Corollary 2 *Let $v\in V$. If $\|f\|_v\leq 1$, we have*

$$|f(\lambda)|\leq\frac{2}{v(z)} \tag{23}$$

for every $\lambda, z\in D_n$. ■

3. Main result.

In this section we construct an atomic decomposition for the space H_V^∞. We shall show that every element f of H_V^∞ can be presented as a linear combination

$$f(z)=\sum_{n=1}^{\infty}\frac{\alpha(n)m(D_n)}{(1-\bar\lambda_n z)^2}, \tag{24}$$

where $\lambda_n \in D_n$ and the coefficient sequence $(\alpha(n))_{n=1}^{\infty}$ belongs to a Köthe sequence space $K_\infty(A)$. Conversely, for every sequence $(\alpha(n))_{n=1}^{\infty} \in K_\infty(A)$, the analytic function defined by (24) belongs to H_V^∞. However, given $f \in H_V^\infty$, the sequence $(\alpha(n))_{n=1}^{\infty}$ is not unique. (The atoms do not form a Schauder basis.)

We now consider the sequence space relevant to the atomic decomposition. Since H_V^∞ is a countable inductive limit of Banach spaces, the same is also true for the sequence space.

Definition 4 *Let us fix, for all $n \in \mathbf{N}$, an interior point λ_n of D_n. For all $k \in \mathbf{N}$, let a_k be the sequence of positive real numbers with*

$$a_k(n) := |\log(1-|\lambda_n|)|^k \quad , \; n \in \mathbf{N}. \tag{25}$$

Denote $A := \{a_k \mid k \in \mathbf{N}\}$, define the Köthe co–echelon space $K_\infty(A)$ corresponding to A as in [1], Section 1.

For a quick definition of $K_\infty(A)$, let us denote by $V_{\mathbf{N}}$ the set of all bounded positive sequences $\tilde{v}$ satisfying

$$\tilde{v}(n) \le C_k/a_k(n) \quad \text{for all } k, n \in \mathbf{N}. \tag{26}$$

Then a sequence $\alpha = (\alpha(n))_{n=1}^{\infty}$ of complex numbers belongs to $K_\infty(A)$ if and only if for every $\tilde{v} \in V_{\mathbf{N}}$ one can find a $C > 0$ such that

$$\sup_{n \in \mathbf{N}} |\alpha(n)| \tilde{v}(n) \le C. \tag{27}$$

The (non–metrizable) topology of $K_\infty(A)$ is defined by the seminorms

$$\|\alpha\|_{\tilde{v}} := \sup_{n \in \mathbf{N}} |\alpha(n)| \tilde{v}(n). \tag{28}$$

Let us next define three operators. Let the sequence $(\lambda_n)_{n=1}^{\infty}$ be as above. Let f be analytic on $\mathbf{D}$ and let $\alpha = (\alpha(n))_{n=1}^{\infty}$ be a sequence of complex numbers. Set

$$Sf(z) \quad := \quad \sum_{n=1}^{\infty} \frac{m(D_n) f(\lambda_n)}{(1-\bar{\lambda}_n z)^2}, \tag{29}$$

$$T(\alpha) \quad := \quad \sum_{n=1}^{\infty} \frac{\alpha(n) m(D_n)}{(1-\bar{\lambda}_n z)^2}, \tag{30}$$

$$(Qf)(n) \quad := \quad f(\lambda_n) \quad \text{for } n \in \mathbf{N}. \tag{31}$$

Hence, formally $S = TQ$. These operators have the following continuity properties.

Lemma 3 *The operators $S : H_V^\infty \to H_V^\infty$, $T : K_\infty(A) \to H_V^\infty$ and $Q : H_V^\infty \to K_\infty(A)$ are continuous.*

PROOF. 1° Consider T. To prove the continuity it suffices to take an arbitrary bounded set $B \in H_W^1$ and show that a neighbourhood of 0 in $K_\infty(A)$ is mapped into the polar B° of B by T, where

$$B^\circ := \{g \in H_V^\infty \mid |\langle f, g\rangle| \le 1 \text{ for every } f \in B\}. \tag{32}$$

We first claim that B is contained in a set $G := \{f \in H_W^1 \mid \int |f| v(z)^{-1} \le 10\}$ for a $v \in V$. To find v we use the boundedness of B to pick the numbers $C_k \ge 1$ such that $B \subset C_k U_k$ for every k; here U_k is the closed unit ball of the continuous seminorm $\|\cdot\|_k$ of H_W^1. This means that every $f \in B$ satisfies, for all k,

$$\int_{\mathbf{D}} |f| |\log(1-|z|)|^k dA \le C_k. \tag{33}$$

Define v by

$$v(z) := \frac{1}{\sum_{k=0}^{\infty} k!^{-1}4^{-k}C_k^{-1}|\log(1-|z|)|^k}. \tag{34}$$

Since

$$\sum_{k=0}^{\infty} k!^{-1}4^{-k}|\log(1-|z|)|^k = e^{-\frac{1}{4}\log(1-|z|)} = (1-|z|)^{-1/4},$$

we get the estimate

$$v(z) \geq (1-|z|)^{1/4}. \tag{35}$$

We clearly have $\sup_z v(z)|\log(1-|z|)|^k \leq k!4^k C_k$ for every k, hence, $v \in V$. Moreover, it follows easily from (33) and (34) that every function in B belongs to the set G:

$$\int_{\mathbf{D}} |f|v^{-1}dA \leq \sum_{k=0}^{\infty} k!^{-1}C_k^{-1} \int_{\mathbf{D}} |f||\log(1-|z|)|^k dA = \sum_{k=0}^{\infty} k!^{-1} \leq 10. \tag{36}$$

Set $\tilde{v}(n) := v(\lambda_n)$ for $n \in \mathbf{N}$; we then have $(\tilde{v}(n))_{n=1}^{\infty} \in V_{\mathbf{N}}$, by definitions. For $\|\alpha\|_{\tilde{v}} \leq C$ we thus have, by the reproducing formula $f(\lambda_n) = \int f(\zeta)(1-\lambda_n\bar{\zeta})^{-2}dA(\zeta)$,

$$\begin{aligned} |\langle T\alpha, f\rangle| &= \Big|\sum_{n=1}^{\infty} \alpha(n)m(D_n)\bar{f}(\lambda_n)\Big| \\ &\leq \sum_{n=1}^{\infty} |\alpha(n)|\tilde{v}(n)v(\lambda_n)^{-1}m(D_n)|f(\lambda_n)| \\ &\leq \|\alpha\|_{\tilde{v}}\Big(\sum_{n=1}^{\infty} v(\lambda_n)^{-1}m(D_n)|f(\lambda_n)|\Big). \end{aligned} \tag{37}$$

We prove in a moment that

$$1/v(\lambda_n) < C/v(z) \tag{38}$$

for every n, for every $z \in D_n$. Using this, Lemma 2.b), the definition of the set G and (35), we can estimate (37) by a constant times

$$\sum_n \int_{D_n} |f(z)|v(z)^{-1}dA + \sum_n \int_{D_n} (1-|z|)^4 v(z)^{-1}dA \leq C. \tag{39}$$

The statement (38) follows from the smallness of diamD_n. First,

$$\frac{\partial v(r)^{-1}}{\partial r} = \sum_{k=1}^{\infty} k!^{-1}k4^{-k}C_k^{-1}|\log(1-r)|^{k-1}(1-r)^{-1} \leq C(1-r)^{-1-1/4},$$

This, diam$D_n \leq C(1-|z|)^4$ (Lemma 1) and the mean value theorem imply $|1/v(z) - 1/v(\lambda_n)| \leq C(1-|z|)^2 \leq C' \leq C'/v(z)$. Hence, (38) follows.

2° Concerning Q, let the weight sequence $\tilde{v} \in V_{\mathbf{N}}$ be given. We define $v \in V$ as follows. First let $(\rho_j)_{j=1}^{\infty}$, $0 \leq \rho_j < 1$, be the unique strictly increasing sequence which satisfies: for every j there exists an n such that $\rho_j = |\lambda_n|$. Let us define the subset N_j of $\mathbf{N}$ by $N_j = \{n \mid |\lambda_n| = \rho_j\}$.

For every j, define for all $z \in \mathbf{D}$ with $|z| = \rho_j$,

$$v(z) = \sup_{n \in N_j} \tilde{v}(n). \tag{40}$$

For every $r \in [0,1[$, extend $v(r)$ piecewise linearly, and then extend v radially to $\mathbf{D}$.

We show that $v \in V$. If $z \in \mathbf{D}$ satisfies $|z| = \rho_j$ for some j, and $k \in \mathbf{N}$ is given, then

$$\begin{aligned} v(z) &= \sup_{n \in N_j} \tilde{v}(n) \le \sup_{n \in N_j} C_k / a_k(n) \\ &= \sup_{n \in N_j} C_k |\log(1 - |\lambda_n|)|^{-k} = C_k |\log(1-|z|)|^{-k}. \end{aligned} \tag{41}$$

Concerning other numbers z, Lemma 1 implies $|\rho_j - \rho_{j+1}| \le C_j(1-\rho_j)^4$, hence, for every k we have

$$|\log(1-\rho_j)| \le |\log(1-\rho_{j+1})| \le c_k |\log(1-\rho_j)| \tag{42}$$

for a constant $c_k > 0$. So, if z satisfies $\rho_j < |z| < \rho_{j+1}$, then the piecewise linearity of v and (41) and (42) imply

$$v(z) \le \max\{v(\rho_j), v(\rho_{j+1})\} \le C'_k |\log(1-\rho_{j+1})|^{-k} \le C'_k |\log(1-|z|)|^{-k}. \tag{43}$$

Hence, $v \in V$.

Now (40) and the definition of Q imply $\|Qf\|_{\tilde{v}} \le C\|f\|_v$, hence Q is continuous.

3° Finally, we have $S = TQ$, hence also S is continuous. ■

The following fact can now be proved using a result of Garnir, De Wilde and Schmets to be found in [5], p. 346.

Lemma 4 *The operator $S : H_V^\infty \to H_V^\infty$ is invertible .*

PROOF. We want to show that the operator $A := I - S$

(i) is bounded (i.e. maps a neighbourhood of 0 of H_V^∞ into a bounded set), and

(ii) for some neighbourhood $U \subset H_V^\infty$ of 0, $A(U) \subset U/2$.

Then S is invertible by [5], p. 346.

Let us take a weight as small as

$$w(z) := \sqrt{1-|z|}. \tag{44}$$

Then, for sure, $w \in V$. Let $U \subset H_V^\infty$ be the unit ball of $\|\cdot\|_w$:

$$U := \Big\{ f \in H_V^\infty \;\Big|\; \sup_{z \in \mathbf{D}} |f(z)|(1-|z|)^{1/2} \le 1 \Big\}; \tag{45}$$

We want to show that

$$\Big| \int_{\mathbf{D}} Af(z)\bar{g}(z)dA(z) \Big| \le \frac{1}{2} \tag{46}$$

for every $f \in U$, g in the unit ball of L_1^1 (see (8)), that is,

$$g \in \tilde{U} := \Big\{ h \in L^1_{\mathrm{loc}}(\mathbf{D}, dA) \;\Big|\; \|h\|_1 := \int_{z \in \mathbf{D}} |h(z)|(1 + |\log(1-|z|)|)dA(z) \le 1 \Big\}. \tag{47}$$

Here $L^1_{\mathrm{loc}}(\mathbf{D}, dA)$ is the space of locally integrable functions on $\mathbf{D}$.

Assuming (46) we show that (i) and (ii) hold. As for (i), let us pick up an arbitrary weight $v \in V$; to prove the boundedness of A it suffices to find a k such that for $f \in H_V^\infty$ and $g \in H_W^1$ with $\|f\|_v \le 1$ and $\|g\|_k \le 1$ we have

$$|\langle Af, g \rangle| \le C. \tag{48}$$

But $v(z) \ge \sqrt{1-|z|}$, see (2) in the definition of V. Hence, $f \in U$. Moreover, for every $k \in \mathbf{N}$, $k \ge 2$, the subset $\{\|g\|_k \le 1\} \subset H_W^1$ is smaller than the set $\tilde{U}$ above. Hence, (48) follows from (46).

Concerning (ii), the identification of the norm of $L^\infty(\mathbf{D},dA)$ as the dual norm of $L^1(\mathbf{D},dA)$ implies the following:
An analytic function φ belongs to $U/2$, if $|\int_{\mathbf{D}}\varphi\bar g|\le 1/2$ for every

$$g\in U^\circ := \Big\{h\in L^1_{\rm loc}(\mathbf{D})\ \Big|\ \int\limits_{\mathbf{D}} |h(z)|(1-|z|)^{-1/2}dA(z)\le 1\Big\}. \tag{49}$$

But U° is a much smaller set than $\tilde U$, hence (ii) follows from (46).

So we want to prove (46). Let now f and g be given. Since Af is analytic and the Bergman projection is self–dual, the identities

$$\int_{\mathbf{D}} Af\bar g = \int (RAf)\bar g = \langle Af, Rg\rangle \tag{50}$$

hold formally. Moreover, by (8),

$$\|\tilde g\|_0 := \|Rg\|_0 \le C_0\|g\|_1\le C_0 \tag{51}$$

We thus have

$$\begin{aligned}\int\limits_{\mathbf{D}} Af\bar g &= \langle Af,\tilde g\rangle = \int_{\mathbf{D}} f(z)\overline{\tilde g(z)}dA(z) - \sum_{n=1}^\infty m(D_n)f(\lambda_n)\overline{\tilde g(\lambda_n)} \\ &= \sum_{n=1}^\infty\int_{D_n}\big(f(z)-f(\lambda_n)\big)\overline{\tilde g(z)}dA(z) + \sum_{n=1}^\infty\int_{D_n} f(\lambda_n)\big(\overline{\tilde g(z)}-\overline{\tilde g(\lambda_n)}\big)dA(z).\end{aligned} \tag{52}$$

Using Lemma 2.b) and (51) we get the following bound for the first sum in (52):

$$\begin{aligned}&\sum_{n=1}^\infty \sup_{z\in D_n}|f(z)-f(\lambda_n)|\int_{D_n}|\tilde g(z)|dA(z)\\ \le\ & 200M^{-1}\int_{\mathbf{D}}|\tilde g(z)|dA(z)\le 2000C_0M^{-1}\le 1/5.\end{aligned} \tag{53}$$

To estimate the second term of (52) from above, we first use Corollary 2 to bound $|f(\lambda_n)|$ by $\frac{2}{w(z)}$ for any $z\in D_n$. Then Lemma 2 a) directly gives the bound

$$\sum_{n=1}^\infty\int_{D_n}\frac{2}{w(z)}|\tilde g(z)-\tilde g(\lambda_n)|dA(z)\le\frac{400C_0}{M}. \tag{54}$$

This completes the proof. ■

We now formulate the main result as follows. We denote $m(n):=m(D_n)$; this number can be estimated, if necessary, from the definition of the sets D_n after (9). Recall that λ_n was chosen in Lemma 2: it is an interior point of the set D_n. The space $K_\infty(A)$ was defined in the beginning of Section 3.

Theorem 1 *For every $(\alpha(n))_{n=1}^\infty\in K_\infty(A)$ the function*

$$f(z)=\sum_{n=1}^\infty\frac{\alpha(n)m(n)}{(1-\bar\lambda_n z)^2} \tag{55}$$

belongs to $H_V^\infty(\mathbf{D})$, and the mapping $(\alpha(n))_{n=1}^\infty\mapsto f$ is continuous between these two spaces.

For every analytic function $f\in H_V^\infty(\mathbf{D})$ there exists a complex sequence $(\alpha(n))_{n=1}^\infty\in K_\infty(A)$ such that

$$f(z)=\sum_{n=1}^\infty\frac{\alpha(n)m(n)}{(1-\bar\lambda_n z)^2}. \tag{56}$$

The mapping $f\mapsto(\alpha(n))_{n=1}^\infty$ can be made linear and continuous from $H_V^\infty(\mathbf{D})$ into $K_\infty(A)$.

PROOF. The first statement follows from the continuity of T. For the second one, given f, choose $g = S^{-1}f$ and then define the sequence $\alpha = Qg$, i.e. $\alpha(n) = g(\lambda_n)$. We then obtain (56) from (29) and (31). The continuity statement follows from the continuity of Q and Lemma 4. ■

Finally, the operator $P := QS^{-1}T$ is a continuous projection in the space $K_\infty(A)$; we have $P^2 = P$. This implies

Proposition 1 *The space H_V^∞ is isomorphic to a complemented subspace of the Köthe sequence space $K_\infty(A)$.* ■

4. The space is not nuclear.

Intuitively it is quite clear that the space H_V^∞ cannot be nuclear, since the weight system is not "steep" enough. Two arbitrary weights differ from each other at most by a factor which grows only logarithmically on the boundary of $\mathbf{D}$.

We give a proof for the nonnuclearity by extracting a subspace which is isomorphic to a nonnuclear Köthe coechelon space. The method of the proof may have some independent interest. It resembles the construction [2], Section 2. However, in [2] the weight system was non–radial, and the essential phenomena occurred on the boundary of the disc. In the present work the weights are radial; consequently, we have to play also with the radii of the points of $\mathbf{D}$. More precisely, the interesting and crucial things happen in the interior points $z_n := e^{i\theta_n}(1-(2n)^{-20})$ of the subdomains D_n (to be chosen below). Of course there is a lot of freedom in the construction; many other points could be used as well.

Notice that the space $K_\infty(A)$ above is not nuclear; however, the results of Section 3 do not imply that H_V^∞ contains a subspace isomorphic to $K_\infty(A)$. The operator T need not be injection, since Q is not a surjection.

Proposition 2 *The space H_V^∞ is not nuclear, since it contains a subspace isomorphic to the Köthe coechelon space $K_\infty(M)$, which is defined as $K_\infty(A)$ in the beginning of Section 3, but $a_k(n)$ replaced by*

$$\mu_k(n) := (\log n)^k \tag{57}$$

So, a sequence α belongs to $K_\infty(M)$, if and only if $\sup_n |\alpha_n|(\log n)^{-k} < \infty$ for some k. It is well known that $K^\infty(M)$ is not nuclear, see e.g. [9], Theorem 6.1.2.

PROOF. Let $\theta_n := 1/n$ and $J_n := [\theta_n - \varepsilon_n, \theta_n + \varepsilon_n]$ for all $n \in \mathbf{N}$, where $\varepsilon_n := 2^{-9}n^{-4}$. Notice that the sets J_n are mutually disjoint. Choose the function $\varphi_n : [0, 2\pi] \to \mathbf{R}^+$ such that $\varphi_n(\theta) = 1$ for $\theta \in J_n$, and $\varphi_n(\theta) = (2n)^{-4}$ for $\theta \notin J_n$. Let e_n be an analytic function on the disc defined by

$$e_n(z) := \exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\varphi_n(\theta)d\theta\Big).$$

The radial limits (on the boundary of the disc) of the functions e_n exist a.e. (we denote them by the same symbol), and we have $|e_n(e^{i\theta})| = \varphi_n(\theta)$ for a.e. θ. Compare with [2] and the references therein. □

Let us define

$$D_n := \{z \in \mathbf{D} \mid |z - e^{i\theta_n}| < 1/n^4\} \ , \ C_n := \mathbf{D} \setminus D_n.$$

We formulate a lemma containing some crucial estimates.

Lemma 5 $1°$ *For all $z \in \mathbf{D}$, all n, we have $|e_n(z)| \le 1$.*

$2°$ *Fix a $z \in \mathbf{D}$; if n is such that $|\arg z - \theta_n| = \min_{m\in\mathbf{N}}\{|\arg z - \theta_m|\}$, then $|e_n(z)| \le 4n^{-4}(1-|z|+n^{-4})^{-1}$. Moreover, if $m \neq n$, then $|e_m(z)| \le 2^{-3}m^{-2}$.*

$3°$ *For every n we have $|e_n(z_n)| \ge 1/2$, where $z_n := e^{i\theta_n}(1-(2n)^{-20})$.*

PROOF. The statement 1° follows immediately from the maximum principle, since the moduli of the radial limits of every e_n are at most 1.

Concerning 2°, if z and n are as in the assumption, the Jensen inequality implies (as in [2])

$$\begin{aligned} |e_n(z)| &= \exp\Big(\frac{1}{2\pi}\mathrm{Re}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\varphi_n(\theta)d\theta\Big) \\ &= \exp\Big(\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\log\varphi_n(\theta)d\theta\Big) \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\varphi_n(\theta)d\theta. \end{aligned} \tag{58}$$

We may assume $|z| \le 1-n^{-4}$; otherwise there is nothing to prove. We have $|z-e^{i\theta}| \ge n^{-4}$, hence, $|z-e^{i\theta}| \ge \max(|z-e^{i\theta}|, n^{-4}) \ge \max(1-|z|, n^{-4}) \ge (1-|z|+n^{-4})/2$. So we have

$$\frac{1-|z|^2}{|e^{i\theta}-z|^2} \le \frac{1-|z|^2}{(1-|z|)|e^{i\theta}-z|} \le \frac{4}{1-|z|+n^{-4}}.$$

Dividing the integration interval $[0,2\pi]$ in (58) into the parts J_n and $[0,2\pi]\setminus J_n$, we thus find that the first part yields the estimate

$$\frac{2}{n^4(1-|z|+n^{-4})},$$

since the lenght of J_n is smaller than $2^{-1}n^{-4}$. The second part has the smaller bound $(2n)^{-4}$, since $\varphi_n = (2n)^{-4}$ on $[0,2\pi]\setminus J_n$.

In the case $m \ne n$ we again have

$$|e_m(z)| \le \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\varphi_m(\theta)d\theta. \tag{59}$$

Dividing the integration interval into the parts J_m and its complement, on J_m we have

$$|e^{i\theta}-z| \ge \frac{1}{2}\Big|\frac{1}{m}-\frac{1}{n}\Big| = \frac{|n-m|}{2mn}.$$

This is always larger than the number $1/(4m^2)$, as seen by considering the cases $n \le 2m$ and $n > 2m$ separately: in the former case one uses $|n-m| \ge 1$ and $2mn \le 4m^2$, and in the latter $|n-m| \ge n/2$ and a cancellation of n. Taking into account the length $2^{-9}m^{-4}$ of the interval J_m, this part yields an estimate $2^{-7}m^{-2}$ in (59). The rest of the integral is bounded by $(2m)^{-4}$, since $\varphi_m = (2m)^{-4}$ on $[0,2\pi]\setminus J_m$.

For the lower bound 3°, we have $\log\varphi_n = 0$ on J_n and $\log\varphi_n = -4\log(2n) < 0$ on the complement of J_n , hence

$$\begin{aligned} |e_n(z_n)| &= \exp\Big(\frac{1}{2\pi}\int_{[0,2\pi]\setminus J_n}\frac{1-|z_n|^2}{|e^{i\theta}-z_n|^2}\log\varphi_n(\theta)d\theta\Big) \\ &\ge \exp\Big(-\frac{4}{2\pi}\int_{[0,2\pi]\setminus J_n}\frac{(2n)^{-20}}{n^{-18}}\log(2n)d\theta\Big) \\ &\ge \exp(-2^{-10}n^{-1/2}) \ge \frac{1}{2}. \quad \blacksquare \end{aligned} \tag{60}$$

We return to the proof of the proposition. We define the mapping ψ from $K_\infty(M)$ into H_V^∞ by $\psi : (\alpha_n)_{n=1}^\infty \mapsto \sum_{n=1}^{\infty} \alpha_n e_n$. We claim that the mapping is an isomorphism onto its image. Since the spaces are strong duals of Fréchet–Schwartz spaces, it suffices to show that, for some constants c_k, $C_k > 0$,

$$c_k \sup_{n\in\mathbf{N}} |\alpha_n|(\log n)^{-k} \le \| \sum \alpha_n e_n\|_{v_k} \le C_k \sup_{n\in\mathbf{N}} |\alpha_n|(\log n)^{-k} \tag{61}$$

for all k. Since the expression $\mu((\alpha_n)_{n=1}^\infty) := \sup_n |\alpha_n| n^{-1/2}$ is obviously a continuous seminorm on $K_\infty(M)$, we may assume that $\mu((\alpha_n)) \le 1$, i.e. $|\alpha_n| \le n^{1/2}$ for all n in (61). We may moreover assume that $|\alpha_n| \ge 1$ there, for all n.

To prove the left hand side inequality, choose N such that

$$|\alpha_N|(\log(N))^{-k} \ge \frac{1}{2} \sup_{n\in\mathbf{N}} |\alpha_n|(\log n)^{-k}. \tag{62}$$

We then have, by 3° of Lemma 5 above, and by $|\alpha_n| \ge 1$,

$$|\alpha_N e_N(z_N)| v_k(z_N) \ge \frac{1}{2} 20^{-k}(\log(2N))^{-k}, \tag{63}$$

and on the other hand

$$\sum_{n\neq N} |\alpha_n e_n(z_N)| v_k(z_N) \le \sum_{n\neq N} n^{1/2} 2^{-3} n^{-2} 20^{-k} (\log(2N))^{-k} \tag{64}$$

Here $\sum_n n^{-3/2} 2^{-3} \le (1 + \int_1^\infty x^{-3/2}) 2^{-3} \le 3/8$. Hence, combining (63) and (64) and using the triangle inequality we get

$$\Big|\sum_{n=1}^\infty \alpha_n e_n(z_N)\Big| v_k(z_N) \ge |\alpha_N e_N(z_N)| v_k(z_N) - \Big|\sum_{n\neq N} \alpha_n e_n(z_N)\Big| v_k(z_N) \ge \frac{1}{8} |\alpha_N e_N(z_N)| v_k(z_N),$$

hence, by (62) and (63), we get

$$\begin{aligned} 20^{-k} \sup_n |\alpha_n|(\log n)^{-k} &\le 2\cdot 20^{-k} |\alpha_N|(\log N)^{-k} \le C|\alpha_N e_N(z_N)| v_k(z_N) \\ \le \quad C' \Big|\sum_{n=1}^\infty \alpha_n e_n(z_N)\Big| v_k(z_N) &\le C' \Big\| \sum \alpha_n e_n \Big\|_{v_k} \end{aligned}$$

We finally prove the right hand side of (61). Let N still be as above, let $z \in \mathbf{D}$ and let n be such that $|\arg z - \theta_n| = \min_{m\in\mathbf{N}} \{|\arg z - \theta_m|\}$. By 2° of Lemma 5 we have (since always $v_k(z) \le 1$)

$$\begin{aligned} \sum_{m\neq n} |\alpha_m e_m(z)| v_k(z) &\le \sum_{m\neq n} |\alpha_m| m^{-2} \\ \le \quad \sum_{m\neq n} C_k m^{-3/2} |\alpha_m| (\log m)^{-k} &\le C_k' |\alpha_N| (\log N)^{-k}. \end{aligned} \tag{65}$$

Concerning the nth term, for $|z| \ge 1 - n^{-3}$ we have by 1°

$$v_k(z)|\alpha_n e_n(z)| \le v_k(1-n^{-3})|\alpha_n e_n(z)| \le |\alpha_n| v_k(1-n^{-3}) \le 3|\alpha_n|(\log n)^{-k} \le 3|\alpha_N|(\log N)^{-k}$$

For $|z| \le 1 - n^{-3}$ we have

$$v_k(z)|\alpha_n e_n(z)| \le \frac{4|\alpha_n|}{n^4(1-|z|+n^{-4})} \le \frac{4|\alpha_n|}{n^4(n^{-3}+n^{-4})}$$
$$\le \quad \frac{4|\alpha_n|}{n} \le C_k|\alpha_n|(\log n)^{-k} \le C_k|\alpha_N|(\log N)^{-k}.$$

This and (65) imply that the right hand side of (61) holds.

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