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Surjective convolution operators on spaces of distributions

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Dedicated to the memory of Klaus Floret

Abstract. We review recent developments in the theory of inductive limits and use them to give a new and rather easy proof for Hörmander's characterization of surjective convolution operators on spaces of Schwartz distributions.

Operadores de convolución sobreyectivos en espacios de distribuciones

Resumen. Se recuerdan avances recientes en la teoría de límites inductivos y se usan para dar una prueba nueva, y bastante elemental, de la caracterización debida a Hörmander de los operadores de convolución sobreyectivos en espacios de distribuciones de Schwartz.

1. Introduction and the surjectivity problem

In [6] L. Hörmander characterized surjective partial differential and convolution operators on $\mathscr{D}'(\Omega)$ by ingenious and rather complicated ad hoc methods. Essentially the same proof can be found in his fundamental treatise [7] where he writes: "We have avoided this terminology [of (LF)-spaces] in order not to encourage the once common misconception that familiarity with (LF)-spaces is essential for the understanding of distribution theory." His work is an impressive proof of this claim that it is possible to avoid (LF)-spaces. However, our aim is to show that recent abstract results about inductive limits of Fréchet spaces can be very helpful to find and to prove surjectivity results for operators on spaces of distributions.

We start with a very general question. Let Ω_1, Ω_2 be open subsets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively, and let $S : \mathscr{D}'(\Omega_2) \to \mathscr{D}'(\Omega_1)$ be a continuous linear operator with dense range such that $S(\mathscr{E}(\Omega_2)) \subseteq \mathscr{E}(\Omega_1)$. When is S onto?

As suggested by Hörmander it is helpful to split the investigation into two parts:

(A) Characterize $\mathscr{E}(\Omega_1) \subseteq S(\mathscr{D}'(\Omega_2))$.

(B) Characterize surjectivity of $\widetilde{S} : \mathscr{D}'(\Omega_2) \to \mathscr{D}'(\Omega_1)/\mathscr{E}(\Omega_1)$.

Step (A) of this program can be done with the aid of classical methods from Fréchet space theory which had been applied since the very beginning of distribution theory e.g. by L. Schwartz [14] or B. Malgrange [9]. These methods are very nicely presented in K. Floret's article [4]. We will indicate below that it is still possible to refine and to simplify as well the methods as the results but we will mainly concentrate on the second step (B).

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Since $\mathscr{E}(\Omega_1)$ is dense in $\mathscr{D}'(\Omega_1)$ the quotient space $\mathscr{D}'(\Omega_1)/\mathscr{E}(\Omega_1)$ is completely useless from the topological point of view but trivially, surjectivity of $\widetilde{S}$ is equivalent to the surjectivity of

$$T: \begin{array}{ccc} \mathscr{D}'(\Omega_2)\times\mathscr{E}(\Omega_1) & \longrightarrow & \mathscr{D}'(\Omega_1) \\ (u,f) & \longmapsto & S(u)+f \end{array}$$

Using the isomorphism $X'\times Y'\to(X\times Y)'$, $(x',y')\mapsto\big((x,y)\mapsto x'(x)+y'(y)\big)$ the transpose of T is easily calculated as

$$T^t: \begin{array}{ccc} \mathscr{D}(\Omega_1) & \longrightarrow & \mathscr{D}(\Omega_2)\times\mathscr{E}'(\Omega_1) \\ \varphi & \longmapsto & (S^t(\varphi),\varphi) \end{array}$$

and by the Hahn-Banach theorem we have that T is surjective if and only if T^t is a weak isomorphism onto its range which, by definition, means that $\mathrm{Im}(T^t)$ is a well-located subspace of the (LF)-space $\mathscr{D}(\Omega_2)\times\mathscr{E}'(\Omega_1)$. This property is reflected by the quotient space $\mathscr{D}(\Omega_2)\times\mathscr{E}'(\Omega_1)/\mathrm{Im}(T^t)$ by a condition which is called weak acyclicity. The classical theory of (LF)-spaces suffered from two problems. The first is that the description of all continuous seminorms on $\mathscr{D}(\Omega_2)\times\mathscr{E}'(\Omega_1)$ is already rather complicated and the seminorms on the quotient look even worse. The second problem is that the classical characterization of weak acyclicity due to Palamodov and Retakh even needs seminorms with peculiar additional properties. Faced with this situation, Hörmander's scepticism against (LF)-spaces seems to be justified, indeed.

We will show that younger results about (LF)-spaces improve the situation significantly.

2. Abstract properties of (LF)-spaces

We briefly recall basic properties and theorems for (LF)-spaces and derive consequences which are suitable to obtain characterizations in the situation explained in the introduction. Much more information can be found e.g. in [1, 12, 16, 17].

By an (LF)-space we mean the union $X=\bigcup_{n\in\mathbb{N}}X_n$ of Fréchet spaces $X_n\subseteq X_{n+1}$ with continuous inclusions endowed with the finest locally convex topology such that all embeddings $X_n\hookrightarrow X$ are continuous. We do not assume a priori that X is Hausdorff. We have the canonical algebraically exact sequence

$$0\to\bigoplus_{n\in\mathbb{N}}X_n\xrightarrow{d}\bigoplus_{n\in\mathbb{N}}X_n\xrightarrow{\sigma}X\to 0$$

where $d((x_n)_{n\in\mathbb{N}})=(x_n-x_{n-1})_{n\in\mathbb{N}}$ (with $x_0=0$) is the difference map and $\sigma((x_n)_{n\in\mathbb{N}})=\sum_{n\in\mathbb{N}}x_n$. According to Palamodov [11] the inductive spectrum $(X_n)_{n\in\mathbb{N}}$ or X is called (weakly) acyclic if d is a (weak) isomorphism onto its range.

The main examples of (weakly) acyclic spaces are strict (LF)-spaces (i.e. the inclusions $X_n\hookrightarrow X_{n+1}$ are (weak) homomorphisms) and (LS)- and (LS$_w$)-spaces (i.e. the inclusions $X_n\hookrightarrow X_{n+1}$ are (weakly) compact). If X, Y, and Z are (LF)-spaces and

$$0\to X\xrightarrow{f}Y\xrightarrow{g}Z\to 0$$

is an algebraically exact complex with continuous maps f and g we obtain from a diagram chase (see e.g. [13] or [15, theorems 1.4 and 1.5]):

(1) If Z is (weakly) acyclic then f is a (weak) homomorphism,

(2) the converse holds if Y is (weakly) acyclic.

This result gives the link to our original problem since f is a weak homomorphism if and only if the transposed map $f^t: Y'\to X'$ is surjective. Note that due to the open mapping theorem g is automatically a quotient map and hence, $g^t: Z'\to\mathrm{Ker}(f^t)$ is always an algebraic isomorphism. We therefore have good knowledge about Z' even though we may fail to have a concrete description of the quotient space $Z = Y/f(X)$.

Till now, we used only the Hahn-Banach theorem to reformulate our problem. This can be useful only if we have evaluable conditions ensuring (weak) acyclicity. A classical characterization is due to Palamodov [11] and Retakh [13]:

> An (LF)-space $Z = \operatorname{ind} Z_n$ is (weakly) acyclic if and only if there are absolutely convex 0-neighbourhoods $U_n \in \mathscr{U}_0(Z_n)$ and integers $m_n \geq n$ such that $U_n \subseteq U_{n+1}$ and the (weak) topologies of Z_k and Z_{m_n} coincide on U_n for all $k \geq m_n$.

As a necessary condition this result is very useful. For instance, it implies that acyclic (LF)-spaces are complete. There are however only very few applications of the sufficiency part. There are many more characterizations of (weak) acyclicity if Z satisfies a certain stability property which typically holds a priori. Z is called boundedly stable if on each set which is bounded in some step Z_n all but finitely many of the topologies of Z_m coincide. Implicitly, this notion appeared at several places, the name was given by Bierstedt [1]. The most obvious example is an inductive limit of Montel spaces, since a compact set does not admit coarser Hausdorff topologies.

For this class we have the following result essentially proved in [16].

Theorem 1 *For a boundedly stable (LF)-space $Z = \operatorname{ind} Z_n$ the following conditions are equivalent.*

1. *Z is acyclic.*
2. *Z is weakly acyclic.*
3. *Z is complete and Hausdorff.*
4. *Z is sequentially retractive, i.e. each null sequence in Z converges to 0 in some step Z_n.*
5. *Z is regular, i.e. each bounded subset of Z is contained and bounded in some step.*
6. *Z is α-regular, i.e. each bounded subset of Z is contained in some step.*
7. *Z is β-regular, i.e. each bounded subset of Z which is contained in some step is bounded in some step.*
8. *The fundamental systems $(\|\cdot\|_{n,N})_{N\in\mathbb{N}}$ of seminorms for Z_n satisfy $(\mathrm{P}_3^\star)$:*

$$\forall\ n \in \mathbb{N}\ \exists\ m \geq n\ \forall\ k \geq m\ \exists\ N \in \mathbb{N}\ \forall\ M \in \mathbb{N}\ \exists\ K \in \mathbb{N}\ \forall\ z \in Z_n$$

$$\|z\|_{m,M} \leq K\left(\|z\|_{n,N} + \|z\|_{k,K}\right).$$

9. *The following closed neighbourhood condition holds:*

$$\forall\ n \in \mathbb{N}\ \exists\ m \geq n\ \forall\ k \geq m\ \exists\ U \in \mathscr{U}_0(Z_n)\quad \overline{U}^{Z_k} \subseteq Z_m. \quad \blacksquare$$

In our situation all conditions of the theorem involve the cokernel of f (i.e. the quotient $Z = Y/f(X)$) and in the next section we will indeed use a description of the quotient which is good enough to check the last item of theorem 1. However, let us state a variant of α-regularity which avoids the quotient.

Corollary 1 *Let $f : X \to Y$ be an injective continuous linear map from an (LF)-space X to an acyclic inductive limit $Y = \operatorname{ind} Y_n$ of Fréchet-Schwartz spaces Y_n such that $\operatorname{Im}(f)$ is stepwise closed, i.e. $\operatorname{Im}(f) \cap Y_n$ is closed in Y_n for each $n \in \mathbb{N}$. The following conditions are equivalent:*

1. *f is a homomorphism.*

2. *f^t is surjective.*

3. *For every sequence $(y_k)_{k\in\mathbb{N}}$ in Y converging pointwise to 0 on $\ker(f^t)$) there are $n \in \mathbb{N}$ and a sequence $(x_k)_{k\in\mathbb{N}}$ in X such that $y_k - f(x_k) \in Y_n$ for each $k \in \mathbb{N}$.* ■

The proof is rather easily obtained from theorem 1 by noting that $Y/f(X)$ is an (LF)-space (since $\mathrm{Im}(f)$ is stepwise closed) which is boundedly stable (since quotients of Fréchet-Schwartz spaces are Montel) and that α-regularity is equivalent to the fact that each weak null sequence is contained in some step.

In our problem from the first section the space Y is is the product of $\mathscr{D}(\Omega_2)$ – which is certainly an acyclic (LF)-space with Fréchet-Schwartz steps – and $\mathscr{E}'(\Omega_1)$ – which is an (LS)-space. It follows from the next lemma (which is probably known, but we indicate the proof since we do not have a precise reference) that (LS)-spaces are acyclic inductive limits of Fréchet-Schwartz spaces and thus, the corollary is applicable in our situation.

Lemma 1 *For every compact subset K of a Fréchet space X there is a Fréchet-Schwartz space $Y \subseteq X$ with continuous inclusion such that K is compact in Y.*

PROOF. There is a null sequence $(x_n)_{n\in\mathbb{N}}$ in X with $K \subseteq \overline{\Gamma(\{x_n : n \in \mathbb{N}\})}$ and there is an increasing sequence of scalars $1 \leq \alpha_n \to \infty$ such that $(\alpha_n x_n)_{n\in\mathbb{N}}$ still converges to 0. For $m \in \mathbb{N}$ we set $K_m = \overline{\Gamma(\{\sqrt[m]{\alpha_n} x_n : n \in \mathbb{N}\})}$ and endow its linear span Y_m with the Banach space topology having K_m as the unit ball. We thus obtain $Y_{m+1} \subseteq Y_m$ and from $\sqrt[k]{\alpha_n}/\sqrt[m]{\alpha_n} \longrightarrow 0$ we obtain that K_k is compact in Y_m for $k > m$, hence $Y = \bigcap_{m\in\mathbb{N}} Y_m$ is a a Fréchet-Schwartz space which is continuously embedded in X. Finally, K is compact in Y since α_n tends to ∞. ■

We finish this general section with two results which are helpful in connection with step (A) of the introduction. The first one is called “Grothendieck-Floret factorization theorem” in [12]. Floret's proof [3] was based on a tricky lemma of Grothendieck but the result can also be proved by straightforward arguments.

Proposition 1 *Let $Z = \mathrm{ind}\, Z_n$ be a regular inductive limit of locally convex spaces and $T : X \to Z$ a (weakly) continuous operator from a semi-metrizable locally convex space X to Z. Then there is $n \in \mathbb{N}$ such that T acts continuously from X to Z_n.*

PROOF. Assuming the contrary and using that X is bornological we find bounded sets B_n in X such that $T(B_n)$ is not a bounded subset of Z_n. Moreover, there are scalars $\alpha_n > 0$ such that $B = \bigcup_{n\in\mathbb{N}} \alpha_n B_n$ is bounded in X. Hence there is $n \in \mathbb{N}$ such that $\alpha_n T(B_n) \subseteq T(B)$ is bounded in Z_n, a contradiction. ■

The last abstract result we need is an improvement of Meise's and Vogt's surjectivity criterion [10, section 26.1] and appeared in [5]:

Proposition 2 *Let $T : X \to Y$ be a continuous linear operator with dense range from a Fréchet space X to a Fréchet-Schwartz space Y and assume that for each $U \in \mathscr{U}_0(X)$ there is $V \in \mathscr{U}_0(Y)$ with*

$$(T^t)^{-1}(U^\circ) \subseteq [V^\circ]$$

(the linear span of V°). Then T is surjective. ■

3. Surjectivity modulo smooth functions

We will now use theorem 1 for step (B) of the program mentioned in the introduction, i.e. to characterize surjectivity of

$$T : \begin{array}{ccc} \mathscr{D}'(\Omega_2) \times \mathscr{E}(\Omega_1) & \longrightarrow & \mathscr{D}'(\Omega_1) \\ (u, f) & \longmapsto & S(u) + f \end{array}$$

In this situation we say that S is surjective mod $\mathscr{E}$. The Hahn-Banach theorem gives the following result.

Lemma 2 *The following conditions are equivalent.*

1. $\{(h, -S(h)) : h \in \mathscr{E}(\Omega_2)\}$ *is dense in* $\ker(T)$.

2. $\{(S^t(\varphi), \varphi) : \varphi \in \mathscr{D}(\Omega_1)\}$ *is dense in* $\{(f, u) \in \mathscr{D}(\Omega_2) \times \mathscr{E}'(\Omega_1) : S^t(u) = f\}$.

3. *If* $w \in \mathscr{D}'(\Omega_2)$ *satisfies* $S(w) \in \mathscr{E}(\Omega_1)$ *then* $\langle S(w), u\rangle = \langle w, S^t(u)\rangle$ *holds for each* $u \in \mathscr{E}'(\Omega_1)$ *with* $S^t(u) \in \mathscr{D}(\Omega_2)$. ■

The equivalent conditions of the lemma are satisfied by convolution operators $S = \mu *$ which can be seen by regularization: If $(f, u) \in \mathscr{D}(\Omega_2) \times \mathscr{E}'(\Omega_1)$ satisfies $S^t(u) = f$ and e_n is an approximate identity then $(S^t(e_n * u), e_n * u) = (e_n * S^t(u), e_n * u)$ converges to to (f, u) in $\mathscr{D}(\Omega_2) \times \mathscr{E}'(\Omega_1)$.

We have a necessary condition for surjectivity mod $\mathscr{E}$ which, for convolution operators, is due to Hörmander:

Proposition 3 *If S is surjective mod $\mathscr{E}$ and satisfies the conditions of lemma 2 then S is invertible, i.e.*

$$\left\{u \in \mathscr{E}'(\Omega_1) : S^t(u) \in \mathscr{D}(\Omega_2)\right\} \subseteq \mathscr{D}(\Omega_1).$$

PROOF. If T is surjective then $\mathrm{Im}(T^t) = \{(S^t(\varphi), \varphi) : \varphi \in \mathscr{D}(\Omega_1)\}$ is closed and dense in $\{(f, u) \in \mathscr{D}(\Omega_2) \times \mathscr{E}'(\Omega_1) : S^t(u) = f\}$. This gives the conclusion. ■

Of course, invertibility implies the conditions of lemma 2. Moreover, it is automatically satisfied if S is a partial differential operator with constant coefficients. On the other hand, the ordinary differential operator $S = xdx$ is surjective on $\mathscr{D}'(\mathbb{R})$ but it is not invertible since $S^t(\delta) = 0$.

Invertibility helps to overcome the first problem mentioned at the end of the introduction, i.e. to find an explicit description of the cokernel of T^t. We define

$$R: \begin{array}{ccc} \mathscr{D}(\Omega_2) \times \mathscr{E}'(\Omega_1) & \longrightarrow & \mathscr{E}'(\Omega_2) \\ (\varphi, u) & \longmapsto & \varphi - S^t(u) \end{array}$$

Then invertibility precisely means $\ker(R) = \mathrm{Im}(T^t)$. The spaces $Y_n = \mathscr{D}(K_n) \times \mathscr{E}'_n(M_n)$ (where $(K_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ are compact exhaustions Ω_2 and Ω_1 respectively, such that $S^t(\mathscr{E}'(M_n)) \subseteq \mathscr{E}'(K_n)$, and $\mathscr{E}'_n(M_n)$ denotes the space of distributions with support in M_n and order less than n) constitute a defining spectrum of $Y = \mathscr{D}(\Omega_2) \times \mathscr{E}'(\Omega_1)$ and if we endow $\mathrm{Im}(R)$ with the (LF)-space topology ind $R(Y_n)$ we obtain an algebraically exact sequence

$$0 \longrightarrow \mathscr{D}(\Omega_1) \xrightarrow{T^t} \mathscr{D}(\Omega_2) \times \mathscr{E}'(\Omega_1) \xrightarrow{R} \mathrm{Im}(R) \longrightarrow 0$$

to which we can apply the abstract results of section 2. In this way we obtain Hörmander's characterization when a convolution operator is surjective modulo smooth functions:

Theorem 2 *A convolution operator $S = \mu *$ is surjective mod $\mathscr{E}$ if and only if it is invertible and (Ω_2, Ω_1) is S-convex for singular supports, i.e. for each compact set $K \subseteq \Omega_2$ there is a compact set $M \subseteq \Omega_1$ such that each $u \in \mathscr{E}'(\Omega_1)$ with $S^t(u)|_{\Omega_2 \setminus K} \in \mathscr{E}(\Omega_2 \setminus K)$ satisfies $u|_{\Omega_1 \setminus M} \in \mathscr{E}(\Omega_1 \setminus M)$.*

PROOF. We first show necessity of the convexity condition for singular supports. Assuming that it does not hold we find a compact set $K \subseteq \Omega_2$ and a sequence $u_k \in \mathscr{E}'(\Omega_1)$ with $S^t(u_k)|_{\Omega_2 \setminus K} \in \mathscr{E}(\Omega_2 \setminus K)$ such that $u_k|_{\Omega_1 \setminus M_k} \notin \mathscr{E}(\Omega_1 \setminus M_k)$. Forming the convolution with the fundamental solution of an appropriate power of the Laplace operator (which does not change the singular support), cutting off, and multiplying with positive constants we may assume $S^t(u_k) \in \mathscr{C}^k(\Omega_2)$, $S^t(u_k) \to 0$ in $\mathscr{C}^m(\Omega_2)$ for $k \to \infty$ and a fixed $m \in \mathbb{N}$ to be determined later, and that $S^t(u_k)|_{\Omega_2 \setminus K} \to 0$.

We choose a cut-off function $\chi \in \mathscr{D}(\Omega_2)$ which takes the value 1 in a neighbourhood of K and set $\varphi_k = (\chi - 1)S^t(u_k) \in \mathscr{D}(\Omega_2)$. To apply item (3) of corollary 1 to the sequence $((\varphi_k, u_k))_{k\in\mathbb{N}}$ we have to check that for each $(w, f) \in \ker(T)$ (i.e. $w \in \mathscr{D}'(\Omega_2)$ satisfies $S(w) = f \in \mathscr{E}(\Omega_1)$) the sequence

$$c_k = \langle w, \varphi_k\rangle + \langle f, u_k\rangle$$

converges to 0. To prove this claim it is enough to justify the *formal* calculation $c_k = \langle S((\chi - 1)w), u_k\rangle + \langle S(w), u_k\rangle = \langle \chi w, S^t(u_k)\rangle$ for large $k \in \mathbb{N}$ since this expression tends to 0 as χw has compact support and thus defines a continuous linear functional on some $\mathscr{C}^m(\Omega_2)$ in which $S^t(u_k)$ is a null sequence.

Note that $S((\chi - 1)w)$ a priori only belongs $\mathscr{D}'(\Omega_1)$ and thus the expression $\langle S((\chi - 1)w), u_k\rangle$ does not make sense. However, if e_n is an approximate identity we have for large $k \in \mathbb{N}$

$$\begin{aligned} c_k &= \lim_{n\to\infty} \langle e_n * w, (\chi - 1)S^t(u_k)\rangle + \langle e_n * S(w), u_k\rangle \\ &= \lim_{n\to\infty} \langle \chi(e_n * w), S^t(u_k)\rangle = \langle \chi w, S^t(u_k)\rangle \end{aligned}$$

since $\chi(e_n * w)$ converges to χw in the *sequentially retractive* (LB)-space $\mathscr{E}'(\Omega_2)$ and thus in some of the steps and in particular in the strong dual of some $\mathscr{C}^m(\Omega_2)$.

From corollary 1 we now obtain elements $f_k \in \mathscr{D}(\Omega_1)$ such that the sequence $(\varphi_k, u_k) - T^t(f_k) = (\varphi_k - S^t(f_k), u_k - f_k)$ belongs to some step of the (LF)-space $\mathscr{D}(\Omega_2) \times \mathscr{E}'(\Omega_1)$ hence the supports of $u_k - f_k$ are contained in some fixed compact set $M \subseteq \Omega_1$ and thus $u_k|_{\Omega_1\setminus M} \in \mathscr{E}(\Omega_1 \setminus M)$ for all $k \in \mathbb{N}$, a contradiction.

To prove the sufficiency part of the theorem we will check item (9) of theorem 1 for the (LF)-space $Z = \operatorname{Im}(R) = \operatorname{ind} R(Y_n)$ as defined above, i.e. for each $n \in \mathbb{N}$ there is $m \geq n$ such that for each $k \geq m$ there is a 0-neighbourhood U in Y_n with $\overline{R(U)}^{R(Y_k)} \subseteq R(Y_{m+1})$.

From the convexity condition for singular supports we obtain

$$\forall\ n \in \mathbb{N}\ \exists\ m \geq n\ \forall\ u \in \mathscr{E}'(\Omega_1) \quad S^t(u)|_{\Omega_2\setminus K_n} \in \mathscr{C}^\infty \Longrightarrow u|_{\Omega_2\setminus M_m} \in \mathscr{C}^\infty$$

Next we use invertibility to deduce for each compact set $M \subseteq \Omega_1$ and $k \in \mathbb{N}$ that the Fréchet space $X = \{u \in \mathscr{E}'_k(M) : S^t(u) \in \mathscr{E}(\Omega_2)\}$ coincides algebraically (and by the closed graph theorem also topologically) with the Fréchet space $\mathscr{D}(M)$ and obtain that for every $r \in \mathbb{N}$ there are $U \in \mathscr{U}_0(\mathscr{E}'_k(M))$ and $s \in \mathbb{N}$ such that $U \cap (S^t)^{-1}(V_s) \subseteq W_r$ (where $(V_s)_{s\in\mathbb{N}}$ and $(W_r)_{r\in\mathbb{N}}$ are the canonical 0-neighbourhood bases of $\mathscr{E}(\Omega_2)$ and $\mathscr{D}(M)$, respectively), and by forming the convolution with an approximate identity we obtain (for $r = 0$)

$$\forall\ k \in \mathbb{N}\ \exists\ s \in \mathbb{N}\ \forall\ u \in \mathscr{E}'_k(M_k) \quad S^t(u) \in \mathscr{C}^s \Longrightarrow u \in \mathscr{C}.$$

For $k \geq m+1$ we choose $t > s = s(k)$ such that the unit ball $B_{n,t}$ of the Banach space $\mathscr{C}^t_0(K_n)$ (of t times continuously differentiable functions on Ω_2 with support in K_n) is relatively compact in $\mathscr{C}^s_0(K_n)$ and set

$$U = B_{n,t} \cap \mathscr{D}(K_n) \times \text{the unit ball of } \mathscr{E}'_n(M_n) \in \mathscr{U}_0(Y_n).$$

Estimating the closure in $R(Y_k)$ by the closure in the relative topology induced by $\mathscr{E}'(\Omega_2)$ we obtain from the fact that the Minkowski sum of two compact sets is closed

$$\overline{R(U)}^{R(Y_k)} \subseteq \overline{R(U)}^{\mathscr{E}'(\Omega_2)} \cap R(Y_k) \subseteq \big(\mathscr{C}^s_0(K_n) + S^t(\mathscr{E}'_n(M_n))\big) \cap R(Y_k).$$

Given g in the set on the left hand side we thus find $f \in \mathscr{C}^s_0(K_n)$, $u \in \mathscr{E}'_n(M_n)$, $\varphi \in \mathscr{D}(K_k)$, and $w \in \mathscr{E}'_k(M_k)$ with

$$g = f + S^t(u) = \varphi + S^t(w)$$

hence $S^t(u-w) = \varphi - f \in \mathscr{C}^s$ which implies $u - w \in \mathscr{C}$, and the singular support of $S^t(u-w)$ is contained in K_n which implies that $u - w$ is smooth outside M_m. Using once more a cut-off function $\psi \in \mathscr{D}(M_{m+1})$

with value 1 in a neighbourhood of M_m we obtain $g = \varphi + S^t((1-\psi)(w-u)) + S^t(\psi(w-u)+u)$, where $v = \psi(w-u) + u \in \mathscr{E}'_n(M_{m+1})$ and $f = \varphi + S^t((1-\psi)(w-u)) \in \mathscr{D}(\Omega_2)$ has support in K_{m+1} since the supports of g and $S^t(v)$ are there. Therefore $g = f + S^t(v) \in R(Y_{m+1})$ and this proves the closed neighbourhood condition of theorem 1. ■

Remark 1 1. The sufficiency part of the theorem used the assumption that S is a convolution operator only to deduce from invertibility the "quantitative" formulation

$$\forall\ k \in \mathbb{N}\ \exists\ s \in \mathbb{N}\ \forall\ u \in \mathscr{E}'_k(M_k) \quad S^t(u) \in \mathscr{C}^s \Longrightarrow u \in \mathscr{C}.$$

If we redefine invertibility for general linear operators by this condition we obtain that any invertible continuous linear operator $S : \mathscr{D}'(\Omega_2) \to \mathscr{D}'(\Omega_1)$ with $S(\mathscr{E}(\Omega_2)) \subseteq \mathscr{E}(\Omega_1)$ is surjective mod $\mathscr{E}$ provided that (Ω_2, Ω_1) is S-convex for singular supports.

2. If $S = \mu*$ is an invertible convolution operator one can use Hörmander's complex analytic results [7, 16.3.9 and 10] and Ehrenpreis' [2] solution of the division problem in $\mathbb{R}^d$ to get a fundamental solution of $\check{\mu}*$. Using this, one can see that already the inital topology on $X = \{u \in \mathscr{E}'(M) : S^t(u) \in \mathscr{E}(\Omega_2)\}$ with respect to $S^t : X \to \mathscr{E}(\Omega_2)$ coincides with the Fréchet space topology of $\mathscr{D}(M)$ and from this one gets the same condition as above without the a priori bound on the order of u:

$$\forall\ k \in \mathbb{N}\ \exists\ s \in \mathbb{N}\ \forall\ u \in \mathscr{E}'(M_k) \quad S^t(u) \in \mathscr{C}^s \Longrightarrow u \in \mathscr{C}. \quad ■$$

4. The Cauchy problem for regular right hand side

We now briefly explain how to proceed with step (A) of the program mentioned in the introduction. We will give an extension and quite simple proofs (at least for some parts) of Hörmander's characterization for the surjectivity of $S : \mathscr{E}(\Omega_2) \to \mathscr{E}(\Omega_1)$. Hörmander proved that this holds if the equation $S(u) = f$ can be solved with $u \in \mathscr{D}'(\Omega_2)$ for each $f \in \mathscr{E}(\Omega_1)$. Here we show that this also equivalent to the solvability of the equation in $\mathscr{D}'(\Omega_2)$ for each $f \in \mathscr{D}(\Omega_1) \cup \mathscr{H}(\Omega_1^*)$ where Ω_1^* is an appropriate complex neighbourhood of Ω_1 (this improves a result of Langenbruch [8] who required solvability of those equations in $\mathscr{E}(\Omega_2)$).

For an open set $\Omega_1 \subseteq \mathbb{R}^d$ we call an open set $\Omega^* \subseteq \mathbb{C}^d$ with $\Omega \subseteq \Omega^*$ *admissible* if $\mathscr{H}(\mathbb{C}^d)$ is dense in $\mathscr{H}(\Omega^*)$ and for every compact set $C \subseteq \Omega^*$ there is a compact set $M \subseteq \Omega$ such that each $u \in \mathscr{E}'(\Omega)$ which is carried by C (see [7, section 9.1]) has its support in M. The following characterization is proved by standard duality:

Proposition 4 *$\Omega^* \subseteq \mathbb{C}^d$ is admissible for $\Omega \subseteq \Omega^* \cap \mathbb{R}^d$ if and only if for each $U \in \mathscr{U}_0(\mathscr{H}(\Omega^*))$ there is a compact set $M \subseteq \Omega$ with $\mathscr{D}(\Omega \setminus M) \subseteq \overline{r(U)}^{\mathscr{E}(\Omega)}$ where $r : \mathscr{H}(\Omega^*) \to \mathscr{E}(\Omega)$ denotes the restriction map.* ■

The main example below for admissible sets is essentially due to Langenbruch [8]. It can be proved by forming the convolution with the Gauß-kernel.

Example 1 *For any open set $\Omega \subseteq \mathbb{R}^d$ the set*

$$\Omega^* = \{x + iy : x \in \Omega \text{ and } |y| < \text{dist}(x, \partial\Omega)\}$$

is admissible for Ω. ■

Theorem 3 *For a convolution operator $S = \mu* : \mathscr{D}'(\Omega_2) \to \mathscr{D}'(\Omega_1)$ the following conditions are equivalent:*

1. *$S : \mathscr{E}(\Omega_2) \to \mathscr{E}(\Omega_1)$ is surjective.*
2. *$\mathscr{E}(\Omega_1) \subseteq S(\mathscr{D}'(\Omega_2))$.*
3. *There is an open set Ω_1^* which is admissible for Ω_1 such that*
$$r(\mathscr{H}(\Omega_1^*)) \cup \mathscr{D}(\Omega_1) \subseteq S(\mathscr{D}'(\Omega_2)).$$
4. *S is invertible and the pair (Ω_2, Ω_1) is S-convex for supports, i.e. for each compact set $K \subseteq \Omega_2$ there is a compact set $M \subseteq \Omega_1$ such that each $\psi \in \mathscr{D}(\Omega_1)$ with $\operatorname{supp} S^t(u) \subseteq K$ satisfies $\operatorname{supp} u \subseteq M$.*

PROOF. Trivially, (3) follows from (2) and (2) from (1), and we have nothing new for the proof that $\mathscr{D}(\Omega_1) \subseteq S(\mathscr{D}'(\Omega_2))$ implies invertibility, see [7, theorem 16.5.1], but we would like to mention Floret's result that invertibility is equivalent to the fact that S^t has stepwise closed range. We will use the Grothendieck-Floret factorization theorem to prove that (Ω_2, Ω_1) is S-convex whenever $r(\mathscr{H}(\Omega_1^*))$ is contained in $S(\mathscr{D}'(\Omega_2))$. For a compact set $K \subseteq \Omega_2$ we endow the space $X = \{\psi \in \mathscr{D}(\Omega_1) : \operatorname{supp} S^t(\psi) \subseteq K\}$ with the (metrizable) initial topology with respect to the (injective) map $S^t : X \to \mathscr{D}(K)$ and consider the map
$$T : X \to \mathscr{H}(\Omega_1^*)'_\beta, \ \psi \mapsto (f \mapsto \int_{\Omega_1} \psi f \, d\lambda).$$
This map is weakly continuous since the restriction to Ω_1 of each element $f \in \mathscr{H}(\Omega_1^*)'' = \mathscr{H}(\Omega_1^*)$ is of the form $f = S(w)$ for some $w \in \mathscr{D}'(\Omega_2)$. Since $\mathscr{H}(\Omega_1^*)'_\beta$ is a regular inductive limit of the Banach spaces $[U^\circ]$, $U \in \mathscr{U}_0(\mathscr{H}(\Omega_1^*))$, we find a 0-neighbourhood U in $\mathscr{H}(\Omega_1^*)$ such that T maps X into $[U^\circ]$. From proposition 4 we get a compact set $M \subseteq \Omega_1$ with $(*)$ $\mathscr{D}(\Omega_1 \setminus M) \subseteq \overline{r(U)}^{\mathscr{E}(\Omega_1)}$.

For $\psi \in \mathscr{D}(\Omega_1)$ with $\operatorname{supp} S^t(\psi) \subseteq K$ and $\varepsilon > 0$ we decompose $\psi = \psi_1 + \psi_2$ with test functions ψ_k such that
$$\operatorname{supp} \psi_1 \subseteq \Omega_1 \setminus M, \ \operatorname{supp} \psi_2 \subseteq \{x \in \Omega_1 : d(x, M) < \varepsilon\}, \text{ and } \int_{\Omega_1} |\psi_1 \psi_2| \, d\lambda < \varepsilon$$
(which can be easily done by $\psi_1 = \chi\psi$ for an appropriate cut-off function χ). We will show $\int_{\Omega_1} |\psi_1|^2 \, d\lambda < \varepsilon$ (since $\varepsilon > 0$ is arbitrary we conclude from this $\int_{\Omega_1 \setminus M} |\psi|^2 \, d\lambda = 0$, hence $\operatorname{supp} \psi \subseteq M$). Using $(*)$ above (and the fact that the left hand side is a linear space) we find $f_n \in \frac{1}{n} U$ with $f_n \to \overline{\psi}_1$ uniformly on the compact set $\operatorname{supp} \psi$. We then get
$$\int_{\Omega_1} |\psi_1|^2 \, d\lambda = \lim_{n \to \infty} \int_{\Omega_1} \psi f_n \, d\lambda - \int_{\Omega_1} \psi_2 \overline{\psi}_1 \, d\lambda \leq \lim_{n \to \infty} \langle T(\psi), f_n \rangle + \varepsilon = \varepsilon.$$

Finally, we use the surjectivity criterion proposition 2 to prove that (4) implies (1). Since $\mu \neq 0$ the convolutions operator S^t is injective on $\mathscr{E}'(\Omega_1)$, hence $S : \mathscr{E}(\Omega_2) \to \mathscr{E}(\Omega_1)$ has dense range. We have to show that for each equicontinuous set U° in $\mathscr{E}'(\Omega_2)$ there is an equicontinuous set V° in $\mathscr{E}'(\Omega_1)$ with with $(S^t)^{-1}(U^\circ) \subseteq [V^\circ]$, i.e. for each compact set $K \subseteq \Omega_2$ and $k \in \mathbb{N}$ there are a compact set $M \subseteq \Omega_1$ and $m \in \mathbb{N}$ with $(S^t)^{-1}(\mathscr{E}'_k(K)) \subseteq \mathscr{E}'_m(M)$.

It follows by regularization that the definition of invertibility does not change if $\mathscr{D}(\Omega_1)$ is replaced by $\mathscr{E}'(\Omega_1)$, we thus get for a compact set $K \subseteq \Omega_2$ a compact set $M \subseteq \Omega_1$ with $(S^t)^{-1}(\mathscr{E}'(K)) \subseteq \mathscr{E}'(M)$. To get control on the orders we use invertibility in the formulation
$$\forall \ M \subseteq \Omega_1 \text{ compact } \ \exists \ s \in \mathbb{N} \ \forall \ u \in \mathscr{E}'(M) \quad S^t(u) \in \mathscr{C}^s \Longrightarrow u \in \mathscr{C}.$$
obtained in the second part of remark 1.

Given $k \in \mathbb{N}$ and $u \in \mathscr{E}'(M)$ with $S^t(u) \in \mathscr{E}'_k(K)$ we take the fundamental solution E_p of an appropriate power p (which only depends on k) of the Laplace operator such that $E_p * S^t(u) \in \mathscr{C}^s$. If $\psi \in \mathscr{D}(\Omega_1)$ takes the value 1 in a neighbourhood of M we obtain
$$S^t(\psi(E_p * u)) = S^t((\psi - 1)(E_p * u)) + E_p * S^t(u) \in \mathscr{C}^s$$

since the first summand is even $\mathscr{C}^\infty$. By invertibility, $\psi(E_p * u) \in \mathscr{C}$ hence $(E_p * u)|_M \in \mathscr{C}$ and therefore $u \in \mathscr{E}'_m(M)$ for some m depending only on p. ■

Remark 2 In condition (3) of the preceding theorem we can replace $\mathscr{H}(\Omega_1^*)$ by any Fréchet space Z such that there is a "restriction operator" $r : Z \to \mathscr{E}(\Omega_1)$ which satisfies the condition of proposition 4, i.e. for each $U \in \mathscr{U}_0(Z)$ there is a compact set $M \subseteq \Omega_1$ with $\mathscr{D}(\Omega_1 \setminus M) \subseteq \overline{r(U)}^{\mathscr{E}(\Omega_1)}$. ■

Remark 3 We have chosen the formulation of the results for convolution operators in such a way that one can read off from the proofs where translation invariance is really needed for regularizations and which parts carry over to general linear operators.

Our approach is not restricted to $\mathscr{D}'(\Omega)$. The techniques developed here allow applications e.g. to spaces $\mathscr{D}'_{(w)}(\Omega)$ of ultradistributions of Beurling type. We will investigate surjectivity questions for convolution operators on such spaces elsewhere. ■

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