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(Ultra)distributions of $\mathrm{L_p}$-growth as Boundary Values of Holomorphic Functions

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To the memory of I. Cioranescu and K. Floret

Abstract. We study the representation of distributions (and ultradistributions of Beurling type) of L_p-growth, $1 \leq p \leq \infty$, on $\mathbb{R}^N$ as boundary values of holomorphic functions on $(\mathbb{C} \setminus \mathbb{R})^N$.

(Ultra)distribuciones de crecimiento $\mathrm{L_p}$ como valor frontera de funciones holomorfas

Resumen. Estudiamos la representación de distribuciones (y ultradistribuciones de tipo Beurling) en $\mathbb{R}^N$ con crecimiento L_p, $1 \leq p \leq \infty$, como valor frontera de funciones holomorfas en $(\mathbb{C} \setminus \mathbb{R})^N$.

1. Introduction

Shortly after Schwartz introduced his theory of distributions, Köthe represented distributions on the unit circle as boundary values of holomorphic functions on its complementary and his results were generalized by Tillmann. Since then, many authors have been concerned with the problem of representing several classes of distributions and ultradistributions as boundary values of holomorphic functions. Let us mention the work of Bengel [1], Carmichael [4], Luszczki and Zielezny [13], Meise [14], Petzsche and Vogt [17], Tillmann [20] and Vogt [22]. See also the section 4 of the recent paper [12].

In 1994, Carmichael and Pilipović [6] represented each ultradistribution of L_p-growth $(1 < p < \infty)$ in $\mathbb{R}^N$ as the boundary value of a holomorphic function satisfying appropriate estimates and conversely, every such a function is shown to have an ultradistribution of L_p-growth as boundary value. The boundary value problem for distributions and ultradistributions of L_∞ or L_1- growth is more involved. In fact, for $p = 1$ or $p = \infty$, Carmichael and Pilipović only obtained partial results and their methods did not permit to prove the surjectivity of the corresponding boundary value operators. They worked in the context of ultradistributions as they were defined by Komatsu [11].

In [8] the authors completely solved the problem of representing bounded distributions and ultradistributions on $\mathbb{R}$ as boundary values of holomorphic functions in $\mathbb{C} \setminus \mathbb{R}$. The case of bounded distributions had not been treated previously in the literature. The lack of nice topological properties of the involved spaces does not permit us to apply the tensor techniques as in [16, 3.6] in order to extend the results obtained in [8] to the several variables setting.

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The results of Petzsche and Vogt [17] and the characterizations of ultradistributions of L_p-growth recently obtained in [2], which are in the spirit of the ones given by Cioranescu [7] and Gómez-Collado [9], allow us to define a class of holomorphic functions on $(\mathbb{C}\setminus\mathbb{R})^N$ having ultradistributions (of Beurling type) of L_p-growth as boundary values and such that the boundary value operator is surjective. We work with ultradistributions in the sense of Braun, Meise and Taylor [3]. Our approach is different from the one of Carmichael and Pilipović and permits a unified treatment of all values of p, including the limit cases $p=1$ and $p=\infty$. Of course the most interesting cases are the extreme values $p=1,\infty$. Our results also cover the classical spaces $(\mathcal{D}_{L_1}(\mathbb{R}^N))'$ and $(\mathcal{D}_{L_\infty}(\mathbb{R}^N))'$. We would like to emphasize that no satisfactory answer to the boundary value problem in these cases was previously known.

2. Preliminaries and statement of the problem

First we introduce the spaces of functions and ultradistributions and most of the notation that will be used in the sequel.

Definition 1 ([3]) *A weight function is an increasing continuous function $\omega : [0,\infty[\longrightarrow [0,\infty[$ with the following properties:*

(α) *there exists $L\geq 1$ with $\omega(2t)\leq L(\omega(t)+1)$ for all $t\geq 0$,*

(β) $\displaystyle\int_1^\infty \frac{\omega(t)}{t^2}\,dt<\infty$,

(γ) $\log(t)=o(\omega(t))$ as t tends to ∞,

(δ) $\varphi : t\to\omega(e^t)$ is convex.

For most of the results of the paper we have to replace the condition (α) by the stronger condition

$$(\alpha_1)\ \sup_{\lambda\geq 1}\limsup_{t\to\infty}\frac{\omega(\lambda t)}{\lambda\omega(t)}<\infty.$$

For a weight function ω we define $\widetilde{\omega} : \mathbb{C}^N\longrightarrow [0,\infty[$ by $\widetilde{\omega}(z)=\omega(|z|)$ and again call this function ω, by abuse of notation. Here $|z|=\sum_{j=1}^N |z_j|$.

The function $\varphi^* : [0,\infty[\to\mathbb{R}$ defined by

$$\varphi^*(s) := \sup_{t\geq 0}\{st-\varphi(t)\}$$

is called the *Young conjugate* of φ.

There is no loss of generality to assume that ω vanishes on $[0,1]$. Then φ^* has only non-negative values and $\varphi^{**}=\varphi$. Examples of weight functions can be found in [3].

Remark 1 If the weight $\omega(t)$ is concave for t large enough then every equivalent weight satisfies (α_1). See [17, 1.1] for details. ■

Definition 2 ([3]) *Let ω be a weight function. For a compact set $K\subset\mathbb{R}^N$ we let*

$$\mathcal{D}_{(\omega)}(K) := \{f\in\mathcal{D}(K) : \| f\|_{K,\lambda}<\infty \quad for\ every\ \lambda>0\},$$

where

$$\| f\|_{K,\lambda} := \sup_{x\in K}\sup_{\alpha\in\mathbb{N}_0^N}|f^{(\alpha)}(x)|\exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right).$$

Then $\mathcal{D}_{(\omega)}(K)$, endowed with its natural topology, is a Fréchet space. For a fundamental sequence $(\mathrm{K}_j)_{j\in\mathbb{N}}$ of compact subsets of $\mathbb{R}^N$ we let

$$\mathcal{D}_{(\omega)}(\mathbb{R}^N) := \underset{j\to}{\operatorname{ind}}\, \mathcal{D}_{(\omega)}(\mathrm{K_j}).$$

The dual $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ of $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ is endowed with its strong topology. The elements of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ are called *ultradistributions of Beurling type*.

We denote by $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ the set of all functions $f \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $\| f \|_{K,\lambda} < \infty$ for every compact K in $\mathbb{R}^N$ and for every $\lambda > 0$.

Definition 3 ([2]) *For every $1 \le p \le \infty$, $k \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{R}^N)$, $\gamma_{k,p}(\phi)$ is defined as follows*

$$\gamma_{k,p}(\phi) = \sup_{\alpha\in\mathbb{N}_0^N} \|\phi^{(\alpha)}\|_p \exp\left(-k\varphi^*\left(\frac{|\alpha|}{k}\right)\right),$$

where $\|.\|_p$ denotes the usual norm in $L_p(\mathbb{R}^N)$.

If $1 \le p < \infty$ the space $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ is the set of all C^∞-functions ϕ on $\mathbb{R}^N$ such that $\gamma_{k,p}(\phi) < \infty$ for each $k \in \mathbb{N}$. A function $\phi \in C^\infty(\mathbb{R}^N)$ is in $\mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ when $\gamma_{k,\infty}(\phi) < \infty$, for every $k \in \mathbb{N}$. We denote by $\mathcal{D}_{L_\infty,(\omega)}(\mathbb{R}^N)$ the subspace of $\mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ consisting of those functions $\phi \in \mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ for which $\lim_{|x|\to\infty} |\phi^{(\alpha)}(x)| = 0$ for each $\alpha \in \mathbb{N}_0^N$.

The topology of $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \le p \le \infty$, is generated by the family of seminorms $\{\gamma_{k,p}\}_{k\in\mathbb{N}}$. Also, we consider on $\mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ the topology associated with $\{\gamma_{k,\infty}\}_{k\in\mathbb{N}}$. Then $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \le p \le \infty$ and $\mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ are Fréchet spaces.

For the definition of the spaces $\mathcal{D}_{L_p}(\mathbb{R}^N)$, $1 \le p \le \infty$, we refer to [19, VI,8].

Remark 2 (a) Clearly $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ is continuously contained in $\mathcal{D}_{L_p}(\mathbb{R}^N)$, $1 \le p \le \infty$. Hence, if $\phi \in \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \le p \le \infty$, then $\lim_{|x|\to\infty} |\phi^{(\alpha)}(x)| = 0$, for each $\alpha \in \mathbb{N}_0^N$.

(b) The inclusions $\mathcal{D}_{(\omega)}(\mathbb{R}^N) \subset \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N) \subset \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ are continuous and dense, and for $1 \le p \le q \le \infty$, $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ is continuously contained in $\mathcal{D}_{L_q,(\omega)}(\mathbb{R}^N)$.

(c) Although the paper [2] was written in the one variable setting, all the results in its section 2 also hold for $N > 1$ with the same proofs. ■

The dual of $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ will be denoted by $(\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N))'$ and it will be endowed with the strong topology. Since $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ is continuously and densely embedded in $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ then $(\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N))'$ can be identified with a subspace of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$. The elements of $(\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N))'$ are known as *ultradistributions of Beurling type of $L_{p'}$-growth* where p' is the conjugate exponent of p. The ultradistributions of L_∞-growth are called *bounded ultradistributions of Beurling type*.

The classical case $\mathcal{D}_{L_p}(\mathbb{R}^N)$ is formally not a particular case of what we present here since $\omega(t) = \log(1+t)$ does not satisfy property (γ). However, all our results also hold in this case after minor modifications.

Let $G \in \mathcal{H}(\mathbb{C}^N)$ be an entire function such that $\log|G(z)| = \mathcal{O}(\omega(|z|))$ as $|z|$ tends to infinity. Then

$$T_G(\varphi) := \sum_{\alpha\in\mathbb{N}_0^N} (-i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} \varphi^{(\alpha)}(0)$$

defines an ultradistribution $T_G \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$. The operator

$$G(D) : \mathcal{D}'_{(\omega)}(\mathbb{R}^N) \to \mathcal{D}'_{(\omega)}(\mathbb{R}^N), \qquad G(D)\nu := T_G * \nu$$

is called an *ultradifferential operator* of (ω)-class. We note that, for every $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^N)$,

$$(G(D)f)(x) = \sum_{\alpha \in \mathbb{N}_0^N} (i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} f^{(\alpha)}(x).$$

It can be easily shown that there are constants $C > 0$ and $k \in \mathbb{N}$ such that $|\frac{G^{(\alpha)}(0)}{\alpha!}| \leq C e^{-k\varphi^*(\frac{|\alpha|}{k})}$. As in [9, Proposition 2.4] it can be shown that each ultradifferential operator $G(D)$ of (ω)-class defines a continuous linear mapping from $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ into itself, for every $1 \leq p \leq \infty$ and also from $\mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ into itself. Thus, $G(D)$ is also a continuous linear operator from $(\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N))'$ into itself.

The **problem** we are concerned with consists in finding a weighted space $\mathcal{H}^N_{\omega^*,p}$ of holomorphic functions on $(\mathbb{C} \setminus \mathbb{R})^N$ such that the map $T : \mathcal{H}^N_{\omega^*,p} \to (\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}^N))'$ given by

$$< T(f), \varphi > := \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} \{ \sum_{\sigma \in \{-1,1\}^N} (\prod_{j=1}^N \sigma_j) f(x + i\sigma\epsilon) \} \, \varphi(x) \, dx$$

is a well-defined, linear, continuous and surjective operator. The description of the appropriate space of holomorphic functions and its basic properties is the aim of the section 3, while the boundary value operator is investigated in section 4.

3. The spaces $\mathcal{H}^{\mathbf{N}}_{\omega^*,\mathbf{p}}$ and $\mathcal{H}^{\mathbf{N}}_{\mathbf{p}}$

In [8] we defined the weighted (LB)-spaces of holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$, $\mathcal{H}_{\omega^*}$, and we showed that $T : \mathcal{H}_{\omega^*} \to (\mathcal{D}_{L_1,(\omega)}(\mathbb{R}))'$ given by

$$< T(f), \varphi > := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} (f(x + i\varepsilon) - f(x - i\varepsilon))\varphi(x) dx$$

is well-defined, continuous, linear and surjective. The natural extension of $\mathcal{H}_{\omega^*}$ to the several variables case and for arbitrary $1 \leq p \leq \infty$ leads to the following definition. In what follows, ω will be either a weight function or $\omega(t) = \log(1 + t)$, $t \geq 0$.

Definition 4 *For $s > 0$, $\omega^*(s)$ is defined by*

$$\omega^*(s) := \sup_{t \geq 0} \{\omega(t) - st\}.$$

The function ω^* is continuous, convex and decreasing. Since each weight function ω satisfies that $\omega(t) = o(t)$ where t tends to ∞, $\omega^*(s) < \infty$ for all $s > 0$.

Given $y \in \mathbb{R}^N$ with $y_j \neq 0$ for all $1 \leq j \leq N$, we denote $\omega^*(y) := \sum_{j=1}^N \omega^*(|y_j|)$.

Definition 5 *For a given ω, $1 \leq p \leq \infty$ and $N \in \mathbb{N}$, we define*

$$\mathcal{H}^N_{\omega^*,p} := \{ f \in \mathcal{H}((\mathbb{C} \setminus \mathbb{R})^N) : \quad |f|_{\omega,p,k} < \infty \ \textit{for some } k \in \mathbb{N} \}$$

where

$$|f|_{\omega,p,k} := \sup_y \| \, f(\cdot + iy) \, \|_p \ \exp\left(-k|y| - k\omega^*(\frac{y}{k}) \right).$$

Then $\mathcal{H}^N_{\omega^*,p}$ is an (LB)-space.

Remark 3 For $\omega(t) = \log(1+t)$, $t \geq 0$, we have that $\omega^*(s) = 0$ for $s \geq 1$ while $\omega^*(s) = s - \log(\mathrm{e}s)$ whenever $0 < s < 1$. Therefore, it is easy to see that in this case $\mathcal{H}_p^N := \mathcal{H}_{\omega^*,p}^N$ can be described as

$$\mathcal{H}_p^N = \{f \in \mathcal{H}((\mathbb{C}\setminus\mathbb{R})^N) : |f|_{p,k} < \infty \text{ for some } k \in \mathbb{N}\}$$

where

$$|f|_{p,k} := \sup_y \| f(\cdot + iy) \|_p \, \mathrm{e}^{-2k|y|} \left(\prod_{j=1}^N |y_j|\right)^k . \quad \blacksquare$$

For any weight ω, $\mathcal{H}_p^N \subset \mathcal{H}_{\omega^*,p}^N$ and given two weights $\sigma \leq \omega$ we have $\mathcal{H}_{\sigma^*,p}^N \subset \mathcal{H}_{\omega^*,p}^N$ with continuous inclusions.

Since we can represent (ultra)distributions of L_p-growth as a (infinite) linear combination of derivatives of functions in L_p, we will show that the spaces just defined are stable under (ultra)differential operators.

Lemma 1 *(1) For each $f \in \mathcal{H}_p^N$ and each $\alpha \in \mathbb{N}_0^N$ we have $f^{(\alpha)} \in \mathcal{H}_p^N$.*

(2) For each $f \in \mathcal{H}_{\omega^*,p}^N$ and each ultradifferential operator $G(D)$ of class (ω) we have $G(D)f \in \mathcal{H}_{\omega^*,p}^N$.

PROOF. Let $f \in \mathcal{H}_{\omega^*,p}(\mathbb{R}^N)$ be given. We fix $y \in \mathbb{R}^N$ such that $y_j \neq 0$ for every $1 \leq j \leq N$ and we put $\rho_j := \frac{1}{2}|y_j|$. Let D_ρ be the polidisc of poliradius $\rho := (\rho_1, \ldots, \rho_N)$. For each $x \in \mathbb{R}^N$ and each $\alpha \in \mathbb{N}_0^N$, by the Cauchy integral formula

$$f^{(\alpha)}(x+iy) = \frac{\alpha!}{(2\pi i)^N} \int_{D_\rho} \frac{f(x+iy+\xi)}{\xi^{\alpha+1}} d\xi.$$

The function $g_\xi(x) := \dfrac{f(x+iy+\xi)}{\xi^{\alpha+1}}$ belongs to $L_p(\mathbb{R}^N)$ and

$$f^{(\alpha)}(\cdot + iy) = \frac{\alpha!}{(2\pi i)^N} \int_{D_\rho} g_\xi(\cdot)\, d\xi.$$

Therefore

$$\| f^{(\alpha)}(\cdot + iy) \|_p \leq \frac{\alpha!}{(2\pi)^N} \max_{\xi \in D_\rho} \| g_\xi \|_p \prod_{j=1}^N (2\pi\rho_j).$$

Since $\| g_\xi \|_p \leq \dfrac{1}{\rho^{\alpha+1}} |f|_{\omega,p,k}\, \mathrm{e}^{2k|y| + k\omega^*(\frac{y}{2k})}$ for some constant $k \in \mathbb{N}$,

$$\| f^{(\alpha)}(\cdot + iy) \|_p \leq |f|_{\omega,p,k}\, \alpha!\, \mathrm{e}^{2k|y|+k\omega^*(\frac{y}{2k})} \prod_{j=1}^N \left(\frac{2}{|y_j|}\right)^{\alpha_j}.$$

If $\omega(t) = \log(1+t)$ ($t \geq 0$) this gives (1). To show (2), let $G(D)$ be an ultradifferential operator of class (ω). Then $G(D)g = \displaystyle\sum_{\alpha \in \mathbb{N}_0^N} a_\alpha g^\alpha$ where $|a_\alpha| \leq C\mathrm{e}^{-m\varphi^*(\frac{|\alpha|}{m})} \leq C\mathrm{e}^{-m\sum_{j=1}^N \varphi^*(\frac{\alpha_j}{m})}$ for some $C > 0$ and some $m \in \mathbb{N}$. Thus

$$\begin{aligned} \| G(D)f(\cdot+iy) \|_p &\leq \sum_{\alpha\in\mathbb{N}_0^N} |a_\alpha| \, \| f^{(\alpha)}(\cdot+iy) \|_p \\ &\leq C|f|_{\omega,p,k} \sum_{\alpha\in\mathbb{N}_0^N} \alpha! \prod_{j=1}^N \left(\frac{2}{|y_j|}\right)^{\alpha_j - m\varphi^*(\frac{\alpha_j}{m})} \mathrm{e}^{2k|y|+k\omega^*(\frac{y}{2k})}. \end{aligned}$$

That is,

$$\| G(D)f(\cdot+iy) \|_p \leq C\, |f|_{\omega,p,k} \prod_{j=1}^{N} \left(\sum_{\alpha_j=0}^{\infty} \alpha_j!\, \mathrm{e}^{-m\varphi^*(\frac{\alpha_j}{m})} \left(\frac{2}{|y_j|}\right)^{\alpha_j} \right) \mathrm{e}^{2k|y_j|+k\omega^*\left(\frac{|y_j|}{2k}\right)}.$$

To finish it is enough to proceed as in [8, Prop.1]. ■

It is clear now that, given $f \in \mathcal{H}_p^N$ and $y \in \mathbb{R}^N$, $y_j \neq 0$ for $1 \leq j \leq N$, the function $f(\cdot + iy)$ belongs to $\mathcal{D}_{L_p}(\mathbb{R}^N)$ for $1 \leq p < \infty$ and $f(\cdot + iy) \in \mathcal{B}_{L_\infty}(\mathbb{R}^N)$ for $p = \infty$. A similar result holds for arbitrary weight functions ω.

Corollary 1 *(1) For every $1 \leq p < \infty$ and each $f \in \mathcal{H}^N_{\omega^*,p}$ we have $f(\cdot + iy) \in \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$.*

(2) Each $f \in \mathcal{H}^N_{\omega^*,\infty}$ satisfies $f(\cdot + iy) \in \mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$.

PROOF. Let $1 \leq p \leq \infty$ and $f \in \mathcal{H}^N_{\omega^*,p}$ be given. Then $G(D)f(. + iy) \in L_p(\mathbb{R}^N)$ for every ultradifferential operator $G(D)$ of $(\omega)-$class. Now it is enough to apply [2, Corollary 2.2] to conclude. ■

Now, it is easy to show that the spaces $\mathcal{H}^N_{\omega^*,p}$ increase with p.

Corollary 2 *For a fixed function ω and $p < q$ we have $\mathcal{H}^N_{\omega^*,p} \subset \mathcal{H}^N_{\omega^*,q}$ with continuous inclusion.*

PROOF. Since $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N) \subset \mathcal{D}_{L_q,(\omega)}(\mathbb{R}^N) \subset \mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ with continuous inclusions ([2]), we deduce from Corollary 1 that for every $f \in \mathcal{H}^N_{\omega^*,p}$ there is a continuous seminorm γ on $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ such that $||f(. + iy)||_q \leq \gamma(f(. + iy))$ for every $y \in \mathbb{R}^N$ with $y_j \neq 0$ for $1 \leq j \leq N$. Hence there is an ultradifferential operator $G(D)$ of $(\omega)-$class and there is a positive constant C satisfying

$$||f(. + iy)||_q \leq C||G(D)f(. + iy)||_p$$

for all $y \in \mathbb{R}^N$ with $y_j \neq 0$ for all $1 \leq j \leq N$ ([2, 2.0.4]). Since $G(D)f \in \mathcal{H}^N_{\omega^*,p}$ then it easily follows that $f \in \mathcal{H}^N_{\omega^*,q}$. ■

4. Boundary values

In this section we will show that each function in $\mathcal{H}^N_{\omega^*,p}$ has an element of $(\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}^N))'$, $\frac{1}{p} + \frac{1}{p'} = 1$, as boundary value and that, conversely, each (ultra)distribution of L_p-growth can be obtained as the boundary value of a suitable f in $\mathcal{H}^N_{\omega^*,p}$. From now on ω will be either a weight function satisfying (α_1) or $\omega(t) = \log(1 + t)$, $t \geq 0$.

We first observe that each $f \in \mathcal{H}^N_{\omega^*,p}$ belongs to $\mathcal{H}^N_{\omega^*,\infty}$ and, after applying [22] for $\omega(t) = \log(1 + t)$ and [17] for ω a weight function satisfying (α_1), we have the following result

Lemma 2 *The boundary value operator $T : \mathcal{H}^N_{\omega^*,p} \to \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ given by*

$$< T(f), \varphi >:= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} \{ \sum_{\sigma \in \{-1,1\}^N} (\prod_{j=1}^{N} \sigma_j) f(x + i\sigma\epsilon) \}\, \varphi(x)\, dx$$

is a well-defined, continuous and linear mapping. Moreover $T(f) \in (\mathcal{D}_{L_1,(\omega)}(\mathbb{R}^N))'$ for each $f \in \mathcal{H}^N_{\omega^*,\infty}$ and T is continuous as a map from $\mathcal{H}^N_{\omega^*,\infty}$ into $(\mathcal{D}_{L_1,(\omega)}(\mathbb{R}^N))'$. Therefore T is also continuous from $\mathcal{H}^N_{\omega^*,p}$ into $(\mathcal{D}_{L_1,(\omega)}(\mathbb{R}^N))'$.

PROOF. In fact, T is nothing else but the restriction of the boundary value operator considered in [17] and [22]. Now it is enough to proceed as in the first part of the proof of Theorem 3 in [8]. ■

Next, we show that the boundary value of a function in $\mathcal{H}^N_{\omega^*,p}$ is an (ultra)distribution of L_p-growth.

Proposition 1 *$T(\mathcal{H}^N_{\omega^*,p})$ is contained in $(\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}^N))'$ with $\frac{1}{p}+\frac{1}{p'}=1$. Moreover,*

$$T(f) = \lim_{\epsilon\to 0^+} \sum_{\sigma\in\{-1,1\}^N} (\prod_{j=1}^N \sigma_j) f(x+i\sigma\epsilon)$$

in the weak topology $\sigma((\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}^N))', \mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}^N))$.

PROOF. First we assume that $N=1$ and that ω is a weight function. Given $f\in\mathcal{H}^1_{\omega^*,p}$ we choose $k\in\mathbb{N}$ and $C>0$ such that

$$\max(\| f(\cdot+iy)\|_\infty, \| f(\cdot+iy)\|_p) \le C\mathrm{e}^{k\omega^*(\frac{y}{k})} \qquad (1)$$

for $0<y<2$. Without loss of generality we may assume that $f\equiv 0$ in the lower half-plane. We will show that $\{f(\cdot+i\epsilon) : 0<\epsilon<1\}$ is a bounded set in $(\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}))'$ and that $T(f)*\varphi\in L_p(\mathbb{R})$ for every $\varphi\in\mathcal{D}_{(\omega)}(\mathbb{R})$. We put $f_{i\epsilon}(x) := f(x+i\epsilon)$. Let $\varphi\in\mathcal{D}_{(\omega)}(\mathbb{R})$ be given and let $b>0$ be such that $\mathrm{supp}\,\varphi\subset]-b,b[$. By [17, 3.4] we find $\phi\in\mathcal{D}((-b,b)\times(-\frac{1}{2},\frac{1}{2}))$ such that
(i) $\phi_{|\mathbb{R}}=\varphi$
(ii) $\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \left|\frac{\partial}{\partial\bar z}\phi(x+iy)\,\mathrm{e}^{k\omega^*(\frac{|y|}{k})}\right| < \infty.$

Applying Stokes' theorem to the function $\theta_x(\xi) := f(\xi+i\epsilon)\phi(x-\xi)$ in the rectangle $D_x := [x-2b, x+2b]\times[0,1]$ we get that

$$(f_{i\epsilon}*\varphi)(x) = 2i\int_D f(x-t+i(v+\epsilon))\frac{\partial}{\partial\bar z}\,\phi(t-iv)\,d(t,v).$$

where $D := [-2b,2b]\times[0,1]$. Therefore

$$\| f_{i\epsilon}*\varphi\|_p \le 2\int_D \| f(\cdot+i(v+\epsilon))\|_p \left|\frac{\partial}{\partial\bar z}\phi(t-iv)\right| d(t,v),$$

from where we conclude that $\{f_{i\epsilon}*\varphi : 0<\epsilon<1\}$ is a bounded set in $L_p(\mathbb{R})$, which shows that $\{f_{i\epsilon} : 0<\epsilon<1\}$ is bounded in $(\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}))'$ ([2]), hence equicontinuos. Moreover, for every null sequence of positive numbers $(\epsilon_n)_n$ one has $(T(f)*\varphi)(x) = \lim_n (f_{i\epsilon_n}*\varphi)(x)$ pointwise and there is $C>0$ with $|(f_{i\epsilon_n}*\varphi)(x)|\le C$ for every $n\in\mathbb{N}$ and each $x\in\mathbb{R}$. Using Lebesgue's dominated convergence theorem we get that $\{(T(f)*\varphi)\chi_{[-n,n]} : n\in\mathbb{N}\}$ is bounded in $L_p(\mathbb{R})$, hence $T(f)*\varphi\in L_p(\mathbb{R})$ and $T(f)\in(\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}))'$. Let us take a 0-neighbourhood V in $\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R})$ such that $T(f)\in V^o$ and $f_{i\epsilon}\in V^o$ for $0<\epsilon<1$ and let τ denote the topology of pointwise convergence on the dense subspace $\mathcal{D}_{(\omega)}(\mathbb{R})$ of $\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R})$. Then the weak topology and τ coincide on the equicontinuous set V^o. Since $< T(f),\varphi >= \lim_{\epsilon\to 0}\int_{\mathbb{R}} f_{i\epsilon}(x)\varphi(x)dx$ for every $\varphi\in\mathcal{D}_{(\omega)}(\mathbb{R})$ we get that $T(f)$ is the limit of $(f_{i\epsilon})$ in the weak topology.

If $\omega(t)=\log(1+t)$, given $\varphi\in\mathcal{D}(\mathbb{R})$ we choose $k\in\mathbb{N}$ satisfying (1) and we put

$$\phi(x,y) := \sum_{j=0}^k \frac{1}{j!}\varphi^{(j)}(x)\,(iy)^j.$$

Then $\frac{\partial\phi}{\partial\bar z}(x,y) = \frac{1}{2}\frac{\varphi^{(k+1)}(x)}{k!}(iy)^k$ and we proceed as above. See [16, 2.2].

For $N > 1$ and ω a weight function, let $\sigma \in \{-1,1\}^N$ and $f \in \mathcal{H}^N_{\omega^*,p}$ be given. We want to show that $\lim_{\epsilon\to 0^+} f(\cdot + i\epsilon\sigma) \in (\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}^N))'$ and that $\{f(\cdot + i\epsilon\sigma) : 0 < \epsilon < 1\}$ is bounded in $(\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}^N))'$. To do this, let A be a real invertible matrix such that $Ae_1 = \sigma$. We put $g := f \circ A$ and $\varphi := \Psi \circ A$ for $\Psi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$ and we denote $f_{i\epsilon\sigma}(x) := f(x + i\epsilon\sigma)$. Then, for each $(x_2, \dots, x_N) \in \mathbb{R}^{N-1}$, $g(\cdot, x_2, \dots, x_N) \in \mathcal{H}(\mathbb{C} \setminus \mathbb{R})$ and

$$(f_{i\epsilon\sigma} * \Psi)(Ax) = |\mathrm{det}A| \int_{\mathbb{R}^N} g(x + t + i\epsilon e_1)\varphi(-t)dt.$$

Let $b > 0$ satisfy $\mathrm{supp}\ \varphi \subset [-b, b]^N$. Since $\{\varphi(\cdot, x_2, \dots, x_N) : (x_2, \dots, x_N) \in \mathbb{R}^{N-1}\}$ is bounded in $\mathcal{D}_{(\omega)}(\mathbb{R})$ we may find $\Phi(z, x_2, \dots, x_N)$, $z \in \mathbb{C}$, $x' := (x_2, \dots, x_N) \in \mathbb{R}^{N-1}$ having compact support such that

(i) $\Phi(\cdot, x_2, \dots, x_N) \in \mathcal{D}((-b, b) \times (-\frac{1}{2}, \frac{1}{2}))$ and $\Phi(\lambda, x_2, \dots, x_N) = \varphi(-\lambda, -x_2, \dots, -x_N), \lambda \in \mathbb{R}$.

(ii) $\frac{\partial}{\partial \bar{z}}\Phi(z, x_2, \dots, x_N)$ is continuous and $\left|\frac{\partial}{\partial \bar{z}}\Phi(z, x')\right| \leq C\mathrm{e}^{-k\omega^*(\frac{|\mathrm{Im} z|}{k})}$ for every $x' \in \mathbb{R}^{N-1}$.

As in the one-dimensional case, one can show that $\{(f_{i\epsilon\sigma} * \Psi)(A\cdot) : 0 < \epsilon < 1\}$ is bounded in $L_p(\mathbb{R}^N)$, therefore $\{(f_{i\epsilon\sigma} * \Psi) : 0 < \epsilon < 1\}$ is bounded in $L_p(\mathbb{R}^N)$ and, again as for $N = 1$, the conclusion follows.

In the case $N > 1$ and $\omega(t) = \log(1 + t)$, we choose $\sigma, A, f, \Psi, \varphi$ and b as above. For $y \in \mathbb{R}$ and $x \in \mathbb{R}^N$ we put $z = x_1 + iy$, $x' = (x_2, \dots, x_N)$ and $\phi(x_1, y, x_2, \dots, x_N) = \sum_{j=0}^{k} \frac{(-1)^j}{j!} \frac{\partial^j}{\partial x_1^j}\varphi(-x)(iy)^j$.

Then

$$\frac{\partial}{\partial \bar{z}}\phi(z, x') = \frac{1}{2}\frac{(-1)^{k+1}}{k!}\frac{\partial^{k+1}}{\partial x_1^{k+1}}\varphi(-x)(iy)^k.$$

Now, we proceed as in the case of a weight function ω. ■

Our next aim is to show that the boundary value map is surjective. Since $\mathcal{H}^N_{\omega*,p}$ is closed under (ultra)differential operators and each (ultra)distribution of L_p-growth is essentially of the form $G(D)f$, where $G(D)$ is a (ultra)differential operator and $f \in L_p(\mathbb{R}^N)$ ([2] and [19]), it suffices to show that $L_p(\mathbb{R}^N) \subset T(\mathcal{H}^N_p)$ and, a fortiori, $L_p(\mathbb{R}^N) \subset T(\mathcal{H}^N_{\omega*,p})$.

Given $N \in \mathbb{N}$ we denote by G_N the Banach space

$$G_N := \{f \in \mathcal{H}((\mathbb{C} \setminus \mathbb{R})^N) : \quad \| f \| := \sup_y \| f(. + iy) \|_1 \mathrm{e}^{-|y|} \prod_{j=1}^{N} |y_j| < \infty\}.$$

Clearly $G_1 \otimes G_N \subset G_{N+1}$ and the canonical bilinear map $G_1 \times G_N \longrightarrow G_{N+1}$ induces a continuous linear map $i : G_1\hat{\otimes}_\pi G_N \longrightarrow G_{N+1}$.

Lemma 3 *There exists a continuous linear map $j : \mathcal{D}(\mathbb{R}^{N+1}) \longrightarrow \mathcal{D}(\mathbb{R})\hat{\otimes}_\pi\mathcal{D}(\mathbb{R}^N)$ such that its restriction to $\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{R}^N)$ is the identity.*

PROOF. From a well known result of Grothendieck [10], for each compact $K \subset \mathbb{R}$ the bilinear map

$$\begin{array}{ccc} \mathcal{D}(K) \times \mathcal{D}(K^N) & \longrightarrow & \mathcal{D}(K^{N+1}), \\ (\phi, \psi) & \rightarrowtail & \phi\psi \end{array}$$

induces an isomorphism of Fréchet spaces between $\mathcal{D}(K)\hat{\otimes}_\pi\mathcal{D}(K^N)$ and $\mathcal{D}(K^{N+1})$, from where we conclude. ■

Proposition 2 *For each $N \in \mathbb{N}$ there exists a continuous linear operator $S_N : \mathcal{D}(\mathbb{R}^N) \longrightarrow G_N$ such that $T \circ S_N$ gives the identity on $\mathcal{D}(\mathbb{R}^N)$.*

PROOF. We proceed by induction on N.

For $N=1$, each $\varphi \in \mathcal{D}(\mathbb{R})$ satisfies $\varphi = -i\mathrm{e}^{-ix}(\mathrm{e}^{ix}\varphi)' + i\varphi'$, hence $\varphi = T(S_1(\varphi))$, where

$$S_1(\varphi)(z) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\varphi(t)}{(t-z)^2}dt - \frac{\mathrm{e}^{-iz}}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{it}\varphi(t)}{(t-z)^2}dt,$$

for $\mathrm{Im} z \neq 0$ [8, 2.3]. For $y \neq 0$ we put $K_y(t) = 1/(2\pi(t+iy)^2)$. Then given $z = x+iy,\ y \neq 0$, we have $S_1(\varphi)(z) = (\varphi * K_y)(x) - \mathrm{e}^{-iz}(\mathrm{e}^{it}\varphi * K_y)(x)$. Thus,

$$\| S_1(\varphi)(\cdot + iy) \|_1 \leq C\frac{\mathrm{e}^{|y|}}{|y|} \| \varphi \|_\infty .$$

This shows that $S_1 : \mathcal{D}(\mathbb{R}) \longrightarrow G_1$ is well-defined, linear and continuous.

Let us assume that the claim is true for N. We define $S_{N+1} := i \circ (S_1 \widehat{\otimes}_\pi S_N) \circ j$, which is linear and continuous. To show that $T \circ S_{N+1}$ gives the identity on $\mathcal{D}(\mathbb{R}^{N+1})$, it is enough to see that $(T \circ S_{N+1})(\varphi_1 \otimes \varphi_2) = (\varphi_1 \otimes \varphi_2)$ as distributions whenever $\varphi_1 \in \mathcal{D}(\mathbb{R})$ and $\varphi_2 \in \mathcal{D}(\mathbb{R}^N)$, and again it is sufficient to check this equality on $\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{R}^N)$, which is very easy. ■

In the sequel, for each compact $K \subset \mathbb{R}^N$ and $m \in \mathbb{N}$, we denote by $\mathcal{D}^m(K)$ the set of all C^m-functions f such that $\mathrm{supp} f \subset K$.

Corollary 3 *For each compact $K \subset \mathbb{R}^N$, there are $m \in \mathbb{N}$ and a continuous linear map $S_{N,K} : \mathcal{D}^m(K) \longrightarrow G_N$ such that for every $\Gamma \in \mathcal{D}^m(K)$ and each $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we have that $\langle T \circ S_{N,K}(\Gamma), \varphi \rangle = \langle \Gamma, \varphi \rangle$.*

PROOF. Let B be the closed unit ball in $\mathbb{R}^N$. We take an even function $\eta \in \mathcal{D}(B)$, $\eta \geq 0$, with $\| \eta \|_1 = 1$. For $n \in \mathbb{N}$ we define $\eta_n(t) := n^N \eta(nt)$ and we put $K_1 := K + B$. Using the continuity of S_N, we find $\ell \in \mathbb{N}$ such that

$$\| S_N(\varphi) \| \leq \| \varphi \|_{K_1,\ell} \tag{2}$$

for each $\varphi \in \mathcal{D}(K_1)$. We put $m = 2\ell$. If $\Gamma \in \mathcal{D}^m(K)$, then $\Gamma * \eta_n \in \mathcal{D}(K_1)$,

$$\| \Gamma * \eta_n \|_{K_1,\ell} \leq \| \Gamma \|_K \tag{3}$$

and $(\Gamma * \eta_n)_n$ converges to Γ in $\mathcal{D}^\ell(K_1)$. It follows from (2) that $(S_N(\Gamma * \eta_n))$ is a Cauchy sequence in G_N, hence it converges and we may define

$$S_{N,K}(\Gamma) = \lim_{n\to\infty} S_N(\Gamma * \eta_n).$$

Thus, $S_{N,K} : \mathcal{D}^m(K) \to G_N$ is well defined, linear and by (2) and (3), it is continuous. ■

From now on, if f is a function, we denote by $\check{f}$ the map defined by $\check{f}(x) = f(-x)$.

For $g \in G_N$ and $f \in L_p(\mathbb{R}^N)$ we put

$$(g * f)(x+iy) := \int_{\mathbb{R}^N} g(x - t + iy) f(t) dt.$$

Then $g * f \in \mathcal{H}_p^N$. Moreover, we have the following result.

Proposition 3 *Let $1 \leq p < \infty$, $\psi \in \mathcal{D}(\mathbb{R}^N)$ and $f \in L_p(\mathbb{R}^N)$ be given. Then $S_N(\psi) * f \in \mathcal{H}_p^N$ and $T(S_N(\psi) * f) = \psi * f$.*

PROOF. Given $\sigma \in \{-1,1\}^N$, $\epsilon > 0$ and $\varphi \in \mathcal{D}(\mathbb{R}^N)$, one has that $\check{f} * \varphi \in \mathcal{D}_{L_p}(\mathbb{R}^N)$ and

$$\langle (S_N(\psi) * f)(\cdot + i\sigma\epsilon), \varphi \rangle = \langle S_N(\psi)(\cdot + i\sigma\epsilon), \check{f} * \varphi \rangle.$$

Since $S_N(\psi) \in \mathcal{H}_1^N \subset \mathcal{H}_{p'}^N$, p' being the conjugate number of p, it suffices to apply Proposition 1 for $\omega(t) = \log(1+t)$ and Proposition 2. ■

The following lemma permit us to get a similar result for $p = \infty$.

Lemma 4 *Let K be a compact set in $\mathbb{R}$ and let $\psi \in \mathcal{D}(K)$ be given. Then $S_1(\psi) \in \mathcal{H}(\mathbb{C} \setminus K)$ and, for some positive constants A and C,*

$$|S_1(\psi)(x \pm i\epsilon)| \leq \frac{C}{|x|^2}$$

whenever $|x| \geq A$ and $0 < \epsilon < 1$.

PROOF. It is clear from the definition of S_1. ■

Proposition 4 *Given $\Gamma \in \mathcal{D}(\mathbb{R}^N)$ and $f \in L_\infty(\mathbb{R}^N)$ we have $S_N(\Gamma) * f \in \mathcal{H}_\infty^N$ and $T(S_N(\Gamma) * f) = \Gamma * f$.*

PROOF. We already know that $S_N(\Gamma) * f \in \mathcal{H}_\infty^N$. To see that $T(S_N(\Gamma) * f) = \Gamma * f$ we proceed in two steps. First, let K be a compact set in $\mathbb{R}$ and let $\varphi_1, \ldots, \varphi_N \in \mathcal{D}(K)$ be given. We consider $\Gamma := \varphi_1 \otimes \cdots \otimes \varphi_N$ and we put $f_j := S_1(\varphi_j) \in \mathcal{H}_1^1$. Then $F := S_N(\Gamma)$ is given by $F(z_1, \ldots, z_N) = \prod_{j=1}^N f_j(z_j)$. Let us check that $T(F * f) = \Gamma * f$. We observe that

$$\sum_{\sigma \in \{-1,1\}^N} \left(\prod_{j=1}^N \sigma_j \right) F(x + i\sigma\epsilon) = \prod_{j=1}^N \left(f_j(x_j + i\epsilon) - f_j(x_j - i\epsilon) \right).$$

From Lemma 4, we choose $\widetilde{K}$ a compact subset in $\mathbb{R}$, $K \subset \widetilde{K}$, and $C > 0$ such that

$$|f_j(x_j \pm i\epsilon)| \ \leq \ \frac{C}{|x|^2}$$

whenever $x \notin \widetilde{K}$ and $0 < \epsilon < 1$. Let $\eta \in \mathcal{D}(\mathbb{R})$ be identically one on a neighborhood of $\widetilde{K}$. For each $\psi \in \mathcal{D}(\mathbb{R}^N)$ and each $0 < \epsilon < 1$ we have,

$$\begin{aligned}
&\langle (\sum_{\sigma \in \{-1,1\}^N} (\prod_{j=1}^N \sigma_j) F(x + i\sigma\epsilon)) * f, \psi \rangle \\
&= \ \langle \prod_{j=1}^N (f_j(x_j + i\epsilon) - f_j(x_j - i\epsilon)), \ \prod_{j=1}^N (1 - \eta(x_j) + \eta(x_j))(\check{f} * \psi) \rangle \\
&= \ I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon),
\end{aligned}$$

where

$$\begin{aligned}
I_1(\epsilon) \ &:= \ \langle \prod_{j=1}^N (f_j(x_j + i\epsilon) - f_j(x_j - i\epsilon)), \ (\prod_{j=1}^N \eta(x_j))(\check{f} * \psi) \rangle, \\
I_2(\epsilon) \ &:= \ \langle \prod_{j=1}^N (f_j(x_j + i\epsilon) - f_j(x_j - i\epsilon)), \ \prod_{j=1}^N (1 - \eta(x_j))(\check{f} * \psi) \rangle,
\end{aligned}$$

and $I_3(\epsilon)$ consists of the remaining terms.

Since $(\prod_{j=1}^N \eta(x_j))(\check{f} * \psi) \in \mathcal{D}(\mathbb{R}^N)$ and $\prod_{j=1}^N \eta(x_j)$ is identically one on a neighborhood of the support of Γ,

$$\lim_{\epsilon \to 0^+} I_1(\epsilon) = \langle \Gamma, \check{f} * \psi \rangle = \langle \Gamma * f, \psi \rangle.$$

Using that $\prod_{j=1}^N (1-\eta(x_j))$ has support in $(\mathbb{R} \setminus \widetilde{K})^N$ and $\prod_{j=1}^N (1-\eta(x_j))(\check{f} * \psi) \in L_\infty(\mathbb{R}^N)$, and on account of the given estimates for the functions f_j's outside of $\widetilde{K}$, we may apply Lebesgue's convergence theorem to get that

$$\lim_{\epsilon \to 0^+} I_2(\epsilon) = 0.$$

Now, we observe that $I_3(\epsilon)$ is a sum of integrals of the form

$$\int_{(\mathbb{R}\setminus \widetilde{K})^m} \prod_{j \in S_m} (f_j(x_j+i\epsilon) - f_j(x_j - i\epsilon))(1-\eta(x_j))\Big(\int_{\mathbb{R}^{N-m}} \prod_{j \notin S_m} (f_j(x_j+i\epsilon) - f_j(x_j-i\epsilon))\eta(x_j)(\check{f} * \psi) d\overline{x}\Big) d\tilde{x},$$

where S_m is a proper subset of $\{1, \ldots, N\}$, $\overline{x}$ denotes the coordinates corresponding to indices in S_m and $\tilde{x}$ are the remaining coordinates. Without loss of generality we take $S_m = \{1, \ldots, m\}$. Since $\check{f} * \psi \in \mathcal{B}_{L_\infty}(\mathbb{R}^N)$, one has that

$$\{(\prod_{j=m+1}^N \eta(x_j))(\check{f} * \psi)(x_1, \ldots, x_m, \ldots) : (x_1, \ldots, x_m) \in \mathbb{R}^m\}$$

is a bounded subset of $\mathcal{D}(\mathbb{R}^{N-m})$. Also

$$\{\prod_{j=m+1}^N (f_j(\cdot + i\epsilon) - f_j(\cdot - i\epsilon)) : 0 < \epsilon < 1\}$$

is bounded in $(\mathcal{D}_{L_1}(\mathbb{R}^{N-m}))'$, hence

$$\left\{ \int_{\mathbb{R}^{N-m}} (\prod_{j=m+1}^N (f_j(x_j + i\epsilon) - f_j(x_j - i\epsilon))\eta(x_j))(\check{f} * \psi) d\overline{x} \ : 0 < \epsilon < 1 \right\}$$

is a bounded set in $L_\infty(\mathbb{R}^m)$. As before we conclude that $\lim_{\epsilon \to 0^+} I_3(\epsilon) = 0$. Consequently $T(F * f) = \Gamma * f$.

To finish, we fix a compact K in $\mathbb{R}$, $f \in L_\infty(\mathbb{R}^N)$, and we consider the continuous linear map

$$R : \mathcal{D}(K^N) \longrightarrow (\mathcal{D}_{L_1}(\mathbb{R}^N))'$$

given by $R(\Gamma) = T(S_N(\Gamma) * f) - \Gamma * f$. Since R vanishes on the dense subset $\mathcal{D}(K) \otimes \ldots \otimes \mathcal{D}(K)$, R must be identically zero, that is, $T(S_N(\Gamma) * f) = \Gamma * f$ for every $\Gamma \in \mathcal{D}(K^N)$. ■

Our next result gives the surjectivity of the boundary value operator in the case of distributions of L_p-growth. For $1 < p < \infty$, this follows from Tillman [21], Luszczki and Zielezny [13] and Bengel [1]. However, their methods did not work for $p = 1, \infty$. Our approach permits a unified treatment of all values of p.

Theorem 1 *For $1 \leq p \leq \infty$, the boundary value operator*

$$T : \mathcal{H}_p^N \longrightarrow (\mathcal{D}_{L_{p'}}(\mathbb{R}^N))'$$

is surjective.

PROOF. Since $\mathcal{H}_p^N$ is stable under differential operators and on account of [19, Th.XXV] it is enough to show that $L_p(\mathbb{R}^N) \subset T(\mathcal{H}_p^N)$. Let B be the closed unit ball in $\mathbb{R}^N$. Then, there are $m \in \mathbb{N}$ and $S_{N,B} : \mathcal{D}^m(B) \longrightarrow G_N$ continuous and linear such that $T \circ S_{N,B} = \mathrm{Id}$. Now, we can find a differential operator $P(D)$ and two functions $\Psi \in \mathcal{D}^m(B)$ and $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $\delta = P(D)\Psi + \varphi$. Therefore, given $f \in L_p(\mathbb{R}^N)$ we have $f = P(D)(\Psi * f) + \varphi * f$. We take $F := P(D)(S_{N,B}(\Psi) * f) + S_N(\varphi) * f$. Then $F \in \mathcal{H}_p^N$ and

$$T(F) = P(D)T(S_{N,B}(\Psi) * f) + T(S_N(\varphi) * f).$$

Clearly, $T(S_N(\varphi) * f) = \varphi * f$. On the other hand, $S_{N,B}(\Psi) = \lim_n S_N(\Psi_n)$, where the convergence is in G_N for a suitable sequence $(\Psi_n) \subset \mathcal{D}(\mathbb{R}^N)$ as in the proof of Corollary 3. Therefore, $S_{N,B}(\Psi) * f = \lim_n S_N(\Psi_n) * f$, where the convergence is in $\mathcal{H}_p^N$, and, since T is continuous, $T(S_{N,B}(\Psi) * f) = \lim_n(\Psi_n * f)$. Here the convergence is in $\mathcal{D}'(\mathbb{R}^N)$, but since $(\Psi_n)_n$ converges to Ψ in $L_1(\mathbb{R}^N)$ then $(\Psi_n * f)_n$ converges to $\Psi * f$ in $L_p(\mathbb{R}^N)$ and we conclude $T(F) = f$. ■

Theorem 2 *Let ω be a weight function satisfying (α_1) and $1 \le p \le \infty$. Then the boundary value operator $T : \mathcal{H}_{\omega^*,p}^N \longrightarrow (\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}^N))'$ is surjective.*

PROOF. For every $\mu \in (\mathcal{D}_{L_{p'},(\omega)}(\mathbb{R}^N))'$ there are an ultradifferential operator $G(D)$ of $(\omega)-$class and $f \in L_p(\mathbb{R}^N)$ such that $\mu = G(D)f$ ([2, 2.1]). By Theorem 1, there is $F \in \mathcal{H}_p^N \subset \mathcal{H}_{\omega^*,p}^N$ with $T(F) = f$. Then $G(D)F \in \mathcal{H}_{\omega^*,p}^N$ on account of Lemma 1 and $T(G(D)F) = \mu$. ■

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