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Ideals of extendable and liftable operators

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Dedicated to the memory of Klaus Floret

Abstract. Ideals of extendable and liftable operators are introduced giving a new approach to the study of the splitting of short exact sequences of Banach spaces. Maximality, duality and closedness with respect to pointwise bounded limits of the ideals are considered. Several examples are summarized and the role of $\mathcal{L}_1$- and $\mathcal{L}_\infty$-spaces is clarified.

Ideales de operadores que admiten extensión o levantamiento

Resumen. Se introducen los ideales de operadores que admiten extensión o levantamiento y se presenta una nueva aproximación al estudio de la escisión de sucesiones exactas cortas de espacios de Banach. Se considera la maximalidad de estos ideales y se investiga si son cerrados respecto de los límites puntuales acotados. Se resumen algunos ejemplos y se clarifica el papel de los espacios $\mathcal{L}_1$ y $\mathcal{L}_\infty$.

1. Introduction

Let

$$0 \longrightarrow X \xrightarrow{\ j\ } W \xrightarrow{\ q\ } Z \longrightarrow 0 \tag{1}$$

be a short exact sequence of Banach (Fréchet, locally convex, ...) spaces and continuous linear maps (= operators). It *splits* if q has a right inverse operator. There is an extensive splitting theory in the Fréchet space setting due to Vogt and his collaborators (see [41], [33], [28], [36], [17], [19], [13] etc.) later on it was extended to some non-metrizable spaces (see [14], [15]). It turns out that for many pairs of natural Fréchet spaces (X, Z) every short exact sequence of the form (1) splits. These results have found many applications, for instance: in the theory of partial differential operators (see, for example, [32], [13], [15]), for problems of linear extension of smooth and holomorphic functions (see, for example, [36], [18]) as well as for linear solutions to the division and composition problems (see, for instance, [2] and the reference list there, [1], [3]). Moreover, the splitting theory is extensively used in the structure theory of Fréchet spaces (see [38], [39], [40], [42], [43], [44], comp. [33]).

In the Banach setting there is a lack of such a theory: for most natural Banach spaces X, Z there are non-splitting sequences of the form (1) (see [16], [22], [23], [24], [25], [27], [26]). That is why I am more and more convinced that the operator ideal approach is the proper setting in the Banach case. Let me explain it by analogy: the class of Fréchet spaces for which every summable sequence is absolutely summable (=

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nuclear spaces) contains many natural, important in analysis, spaces. The same class in the Banach case contains only trivial spaces, nevertheless the operator ideal of absolutely summing operators is of great significance in the theory and applications.

In the same spirit, we introduce the ideals of extendable and liftable operators. In the present paper we give their definitions as well as the definitions of their natural operator ideal norms. Moreover, we present several basic properties, some examples and, finally, we explain the role of $\mathcal{L}_1$- and $\mathcal{L}_\infty$-spaces in the theory. The idea of extendable and liftable operators goes back to our Ph.D. thesis (1987) — for a source of inspiration see also [23, 8.2]. Some of the results were contained in a never published manuscript [11] based on my Ph.D. thesis, few results were announced without proof in [12]. Recently the interest in short exact sequences of Banach spaces has revived (see [26], [20], [5], [4], [6]), moreover, a corresponding set of problems is considered in the vividly developing area of operator spaces (see, for instance, [37], [34], comp. [21]). That is why I decided to come back to this area and the paper is a consequence of that recent research.

I believe that the operator ideals of extendable and liftable operators deserve some more attention and some further research. Any attentive reader will find plenty of natural open questions related to the introduced notions.

2. Preliminaries

We denote by L, $\mathcal{A}$ and $\mathcal{F}$ the class of all (continuous linear) operators, the ideals of approximable and finite dimensional operators, respectively. If $\mathcal{I}$ is an operator ideal, then $\mathcal{I}\circ\mathcal{A}^{-1}$ and $\mathcal{A}^{-1}\circ\mathcal{I}$ denote the classes of operators T such that $T\circ S$ and $S\circ T$, respectively, are in $\mathcal{I}$ for every $S\in\mathcal{A}$. By B_X we denote the unit ball of the Banach space X and $i_X : X \to X''$ denotes the canonical embedding. A short exact sequence (1) is *isometric* if j is an isometric embedding and

$$\|q(x)\| = \inf\{\|w\| : qw = qx, w \in W\} \qquad \text{for all } x \in W.$$

We will use very often the classical, and well-known for a long time, *pull-back* and *push-out* procedures summarized in the following result (comp. [10, Cor. 3.2], [14, Prop. 1.7], [5, 1.2, 1.3]).

Proposition 1 *Let (1) be a given exact sequence.*

(a) (pull-back) Let $T : Y \to Z$ be an operator, then there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{j} & W & \xrightarrow{q} & Z & \longrightarrow & 0\\
 & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle T_1}\uparrow & & {\scriptstyle T}\uparrow & & \\
0 & \longrightarrow & X & \xrightarrow{j_1} & W_1 & \xrightarrow{q_1} & Y & \longrightarrow & 0.
\end{array}$$

If (1) is isometric, then the second row is isometric as well and $\|T_1\| \le \|T\|$.

(b) (push-out) Let $T : X \to Y$ be an operator, then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{j_1} & W_1 & \xrightarrow{q_1} & Z & \longrightarrow & 0\\
 & & {\scriptstyle T}\uparrow & & {\scriptstyle T_1}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & \\
0 & \longrightarrow & X & \xrightarrow{J} & W & \xrightarrow{q} & Z & \longrightarrow & 0.
\end{array}$$

If (1) is isometric then the first row is isometric as well and $\|T_1\| \le \|T\|$.

(c) Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{j_1} & W_1 & \xrightarrow{q_1} & Z_1 & \longrightarrow & 0\\
 & & {\scriptstyle T}\uparrow & & {\scriptstyle S}\uparrow & & {\scriptstyle R}\uparrow & & \\
0 & \longrightarrow & X & \xrightarrow{j} & W & \xrightarrow{q} & Z & \longrightarrow & 0
\end{array}$$

be a commutative diagram with exact rows. There is a lifting of R to W_1 (i.e., an operator $R_1 : Z \to W_1$ such that $q_1 \circ R_1 = R$) if and only if there is an extension of T onto W (i.e., an operator $T_1 : W \to X_1$ such that $T_1 \circ j = T$). If both rows are isometric then we can choose T_1 such that $\|T_1\| \le \|S\| + \|R_1\|$ and we can choose R_1 such that $\|R_1\| \le \|S\| + \|T_1\|$. ■

For some additional information on operator ideals, absolutely summing operators, $\mathcal{L}_p$-spaces, the Radon-Nikodym property and representable operators we refer to [35], [8], [31], [9], respectively.

3. Ideals of extendable and liftable operators

Let X, Y, Z be given Banach spaces and let $T : X \to Y$ be an operator. We call T to be Z-*extendable* ($T \in \mathcal{E}_Z$) if and only if for every short exact sequence of Banach spaces (1) the map T extends to $T_1 : W \to Y$ (i.e., $T = T_1 \circ j$). analogously, T is Z-liftable ($T \in \mathcal{L}_Z$) if for every short exact sequence of Banach spaces

$$0 \longrightarrow Z \xrightarrow{\;j\;} W \xrightarrow{\;q\;} Y \longrightarrow 0 \tag{2}$$

the map T lifts to $T_2 : X \to W$ (i.e., $T = q \circ T_2$). Let us find for a fixed isometric sequence (1) the infimum of norms of T_1, then $e_Z(T)$ is defined to be the supremum over all (1) of such infima. Moreover, $l_Z(T)$ is defined to be the supremum over all isometric exact sequences (2) of the infima of norms of possible liftings T_2.

Theorem 1 *The pairs $(\mathcal{E}_Z, e_Z)$ and $(\mathcal{L}_Z, l_Z)$ are Banach operator ideals. Moreover, $T \in \mathcal{E}_Z$ if and only if $e_Z(T) < \infty$ and $T \in \mathcal{L}_Z$ if and only if $l_Z(T) < \infty$.* □

Remark 1 Of course, $\mathcal{L}_Z$ and $\mathcal{E}_Z$ contain the class of nuclear operators (as the smallest Banach operator ideal). ■

Before we prove the result we need the following observation.

Lemma 1 *(a) Let*

$$0 \longrightarrow \ker Q \xrightarrow{\;J\;} l_1(I) \xrightarrow{\;Q\;} Z \longrightarrow 0$$

be an isometric short exact sequence, then for

$$\tilde{e}_Z(T) := \sup\{\inf\{\|U_1\| : U_1 \circ J = T \circ R, U_1 : l_1(I) \to Y\} : R : \ker Q \to X, \|R\| \le 1\}$$

we have

$$\tilde{e}_Z(T) \le e_Z(T) \le 3\tilde{e}_Z(T).$$

(b) Let

$$0 \longrightarrow Z \xrightarrow{\;J\;} l_\infty(I) \xrightarrow{\;Q\;} l_\infty(I)/Z \longrightarrow 0$$

be an isometric short exact sequence, then for

$$\tilde{l}_Z(T) := \sup\{\inf\{\|U_2\| : Q \circ U_2 = S \circ T, U_2 : X \to l_\infty(I)\} : S : Y \to l_\infty(I)/Z, \|S\| \le 1\}$$

we have

$$\tilde{l}_Z(T) \le l_Z(T) \le 3\tilde{l}_Z(T).$$

PROOF. We will prove only part (b), the other one is analogous.

Let us consider an isometric short exact sequence

$$0 \longrightarrow Z \xrightarrow{j} W \xrightarrow{q} Y \longrightarrow 0.$$

clearly, Z embeds isometrically into some $l_\infty(I)$ so we get the following commutative diagram where rows are isometric exact and the lowest row is obtained via the pull-back procedure (see Proposition 1):

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "J"] & l_\infty(I) \arrow[r, "Q"] & l_\infty(I)/Z \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] \arrow[u, "\mathrm{id}"] & W \arrow[r, "q"] \arrow[u, "S_1"] & Y \arrow[r] \arrow[u, "S"] & 0 \\
0 \arrow[r] & Z \arrow[r, "j_1"] \arrow[u, "\mathrm{id}"] & W_1 \arrow[r, "q_1"] \arrow[u, "T_1"] & X \arrow[r] \arrow[u, "T"] & 0,
\end{tikzcd}$$

where S_1 extends J onto W, $\|S_1\| \leq ||J|| = 1$ ($l_\infty(I)$ is injective), S is the map induced by S_1 on quotients, $||S|| \leq 1$. Let $U_2 : X \to l_\infty(I)$ be a lifting of $S \circ T$. By Proposition 1 (c), there is a map $V_1 : W_1 \to Z$ such that $\mathrm{id}_Z = V_1 \circ j_1$ and, again by Proposition 1 (c), T has a lifting $T_2 : X \to W$. It is easily seen (Proposition 1 (c)) that

$$||T_2|| \leq ||T_1|| + ||T_1|| + ||U_2||.$$

Since $\tilde{l}_Z(T)$, $||T_1|| \geq ||T||$ we get $l_Z(T) \leq ||T_2|| \leq 3\tilde{l}_Z(T)$.

Let us take $S : Y \to l_\infty(I)/Z$, $||S|| \leq 1$. By Proposition 1 (the pull-back procedure) we construct the following commutative diagram with isometric exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "J"] & l_\infty(I) \arrow[r, "Q"] & l_\infty(I)/Z \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] \arrow[u, "\mathrm{id}"] & W \arrow[r, "q"] \arrow[u, "S_1"] & Y \arrow[r] \arrow[u, "S"] & 0.
\end{tikzcd}$$

Thus there is $T_1 : X \to W$, $q \circ T_1 = T$, $||T_1|| \leq l_Z(T) + \varepsilon$. Since $||S_1|| \leq 1$, $||S_1 \circ T_1|| \leq l_Z(T) + \varepsilon$ and therefore $\tilde{l}_Z(T) \leq l_Z(T)$. ■

PROOF OF THEOREM 1. We concentrate on the class $\mathcal{L}_Z$, the proof for the class of extendable operators is analogous.

First, we observe that the class $\mathcal{L}_Z$ is an operator ideal. Clearly, $\mathcal{L}_Z(X, Y)$ is a linear space. Moreover, if $T \in \mathcal{L}_Z(X, Y)$ and $U \in L(W, X)$ then obviously $T \circ U \in \mathcal{L}_Z(W, Y)$. Now, assume that $S : Y \to W$ and let

$$0 \longrightarrow Z \xrightarrow{j_1} W_1 \xrightarrow{q_1} W \longrightarrow 0$$

be an exact sequence. Then we construct the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "j_1"] & W_1 \arrow[r, "q_1"] & W \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r, "j_2"] \arrow[u, "\mathrm{id}"] & W_2 \arrow[r, "q_2"] \arrow[u, "S_1"] & Y \arrow[r] \arrow[u, "S"] & 0,
\end{tikzcd}$$

where the lower row is obtained via the pull-back procedure (see Proposition 1). Clearly, if $T \in \mathcal{L}_Z(X, Y)$ then there exists a lifting $T_1 : X \to W_2$, $q_2 \circ T_1 = T$ and $S_1 \circ T_1$ is a lifting of $S \circ T$. Thus $S \circ T \in \mathcal{L}_Z(X, W)$.

Looking at the construction above, it is easily seen that both l_Z and $\tilde{l}_Z$ are operator ideal norms whenever they are defined (and, by Lemma 1, they are defined on the same classes of operators).

Assume that $l_Z(T) < \infty$, we will prove that $T \in \mathcal{L}_Z(X,Y)$. Let us take an arbitrary exact sequence

$$0 \longrightarrow Z \xrightarrow{\ j\ } W \xrightarrow{\ q\ } Y \longrightarrow 0.$$

By renorming we may assume that Z is an isometric subspace of W. The image of the unit ball B_W via q contains some ball of Y. Thus defining a new norm on Y as some multiplicity of the original norm we may assume that $q(B_W)$ contains the unit ball B_Y. Taking as the new unit ball of W the set $B_W \cap q^{-1}(B_Y)$ we get an isometric short exact sequence (and Y with the new norm is isometric to Y with the previous norm). Therefore T has a lifting $T_1 : X \to W$, $T_1 \circ q = T$, and, of course, T_1 is a lifting of T for the original short exact sequence.

Assume that $T \in \mathcal{L}_Z(X,Y)$. We define on the space $L(Y, l_\infty(I)/Z)$ a new norm:

$$\Delta(S) := \inf ||U_1|| + ||S||,$$

where U_1 is a lifting of $S \circ T$, i.e., $Q \circ U_1 = S \circ T$. Since $S \circ T \in \mathcal{L}_Z$, thus $\Delta(S)$ is a well defined norm. One proves easily that, $L(Y, l_\infty(I)/Z)$ is complete equipped with the norm $\Delta(\cdot)$. Therefore, by the Open Mapping Theorem, for some constant C independent of S:

$$\inf ||U_1|| \le \Delta(S) \le C||S||$$

and

$$\tilde{l}_Z(T) \le C < \infty.$$

Using the direct sum

$$0 \longrightarrow Z \longrightarrow Z \oplus Y \longrightarrow Y \longrightarrow 0$$

one observes easily that $l_Z(T) \ge ||T||$. Thus if (T_n) is a Cauchy series with respect to l_Z, then it has a norm limit T and clearly we will find a sequence of liftings (U_n) (where U_n is a lifting of T_n for each $n \in \mathbb{N}$) which is also Cauchy in $||\cdot||$ and converges to a lifting U of T. Therefore $\mathcal{L}_Z$ is complete with respect to l_Z. ■

4. Properties of the ideals

First, we show that, under some natural assumptions, both ideals are closed with respect to pointwise limits of bounded nets.

Theorem 2 *(a) Let $(T_i)_{i\in I}$ be a e_Z-bounded net of operators $T_i : X \to Y$ pointwise convergent to an operator T. If Y is complemented in its bidual, then T is Z-extendable,*

$$e_Z(T) \le \lambda \sup_{i\in I} e_Z(T_i),$$

where λ is the projection constant of Y in Y''.

(b) Let $(T_i)_{i\in I}$ be a l_Z-bounded net of operators $T_i : X \to Y$ pointwise convergent to an operator T. If Z is complemented in its bidual, then T is Z-liftable,

$$l_Z(T) \le 3\lambda \sup_{i\in I} l_Z(T_i),$$

where λ is the projection constant of Z in Z''.

Remark 2 The part (b) was announced in the author's paper [12] without proof and the proof is contained in the author's preprint [11]. A "space" version was published in [26, Prop. 3.3]. ■

PROOF. (a) Let

$$0 \longrightarrow X \xrightarrow{j} W \xrightarrow{q} Z \longrightarrow 0$$

be an isometric short exact sequence. Of course, for every $i \in I$ there is an extension $U_i : W \to Y$, $||U_i|| \leq (1+\varepsilon)e_Z(T_i)$. By the classical Lindenstrauss compactness argument (see [29], [30, proof of Th. 2.1, page 12]), there is a subnet of $(U_i)_{i\in I}$ pointwise weak-star convergent to $U : W \to Y''$, $||U|| \leq (1+\varepsilon)\sup_{i\in I} e_Z(T_i)$. Clearly, U is an extension of $T : X \to Y''$. If $P : Y'' \to Y$, $||P|| \leq (1+\varepsilon)\lambda$, is a projection, then $P \circ U$ is an extension of T we are looking for.

(b) Let

$$0 \longrightarrow Z \xrightarrow{j} W \xrightarrow{q} Y \longrightarrow 0$$

be an isometric short exact sequence. Clearly, one can produce the following commutative diagram where the lowest row is obtained from the middle one by the pull-back procedure (see Proposition 1)

$$\begin{tikzcd}
0 \arrow[r] & Z'' \arrow[r, "j''"] & W'' \arrow[r, "q''"] & Y'' \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] \arrow[u, "i_Z"] & W \arrow[r, "q"] \arrow[u, "i_W"] & Y \arrow[r] \arrow[u, "i_Y"] & 0 \\
0 \arrow[r] & Z \arrow[r, "j_1"] \arrow[u, "\mathrm{id}"] & W_1 \arrow[r, "q_1"] \arrow[u] & X \arrow[r] \arrow[u, "T"] & 0.
\end{tikzcd}$$

For every $i \in I$ there is a lifting $U_i : X \to W$ of T_i, $||U_i|| \leq (1+\varepsilon)l_Z(T_i)$. By the Lindenstrauss compactness argument as in (a), there is a subnet of $(U_i)_{i\in I}$ pointwise weak-star convergent to $U : X \to W''$, $||U|| \leq (1+\varepsilon)\sup_{i\in I} l_Z(T_i)$. By Proposition 1 (c), i_Z has an extension R onto W_1. If $P : Z'' \to Z$ is a projection, then $P \circ R$ is a projection of W_1 onto Z and again, by Proposition 1 (c), T has a lifting into W. Analyzing the estimates in the used proposition we get the estimates on the norm of the obtained lifting. ■

Let us recall that for Banach operator ideal $\mathcal{I}$ we define $\mathcal{I}^{\max} := \mathcal{A}^{-1} \circ \mathcal{I} \circ \mathcal{A}^{-1}$.

Corollary 1 *Let Y be a complemented subspace of its bidual.*

(a) If X or Z has the bounded approximation property then

$$\mathcal{E}_Z \circ \mathcal{A}^{-1}(X,Y) = \mathcal{E}_Z(X,Y).$$

(b) If Y has the the bounded approximation property then

$$\mathcal{A}^{-1} \circ \mathcal{E}_Z(X,Y) = \mathcal{E}_Z(X,Y).$$

(c) If Y and X or Z have the bounded approximation property then

$$\mathcal{E}_Z(X,Y) = \mathcal{E}_Z^{\max}(X,Y).$$

Corollary 2 *Let Z be a complemented subspace of its bidual.*

(a) $\mathcal{L}_Z \circ \mathcal{A}^{-1} = \mathcal{L}_Z$.

(b) If Y has the bounded approximation property then

$$\mathcal{A}^{-1} \circ \mathcal{L}_Z(X,Y) = \mathcal{L}_Z(X,Y).$$

(c) If Y has the bounded approximation property then

$$\mathcal{L}_Z(X,Y) = \mathcal{L}_Z^{\max}(X,Y).$$

PROOF OF COROLLARY 1 AND 2. Assume that X has the bounded approximation property then there is a bounded family of finite dimensional operators $(S_i)_{i\in I} \subseteq L(X,X)$, $||S_i|| \le C$ for every $i \in I$, pointwisely tending to identity. If $T \in \mathcal{E}_Z \circ \mathcal{A}^{-1}$ then $e_Z(T \circ S_i) \le C_1$ for some fixed C_1 and every $i \in I$ (comp. [35, 8.7.5]). By Theorem 2 (a), $T \in \mathcal{E}_Z$ and this proves 1 (a) for X having the bounded approximation property. The proof for 1 (b) and 2 (b) is completely analogous. Conditions (c) follow immediately from (a) and (b) in both corollaries

1 (a), Z has the bounded approximation property: Let $T \in \mathcal{E}_Z \circ \mathcal{A}^{-1}(X,Y)$. There exists a family of finite dimensional operators $(S_i)_{i\in I}$ on Z tending pointwisely to the identity, $||S_i|| \le C$ for every $i \in I$. We define $Z_i := \mathrm{Im} S_i$, $R_i : Z_i \to Z$ the natural embedding. Let us consider an isometric short exact sequence

$$0 \longrightarrow X \xrightarrow{j} W \xrightarrow{q} Z \longrightarrow 0$$

For every $i \in I$ there is a finite dimensional space $\tilde{W}_i \subseteq W$ such that $q(B_{\tilde{W}_i}) \supseteq (1 - (3\dim Z_i)^{-1})B_{Z_i}$. Thus taking $\tilde{X}_i := j^{-1}(\tilde{W}_i \cap j(X))$ we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "j"] & W \arrow[r, "q"] & Z \arrow[r] & 0 \\
0 \arrow[r] & \tilde{X}_i \arrow[r] \arrow[u, "A_i"] & \tilde{W}_i \arrow[r] \arrow[u, "B_i"] & Z_i \arrow[r] \arrow[u, "R_i"] & 0.
\end{tikzcd}$$

Modifying the norm on $\tilde{W}_i$ and $\tilde{X}_i$ we may obtain new finite dimensional spaces W_i, X_i and the following diagram with both rows isometric exact and $||A_i||, ||B_i|| \le (1 - (2\dim Z_i)^{-1})$

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "j"] & W \arrow[r, "q"] & Z \arrow[r] & 0 \\
0 \arrow[r] & X_i \arrow[r] \arrow[u, "A_i"] & W_i \arrow[r] \arrow[u, "B_i"] & Z_i \arrow[r] \arrow[u, "R_i"] & 0.
\end{tikzcd}$$

Finally, we construct the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Y'' \arrow[r] & W_1'' \arrow[r] & Z'' \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, "i_Y"] & W_1 \arrow[r] \arrow[u, "i_{W_1}"] & Z \arrow[r] \arrow[u, "i_Z"] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[u, "T"] & W \arrow[r] \arrow[u, "T_1"] & Z \arrow[r] \arrow[u, "\mathrm{id}"] & 0 \\
0 \arrow[r] & X_i \arrow[r] \arrow[u, "A_i"] & W_i \arrow[r] \arrow[u, "B_i"] & Z_i \arrow[r] \arrow[u, "R_i"] & 0 \\
0 \arrow[r] & X_i \arrow[r] \arrow[u, "\mathrm{id}"] & V_i \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u, "S_i"] & 0,
\end{tikzcd}$$

where the second row is obtained from the third one via the push-out procedure and the last one from the fourth one via the pull-back procedure (see Proposition 1). Since $T \in \mathcal{E}_Z \circ \mathcal{A}^{-1}(X,Y)$, we have $e_Z(T \circ A_i) \le C_1$, where C_1 does not depend on i (comp. [35, 8.7.5]). By Proposition 1 (c), $R_i \circ S_i : Z \to Z$ has a lifting $\tilde{S}_i : Z \to W_1$, $||\tilde{S}_i|| \le C_2$ for every $i \in I$. Since S_i tends pointwisely to id_Z, by the Lindenstrauss compactness argument (see [29], cite[proof of Th. 2.1, page 12]L1), there is a weak-star pointwise cluster point of $(\tilde{S}_i)$ denoted by $U : Z \to W_1''$ which is a lifting of $i_Z : Z \to Z''$. Clearly, by

Proposition 1 (c), $i_Y \circ T$ has an extension $U_1 : W \to Y''$ and if $P : Y'' \to Y$ is a projection, $P \circ U_1$ is an extension of T. Therefore $T \in \mathcal{E}_Z(X,Y)$.

2 (a): Let

$$0 \longrightarrow Z \xrightarrow{j} W \xrightarrow{q} Y \longrightarrow 0$$

be an isometric exact sequence. We construct the following commutative diagram with isometric exact rows

$$\begin{tikzcd}
0 \arrow[r] & Z'' \arrow[r, "j''"] & W'' \arrow[r, "q''"] & Y'' \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] \arrow[u, "i_Z"] & W \arrow[r, "q"] \arrow[u, "i_W"] & Y \arrow[r] \arrow[u, "i_Y"] & 0 \\
0 \arrow[r] & Z \arrow[r] \arrow[u, "\mathrm{id}_Z"] & W_1 \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "T"] & 0,
\end{tikzcd}$$

where the lowest row is obtained via the pull-back procedure (see Proposition 1). If $T \in \mathcal{L}_Z \circ \mathcal{A}^{-1}(X,Y)$ then for every finite dimensional subspace $X_i \subseteq X$ the map $T|_{X_i}$ has a lifting $S_i : X_i \to W$ with $||S_i|| \leq C$, C does not depend on i. Defining $S_i(x) = 0$ for $x \notin X_i$ we may assume that $S_i : X \to W$. By the Lindenstrauss compactness argument as in the proof of 1 (a), there is a weak-star pointwise cluster point $S \in L(X, W'')$ which is a lifting of $i_Y \circ T$. By Proposition 1, i_Z has an extension $U : W_1 \to Z''$ and using a projection $P : Z'' \to Z$, the operator $P \circ U : W_1 \to Z$ extends id_Z. Again by Proposition 1, T lifts to W and $T \in \mathcal{L}_Z(X,Y)$. ■

Let us recall that for any operator ideal $\mathcal{I}$ we define $\mathcal{I}^{\mathrm{dual}} := \{T : T' \in \mathcal{I}\}$.

Theorem 3 *For every Banach space Z the following equality holds:*

$$\mathcal{L}_{Z'} = (\mathcal{E}_Z)^{\mathrm{dual}}.$$

Remark 3 In general, $\mathcal{L}_Z \neq (\mathcal{E}_{Z'})^{\mathrm{dual}}$. For instance,

$$\mathrm{id}_{l_\infty/c_0} \in \mathcal{L}_{l_\infty} = (\mathcal{E}_{l_1})^{\mathrm{dual}} \qquad \text{but} \qquad \mathrm{id}_{l_\infty/c_0} \notin \mathcal{L}_{c_0}. \quad ■$$

PROOF. Let $T \in \mathcal{L}_{Z'}(X,Y)$ and let

$$0 \longrightarrow Y' \xrightarrow{j} W \xrightarrow{q} Z \longrightarrow 0$$

be an isometric short exact sequence. We have $i_Y \circ T : X \to Y'' \in \mathcal{L}_{Z'}$ and it has a lifting $T_1 : X \to W'$ with respect to $j' : W' \to Y''$ ($\ker j' \simeq Z'$). Since $T' \circ i'_Y = T'_1 \circ j''$, we get

$$T' = T' \circ i'_Y \circ i_{Y'} = T'_1 \circ j'' \circ i_{Y'} = T'_1 \circ i_W \circ j \quad \text{and} \quad T' \in \mathcal{E}_Z.$$

To prove the other inclusion, we assume that $T \in (\mathcal{E}_Z)^{\mathrm{dual}}(X,Y)$, i.e., $T' \in \mathcal{E}_Z(Y', X')$. Let

$$0 \longrightarrow Z' \xrightarrow{j} W \xrightarrow{q} Y \longrightarrow 0$$

be an isometric short exact sequence. By duality and the pull-back procedure applied to the operator $i_Z : Z \to Z''$ (see Proposition 1) we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r, "q'"] & W' \arrow[r, "j'"] & Z'' \arrow[r] & 0 \\
0 \arrow[r] & Y' \arrow[r, "j_1"] \arrow[u, "\mathrm{id}"] & W_1 \arrow[r, "q_1"] \arrow[u, "i"] & Z \arrow[r] \arrow[u, "i_Z"] & 0.
\end{tikzcd}$$

Clearly T' extends to $T_1 : W_1 \to X'$, $T_1 \circ j_1 = T'$. Dualizing once again we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z' & \xrightarrow{q_1'} & W_1' & \xrightarrow{j_1'} & Y'' & \longrightarrow & 0 \\
 & & {\scriptstyle (i_Z)'}\Big\uparrow & & {\scriptstyle i'}\Big\uparrow & & {\scriptstyle \mathrm{id}}\Big\uparrow & & \\
0 & \longrightarrow & Z''' & \xrightarrow{j''} & W'' & \xrightarrow{q''} & Y'' & \longrightarrow & 0 \\
 & & {\scriptstyle i_{Z'}}\Big\uparrow & & {\scriptstyle i_W}\Big\uparrow & & {\scriptstyle i_Y}\Big\uparrow & & \\
0 & \longrightarrow & Z' & \xrightarrow{j} & W & \xrightarrow{q} & Y & \longrightarrow & 0.
\end{array}$$

It is easily seen that

$$(i_Z)' \circ i_{Z'} = \mathrm{id}_{Z'} \qquad \text{and} \qquad (i' \circ i_W)(W) = (j_1')^{-1}(i_Y(Y)).$$

Moreover, if $x \in \ker(i' \circ i_W)$ then $x \in \ker q$, thus $x = j(z)$ for some $z \in Z'$ and

$$q_1' \circ (i_Z)' \circ i_{Z'}(z) = i' \circ i_W(x) = 0$$

which implies that $z = 0$. We have proved that $i' \circ i_W$ is an embedding onto a closed subspace of W_1', hence it is a topological embedding. Since $j_1' \circ T_1' = T''$, we obtain

$$T_1'(X) \subseteq (j_1')^{-1}(i_Y(Y)) = (i' \circ i_W)(W)$$

and we observe that

$$T = q \circ (i' \circ i_W)^{-1} \circ T_1'|_X.$$

Therefore $T \in \mathcal{L}_{Z'}$. ∎

5. Examples

In this section we summarize several examples of extendable and liftable operators.

Example 1 *For an arbitrary Banach space Z every map factorizing through an injective space (for instance, $l_\infty(I)$, $L_\infty(\mu)$, $C(K)$, K extremally disconnected) belongs to $\mathcal{E}_Z$. In particular, $\mathcal{E}_Z$ contains:*

- *all 2-absolutely summing operators (comp. [8, Cor. 2.16]).* ∎

Let us recall that $T : X \to Y$ is *Radon-Nikodym* if it maps each μ-continuous X-valued measure of finite variation into a μ-differentiable Y-valued measure, where μ is an arbitrary probability measure (comp. also [35, 24.2.6]).

Example 2 *For an arbitrary Banach space Z every map factorizing through $l_1(I)$ belongs to $\mathcal{L}_Z$. In particular, $\mathcal{L}_Z$ contains:*

- *all Radon-Nikodym operators $T : X \to Y$, where X is an abstract L_1-space (comp. [26, Fact]);*
- *every composition $T \circ S$, where T is a 2-absolutely summing operator and S is an absolutely summing operator.* ∎

Indeed, if $X \simeq L_1(\mu)$, μ finite, then every Radon-Nikodym operator is representable and, by the Lewis-Stegall theorem (see the proof of [9, Th. III.1.8]) it factorizes through l_1. The general case follows from the Kakutani representation theorem (in fact, every abstract L_1-space is an l_1-sum of a family of spaces $(L_1(\mu_i))_{i\in I}$ with μ_i finite).

If $T \circ S$ then, by the Pietsch Factorization Theorem [8, Th. 2.13], S factorizes through a subspace of $L_1(\mu)$. Since every 2-absolutely summing operator is extendable (see Example 1 above) $T \circ S$ factorizes through a map $R : L_1 \to L_2$ which is clearly Radon-Nikodym [9, Cor. III.2.13].

Example 3 *For an arbitrary Banach space Z every weakly compact map $T : X \to Y$, X an $\mathcal{L}_\infty$-space, belongs to $\mathcal{E}_Z$. In particular, $\mathcal{E}_Z$ contains:*

- *all p-absolutely summing operators $T : C(K) \to Y$ ($1 \le p < \infty$).* ■

Indeed, by [7, Cor. 1], T factorizes through a map $S : X \to W$, W reflexive. Moreover, S is a pointwise limit of maps factorizing through finite dimensional l_∞ spaces (with an upper bounds on norms). Since W is reflexive then Theorem 2 (a) implies that $S \in \mathcal{E}_Z$. It is known that p-absolutely summing operators are weakly compact [8, Th. 2.17].

Example 4 *(Comp. Lindenstrauss [29]) For an arbitrary Banach space Z complemented in its bidual every map T factorizing through an $\mathcal{L}_1$-space belongs to $\mathcal{L}_Z$. In particular, $\mathcal{L}_Z$ contains:*

- *all absolutely summing operator $T : C(K) \to Y$.* ■

Indeed, T is a pointwise limit of maps factorizing through finite dimensional l_1 spaces. Then the result follows from Theorem 2 (b). If $T : C(K) \to Y$ is absolutely summing then T factorizes through L_1 (see [8, Cor. 2.15]).

Example 5 *For every $\mathcal{L}_1$-space Z every weakly compact map belongs to $\mathcal{E}_Z$.* ■

Indeed, every weakly compact map factorizes through a reflexive space [7, Cor. 1] and we apply Lindenstrauss [29] ($\mathrm{id}_W \in \mathcal{E}_Z$ for every reflexive W).

Example 6 *Every separable map belongs to $\mathcal{L}_{c_0(I)}$ for any set I.* ■

To prove the above statement, let us consider a short exact sequence

$$0 \longrightarrow c_0(I) \xrightarrow{j} W \xrightarrow{q} Y \longrightarrow 0.$$

We consider a separable map $T : X \to Y$ where $T(X) \subseteq Y_1$, Y_1 separable Banach subspace of Y. Clearly, there is a separable subspace W_1 of W such that $q(B_{W_1}) \supseteq (1-\varepsilon)B_{Y_1}$. Taking $Z := j^{-1}(W_1)$ we obtain a separable subspace of $c_0(I)$. Without loss of generality (enlarging Z if necessary) we may assume that $Z \simeq c_0$. We get the following commutative diagram where the lowest row is obtained via the pull-back procedure (see Proposition 1)

$$\begin{tikzcd}
0 \arrow[r] & c_0(I) \arrow[r, "j"] & W \arrow[r, "q"] & Y \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r] \arrow[u] & W_1 \arrow[r] \arrow[u] & Y_1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Z \arrow[r] \arrow[u, "\mathrm{id}"] & W_2 \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "T"] & 0.
\end{tikzcd}$$

By Sobczyk's Theorem, the middle row splits, therefore T has a lifting.

The classes of $\mathcal{L}_1$- and $\mathcal{L}_\infty$-spaces play a special role in the theory. In particular, only for Z belonging to these classes the ideals $\mathcal{E}_Z$ or $\mathcal{L}_Z$ can be closed.

Theorem 4 *(a) The following assertions are equivalent:*

(i) Z *is an* $\mathcal{L}_1$*-space;*

(ii) $\mathcal{E}_Z$ *contains all weakly compact operators;*

(iii) $\mathcal{E}_Z \supseteq \mathcal{A}$*;*

(iv) e_Z *and* $||\cdot||$ *are equivalent on* $\mathcal{A}$*;*

(v) e_Z *and* $||\cdot||$ *are equivalent on* $\mathcal{F}$*.*

(b) The following assertions are equivalent:

(i) Z *is an* $\mathcal{L}_\infty$*-space;*

(ii) $\mathcal{L}_Z \supseteq \mathcal{A}$*;*

(iii) l_Z *and* $||\cdot||$ *are equivalent on* $\mathcal{A}$*;*

(iv) l_Z *and* $||\cdot||$ *are equivalent on* $\mathcal{F}$*.*

PROOF. By [35, Th. 6.1.6] it follows that three last conditions are equivalent for any Banach operator ideal.

(a) (i)⇒(ii): See Example 5. (ii)⇒(iii): Obvious.

(iii)⇒(i): Let Y be an arbitrary Banach space such that Y' has the bounded approximation property. then, by Corollary 1,

$$\mathcal{E}_Z(Y', Y') = \mathcal{E}_Z \circ \mathcal{A}^{-1}(Y', Y') = L(Y', Y').$$

In particular, $\mathrm{id}_{Y'} \in \mathcal{E}_Z$, thus, by Theorem 3, $\mathrm{id}_Y \in \mathcal{L}_{Z'}$. We have proved that every short exact sequence

$$0 \longrightarrow Z' \longrightarrow W \longrightarrow Y \longrightarrow 0$$

splits whenever the dual Y' of Y has the bounded approximation property.

Let us take Y arbitrary and define as Y_0 an l_2-direct sum of all finite dimensional subspaces of Y. Clearly, Y_0' has the bounded approximation property and $\mathrm{id}_{Y_0} \in \mathcal{L}_{Z'}$ which implies that $l_{Z'}(\mathrm{id}_W) \leq l_{Z'}(\mathrm{id}_{Y_0})$ for every finite dimensional subspace W of Y. Using again the Lindenstrauss compactness arguments (comp. [29], [30]), we show that also $\mathrm{id}_Y \in \mathcal{L}_{Z'}$, thus Z' is injective (and $\mathcal{L}_\infty$-space) and therefore Z is an $\mathcal{L}_1$-space.

(b) (i)⇒(iv): Let

$$0 \longrightarrow Z \longrightarrow W \longrightarrow Y \longrightarrow 0$$

be an isometric short exact sequence. If Y is finite dimensional, we observe as in the proof of Corollary 1 (a), that there is commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \xrightarrow{j} & W & \xrightarrow{q} & Y & \longrightarrow & 0 \\ & & {\scriptstyle J}\Big\uparrow & & \Big\uparrow & & {\scriptstyle \mathrm{id}}\Big\uparrow & & \\ 0 & \longrightarrow & Z_1 & \longrightarrow & W_1 & \longrightarrow & Y & \longrightarrow & 0, \end{array}$$

where Z_1 is finite dimensional. Since Z is an $\mathcal{L}_\infty$-space, we may assume that Z_1 is a finite dimensional l_∞ space. Thus the lower row splits and, by Proposition 1 (c), the upper row splits as well (we have a control on the norms of the right inverse for q).

(iv)⇒(i): We apply [24, Th. 6.1 (5) and remarks before]. ■

References

[1] Bierstone, E., Milman, P. (1987). Local analytic invariants and splitting theorems in differential analysis, *srael J. Math.*, **60**, 257–280.

[2] Bierstone, E., Milman, P. (1998). Geometric and differential properties of subanalytic sets, *Annals Math.*, **147**, 731–785.

[3] Bierstone, E., Schwarz, G. W. (1983). The extension problem and related themes in differential analysis, *Proc. Symp. Pure Math.*, **40**, 137–143.

[4] Castillo, J. M. F. (2000). Banach spaces, á la recherche du temps perdu, in: *III Congress on Banach Spaces (Jarandilla de la Vera, 1998)*, Extracta Math. **15**, 373–390.

[5] Castillo, J. M. F., González, M. (1997). *Three-space Problems in Banach Space Theory*, Lecture Notes Math. **1667**, Springer, Berlin.

[6] Castillo, J. M. F., González, M., Plichko, A. M., Yost, D. (2001). Twisted properties of Banach spaces, *Math. Scand.*, **89**, 217–244.

[7] Davis, W. J., Figiel, T., Johnson, W. B., Pełczyński, P. (1974). Factoring weakly compact operators, *J. Funct. Anal.*, **17**, 311–327.

[8] Diestel, J., Jarchow, H., Tonge, A. (1995). *Absolutely Summing Operators*, Cambridge Univ. Press, Cambridge.

[9] Diestel, J., Uhl, J. J. (1977). *Vector Measures*, Amer. Math. Soc., Providence.

[10] Domański, P. (1985). On the splitting of twisted sums and the three space problem for local convexity, *Studia Math.*, **82**, 155–189.

[11] Domański, P. (1987). *Extension and liftings of linear operators*, Preprint, Poznań.

[12] Domański, P. (1986). On the splitting of twisted sums and liftable operators, in: *Proc. Meeting on Probability and Banach Spaces, Zaragoza 1985, Spain*, Publ. Sem. Mat. Garcia de Galdeano Ser. II, Sec. 1, **85**, 39–45.

[13] Domański, P., Vogt, D. (1998). A splitting theorem for the space of smooth functions, *J. Funct. Anal.*, **153**, 203–248.

[14] Domański, P., Vogt, D. (2000). A splitting theory for the space of distributions, *Studia Math.*, **140**, 57–77.

[15] Domański, P., Vogt, D. (2000). Distributional complexes split for positive dimensions, *J. reine angew. Math.*, **522**, 63–79.

[16] Enflo, P., Lindenstrauss, J., Pisier, G. (1975). On the three space problem, *Math. Scand.*, **36**, 199–210.

[17] Frerick, L. (1996), A splitting theorem for nuclear Fréchet spaces, in: *Functional analysis, Proc. First International Workshop held at Trier University, Germany, 1994*, (S. Dierolf, S. Dineed, P. Domański, eds.), de Gruyter, Berlin, pp. 163–167.

[18] Frerick, L. (2001), *Extension Operators for Spaces of Infinitely Differentiable Whitney Functions*, Habilitationsschrift, Wuppertal.

[19] Frerick, L., Wengenroth, J. (1996). A sufficient condition for vanishing of the derived projective limit functor, *Arch. Math. (Basel)*, **67**, 296–301.

[20] Johnson, W. B. (1997). Extensions of c_0, *Positivity*, **1**, 55–74.

[21] Johnson, W. B., Oikhberg, T. (2001). Separable lifting property and extensions of local reflexivity, *Illinois J. Math.*, **45**, 123–137.

[22] Kalton, N. J. (1978). The three space problem for locally bounded F-spaces, *Comp. Math.*, **37**, 243–276.

[23] Kalton, N. J. (1981). Convexity, type and the three space problem, *Studia Math.*, **69**, 247–287.

[24] Kalton, N. J. (1984). Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p \leq 1$, *Math. Nachr.*, **115**, 71–97.

[25] Kalton, N. J., Peck, N. T. (1979). Twisted sums of sequence spaces and the three space problem, *Trans. Amer. Math. Soc.*, **255**, 1–30.

[26] Kalton, N. J., Pełczyński, A. (1997). Kernels of surjections from $\mathcal{L}_1$-spaces with an application to Sidon sets, *Math. Ann.*, **309**, 135–158.

[27] Kalton, N. J., Roberts, W. (1983). Uniformly exhaustive submeasures and nearly additive set functions, *Trans. Amer. Math. Soc.*, **278**, 803–815.

[28] Krone, J., Vogt, D. (1985). The splitting relation for Köthe spaces, *Math. Z.*, **190**, 387–400.

[29] Lindenstrauss, J. (1964). On certain subspaces of l_1, *Bull. Acad. Sci. Polon. Ser. Sci. Math.*, **12**, 539–542.

[30] Lindenstrauss, J. (1964). *Extension of compact operators*, *Mem. Amer. Math. Soc.*, **48**.

[31] Lindenstrauss, J., Tzafriri, L. (1973). *Classical Banach Spaces*, Lecture Notes Math. **338**, Springer, Berlin.

[32] Meise, R., Taylor, B. A., Vogt, D. (1990). Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse, *Ann. Inst. Fourier (Grenoble)*, **40**, 619–655.

[33] Meise, R., Vogt, D. (1997). *Introduction to Functional Analysis*, Clarendon Press, Oxford.

[34] Oikhberg, T., Rosenthal, H. P. (2001). Extension properties for the space of compact operators, *J. Funct. Anal.*, **179**, 251–308.

[35] Pietsch, A. (1978). *Operator Ideals*, DVW, Berlin.

[36] Poppenberg, M., Vogt, D. (1995). A tame splitting theorem for exact sequences of Fréchet spaces, *Math. Z.*, **219**, 141–161.

[37] Rosenthal, H. P. (2000). The complete separable extension property, *J. Operator Theory*, **43**, 329–374.

[38] Vogt, D. (1977). Charakterisierung der Unterräume von s, *Math. Z.*, **155**. 109–117.

[39] Vogt, D. (1977). Subspaces and quotient spaces of (s). Functional analysis: surveys and recent results, in: *Proc. Conf., Paderborn, 1976* (K. D. Bierstedt, B. Fuchssteiner, eds.), North-Holland Math. Studies **27**; Notas de Mat. **63**, North-Holland, Amsterdam, pp. 167–187.

[40] Vogt, D. (1981/82). Charakterisierung der Unterräume eines nuklearen stabilen Potenzreihenraumes von endlichem Typ, *Studia Math.*, **71**, 251–270.

[41] Vogt, D. (1987). On the functors $\mathrm{Ext}^1(E, F)$ for Fréchet spaces, *Studia Math.*, **85**, 163–197.

[42] Vogt, D. (1988). On the characterization of subspaces and quotient spaces of stable power series spaces of finite type, *Arch. Math. (Basel)*, **50**, 463–469.

[43] Vogt, D., Wagner, M. J. (1980). Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau, *Studia Math.*, **67**, 225–240.

[44] Vogt, D., Wagner, M. J. (1981). Charakterisierung der Unterräume und Quotientenräume der nuklearen stabilen Potenzreihenräume von unendlichem Typ, *Studia Math.*, **70**, 63–80.

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