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The three-space-problem for locally-m-convex algebras

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Abstract. We prove that a locally convex algebra A with jointly continuous multiplication is already locally-m-convex, if A contains a two-sided ideal I such that both I and the quotient algebra A/I are locally-m-convex. An application to the behaviour of the associated locally-m-convex topology on ideals is given.

El problema de los tres espacios para álgebras localmente-m-convexas

Resumen. Probamos que un álgebra localmente convexa con multiplicación continua es automáticamente localmente-m-convexa, si contiene un ideal bilátero I tal que tanto I como el álgebra cociente A/I son localmente-m-convexas. Se presenta una aplicación al comportamiento de la topología localmente-m-convexa asociada en los ideales.

1. Introduction

There are three reasonable possibilities to extend the classical notion of a normed algebra to algebras provided with a locally convex topology. In fact, let A be an associative real or complex algebra and $\mathscr{T}$ a locally convex topology on A. The weakest requirement on the compatibility of the algebraic and topological structures on A is

1. multiplication $m : (A, \mathscr{T}) \times (A, \mathscr{T}) \longrightarrow (A, \mathscr{T})$ is separately continuous.

   A stronger requirement would be

2. the above multiplication m is jointly continuous, or - equivalently - for every 0-nbhd U in $(A, \mathscr{T})$ there is another 0-nbhd V in $(A, \mathscr{T})$ satisfying

$$V^2 := \{xy : x, y \in V\} \subset U.$$

   The following requirement is still stronger

3. The 0-nbhd-filter in $(A, \mathscr{T})$ has a basis consisting of sets that are stable w.r. to multiplication, i.e. for each 0-nbhd U in $(A, \mathscr{T})$ there is a 0-nbhd V in $(A, \mathscr{T})$ satisfying $V^2 \subset V \subset U$.

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If $\mathscr{T}$ is metrizable and complete, (1) and (2) are equivalent. We will call an algebra A provided with a locally convex topology satisfying condition (2), a *locally convex algebra* ; if $(A, \mathscr{T})$ satisfies (3), we will call $(A, \mathscr{T})$ *locally-m-convex* (loc-m-conv.). Whereas it is easy to find algebras $(A, \mathscr{T})$ with (1) but without (2), the few complete metrizable locally convex algebras known to fail (3) are nontrivial and rather famous (see e.g. [1], [7], [8]).

Although many authors deal with algebras satisfying just condition (1), we consider condition (2) a very natural extension of normed algebras, which leads to a category with good properties.

On the other hand, loc-m-conv. algebras (which are nothing else but dense subalgebras of projective limits of normed algebras) are easier to handle. Therefore conditions are welcome, under which a locally convex algebra will "automatically" be already loc-m-conv. We will provide such a condition in form of a three-space- statement.

Notations. Let A be an algebra (always assumed to be associative and real or complex). For a subset C in A let $\Gamma\, C$ denote the convex balanced hull. Given $n \in \mathbb{N}$ and subsets $C_1, \ldots, C_n$ in A, we define $C_1 \cdots C_n := \{x_1 \cdots x_n : x_j \in C_j$ for all $1 \leq j \leq n\}$ and abbreviate $C^n := C \cdots C$ (n factors). It is easy to see that $C_1(\Gamma C_2) \cup (\Gamma C_1)C_2 \subset \Gamma(C_1C_2)$ and $\Gamma((\Gamma C_1) \cdot (\Gamma C_2)) = \Gamma(C_1C_2)$ for all $C_1, C_2 \subset A$.

2. The three-space-theorem

Theorem 1 *Let $(A, \mathscr{T})$ be a locally convex algebra containing a (two-sided) ideal I, such that I provided with the relative topology $\mathscr{T} \cap I$ and the quotient algebra A/I provided with the quotient topology $\mathscr{T}/I$ are both loc-m-conv. algebras. Then also $(A, \mathscr{T})$ is loc-m-convex.*

PROOF. Let $U = \Gamma\, U$ be a 0-nbhd in $(A, \mathscr{T})$. Then we find a 0-nbhd V in $(I, \mathscr{T} \cap I)$ such that $V^2 \subset V = \Gamma\, V \subset U \cap I$. As the restricted multiplication

$$(A, \mathscr{T}) \times (I, \mathscr{T} \cap I) \longrightarrow (I, \mathscr{T} \cap I)$$

is continuous, there are 0-nbhds $U_1 = \Gamma U_1$ in $(A, \mathscr{T})$ and $V_1 = \Gamma V_1$ in $(I, \mathscr{T} \cap I)$ such that $U_1V_1 \subset V$.

First we may assume that $V_1^2 \subset V_1 \subset V$; next we may assume that $U_1 \subset U$ and that $(U_1^2 + U_1) \cap I \subset V_1$ (observe that there is a 0-nbhd $\check{U} = \Gamma\, \check{U}$ in $(A, \mathscr{T})$ such that $\check{U} \cap I \subset V_1$ and we may choose U_1 so small that $U_1^2 \cup U_1 \subset \frac{1}{2}\, \check{U}$).

Let $q : A \to A/I$ denote the quotient map. As $(A/I, \mathscr{T}/I)$ is loc-m-conv., there is a 0-nbhd $W = \Gamma W$ in $(A/I, \mathscr{T}/I)$ satisfying $W^2 \subset W \subset q(U_1)$.

$U_2 := \frac{1}{2}(q^{-1}(W) \cap U_1)$ is a balanced convex 0-nbhd in $(A, \mathscr{T})$ satisfying $q(U_2) = \frac{1}{2}W$, hence

$$U_2 + I = q^{-1}(q(U_2)) = q^{-1}(\frac{1}{2}W) \text{ and}$$

$$U_2^2 \subset U_2^2 + I = q^{-1}(q(U_2^2)) = q^{-1}((q(U_2))^2) = q^{-1}(\frac{1}{4}W^2) \subset \frac{1}{2}q^{-1}(\frac{1}{2}W) = \frac{1}{2}U_2 + I.$$

As a consequence we obtain

$$\begin{aligned} U_2^2 &\subset \frac{1}{2}U_2 + I \cap (U_2^2 - \frac{1}{2}U_2) \subset \frac{1}{2}U_2 + I \cap ((\frac{1}{2}U_1)^2 + \frac{1}{2}U_1) \\ &= \frac{1}{2}U_2 + \frac{1}{2}I \cap (U_1^2 + U_1) \subset \frac{1}{2}U_2 + \frac{1}{2}V_1 \subset \Gamma(U_2 \cup V_1). \end{aligned}$$

Now we prove by induction that $U_2^n \subset \Gamma(U_2 \cup U_2V_1 \cup V_1)$ for all $n \in \mathbb{N}$. In fact, the case $n = 1$ is clear; assume that the inclusion is true for some $n \in \mathbb{N}$; then, by induction hypothesis,

$$\begin{aligned} U_2^{n+1} &= U_2U_2^n \subset U_2\Gamma(U_2 \cup U_2V_1 \cup V_1) \subset \Gamma(U_2^2 \cup U_2^2V_1 \cup U_2V_1) \\ &\subset \Gamma(U_2 \cup V_1 \cup \Gamma(U_2 \cup V_1)V_1 \cup U_2V_1) \subset \Gamma(U_2 \cup V_1 \cup U_2V_1 \cup V_1^2) \\ &\subset \Gamma(U_2 \cup U_2V_1 \cup V_1). \end{aligned}$$

Finally, the set $\tilde{U} := \Gamma\left(\bigcup_{n\in\mathbb{N}} U_2^n\right)$ is clearly a 0-nbhd in $(A, \mathscr{T})$ satisfying $\tilde{U}^2 \subset \tilde{U}$; moreover

$$\tilde{U} \subset \Gamma(U_2 \cup U_2V_1 \cup V_1) \subset \Gamma(U_2 \cup V) \subset \Gamma(U_1 \cup V) \subset \Gamma U \subset U,$$

which finishes the proof. ■

Remark 1 Without the assumption of $(A, \mathscr{T})$ having continuous multiplication, the 3-space-statement of the theorem becomes wrong, as has been shown in [4]. ■

3. An application

As local-m-convexity is stable under the formation of initial topologies w.r. to linear multiplicative maps, for every locally convex algebra $(A, \mathscr{T})$ there is a strongest loc-m-conv. topology $\mathscr{T}_m$ on A coarser that $\mathscr{T}$. The formation of this ”associated loc-m-convex topology” induces a functor from the category of locally convex algebras and linear multiplicative continuous maps into the category of loc-m-convex algebras and linear multiplicative continuous maps:

$$(A, \mathscr{T}) \rightsquigarrow (A, \mathscr{T}_m);$$

$$(A, \mathscr{T}) \xrightarrow{f} (B, \mathscr{S}) \ \rightsquigarrow\ (A, \mathscr{T}_m) \xrightarrow{f} (B, \mathscr{S}_m).$$

If $(A, \mathscr{T})$ is a locally convex algebra and $I \subset A$ an ideal, then $\mathscr{T}_m/I = (\mathscr{T}/I)_m$ (by a general device and also by immediate verification).

On the other hand, the functional property only yields that $(\mathscr{T} \cap I)_m \supset \mathscr{T}_m \cap I$. The following example shows that this inclusion may be strict.

Example 1 (see [5, 4.12])

Let $(A, \mathscr{T}) = (A, \cdot, \mathscr{T})$ be a locally convex algebra with unit element $e \neq 0$, and let A_{nil} denote the linear space A provided with 0-multiplication (so that $(A_{\text{nil}}, \mathscr{T})$ is even loc-m-convex). The maps

$$A \longrightarrow L(A_{\text{nil}}) := \{f : A \to A \text{ linear}\},$$
$$a \mapsto (b \mapsto ab) \text{ and } a \mapsto (b \mapsto b \cdot a),$$

respectively, satisfy the conditions of the second proposition in [4] for $B := (A, \cdot)$ and $C := A_{\text{nil}}$. Consequently by loc.cit., the corresponding semidirect product $E := A_{\text{nil}} \times_s (A, \cdot)$ is an algebra, which is a locally convex algebra w.r. to the product topology $\mathscr{S} := \mathscr{T} \times \mathscr{T}$. $I := A \times \{0\}$ is an ideal in E and has 0-multiplication, whence $(I, \mathscr{S} \cap I)$ is loc-m-convex, i.e. $(\mathscr{S} \cap I)_m = \mathscr{S} \cap I$. On the other hand, we will show that $\mathscr{S}_m = \mathscr{T}_m \times \mathscr{T}_m$.

In fact, $\mathscr{T}_m \times \mathscr{T}_m$ is loc-m-convex by loc. cit, hence $\mathscr{T}_m \times \mathscr{T}_m \subset \mathscr{S}_m$.
Conversely, let p be a submultiplicative continuous seminorm on $(E, \mathscr{S})$, then $p(a, 0) = p((e, 0)(0, a)) \leq p(e, 0)p(0, a)$ for all $a \in A$. Thus $p(\cdot, 0)$ is dominated by $p(0, \cdot)$ which is a submultiplicative continuous seminorm on $(A, \cdot, \mathscr{T})$, hence $\mathscr{T}_m$-continuous. From this we obtain that $\mathscr{S}_m \cap I = (\mathscr{T}_m \times \mathscr{T}_m) \cap I$.
Since $(E/I, \mathscr{S}_m/I) = (E/I, (\mathscr{S}/I)_m)$ is canonically topologically algebra-isomorphic to $(A, \mathscr{T}_m)$, we obtain that $\mathscr{S}_m/I = (\mathscr{T}_m \times \mathscr{T}_m)/I$.

Now, [3, lemma 1] yields $\mathscr{S}_m = \mathscr{T}_m \times \mathscr{T}_m$. In particular we have

$$(\mathscr{S} \cap I)_m = \mathscr{S}_m \cap I \iff \mathscr{T} = \mathscr{T}_m.$$

So any locally convex unital algebra $(A, \mathscr{T})$ that fails to be loc-m-convex, provides a counter example of the announced kind. ■

In contrast to this example we have the following

Proposition 1 *Let $(A, \mathscr{T})$ be a locally convex algebra containing an ideal I such that $(A/I, \mathscr{T}/I)$ is loc-m-convex. Then $\mathscr{T}_m \cap I = (\mathscr{T} \cap I)_m$.*

PROOF. The set

$$\{\Gamma(U \cup V) : U \text{ a } 0-\text{nbhd in } (A, \mathscr{T}), V \text{ a } 0-\text{nbhd in } (I, (\mathscr{T} \cap I)_m)\}$$

is a 0-basis for a locally convex topology $\mathscr{R}$ on A, satisfying $\mathscr{R} \cap I = (\mathscr{T} \cap I)_m$ and $\mathscr{R}/I = \mathscr{T}/I$; in fact, $\mathscr{R}$ is the strongest locally convex topology on A, satisfying $\mathscr{R} \subset \mathscr{T}$ and $\mathscr{R} \cap I \subset (\mathscr{T} \cap I)_m$ (cf. [2], [6]).

$(A, \mathscr{R})$ is even a locally convex algebra. In order to show this, let 0-nbhds $U = \Gamma U$ in $(A, \mathscr{T})$ and $V = \Gamma V$ in $(I, (\mathscr{T} \cap I)_m)$ be given; we may assume that $V^2 \subset V$. By continuity of multiplication there are 0-nbhds $U_1 = \Gamma U_1 \subset U$ in $(A, \mathscr{T})$ and $V_1 = \Gamma V_1 \subset V$ in $(I, \mathscr{T} \cap I)$ such that $U_1 V_1 \cup V_1 U_1 \subset V$, where we may assume that $U_1^2 \subset U$.

As $V_1 \subset V$ and $V^m \subset V$ for all $m \in \mathbb{N}$, we obtain that $U_1 V_1^n \cup V_1^n U_1 \subset V$ for all $n \in \mathbb{N}$. Therefore the 0-nbhd

$$\tilde{V} := \Gamma \bigcup_{n \in \mathbb{N}} V_1^n \text{ in } (I, (\mathscr{T} \cap I)_m)$$

(observe that $\tilde{V}^2 \subset \tilde{V}$ and $\tilde{V}$ is a 0-nbhd in $(I, \mathscr{T} \cap I)$) satisfies $\tilde{V} U_1 \cup U_1 \tilde{V} \subset V$.

Now $W := \Gamma(U_1 \cup \tilde{V})$ is a 0-nbhd in $(A, \mathscr{R})$ and $W^2 \subset \Gamma(U_1^2 \cup U_1 \tilde{V} \cup \tilde{V} U_1 \cup \tilde{V}^2) \subset \Gamma(U \cup V)$.

From the theorem we now obtain that $(A, \mathscr{R})$ is even loc-m-convex, hence $\mathscr{R} \subset \mathscr{T}_m$, which implies

$$(\mathscr{T} \cap I)_m = \mathscr{R} \cap I \subset \mathscr{T}_m \cap I \subset (\mathscr{T} \cap I)_m.$$

■

Remark 2 A different, easier sufficient condition for $(\mathscr{T} \cap I)_m = \mathscr{T}_m \cap I$ would be the hypothesis that $(\mathscr{T} \cap I)_m$ is the initial topology on I w.r. to a family of multiplicative linear surjections from I onto loc-m-convex algebras with unit element.

In fact, given a locally convex algebra $(A, \mathscr{T})$, containing an ideal I, a loc-m-convex algebra $(B, \mathscr{S})$ with unit $e \neq 0$ and a linear multiplicative continuous surjection $f : (I, (\mathscr{T} \cap I)_m) \longrightarrow (B, \mathscr{S})$, there is $y \in I$ such that $f(y) = e$. Define $\tilde{f} : A \longrightarrow B$, $x \mapsto f(xy)$. It is easy to see that $\tilde{f}$ is a linear multiplicative $\mathscr{T}$-$\mathscr{S}$-continuous extension of f, uniquely determined by f.

If $(\mathscr{T} \cap I)_m$ is the initial topology on I w.r. to $(f_c : I \longrightarrow (B_c, \mathscr{S}_c))_{c \in J}$ as above, then the initial topology $\mathscr{S}$ on A w.r. to $(\tilde{f}_c : A \longrightarrow (B_c, \mathscr{S}_c))_{c \in J}$ is loc-m-conv., $\mathscr{S} \subset J$, $\mathscr{S} \cap I = (\mathscr{T} \cap I)_m$, hence $\mathscr{S} \subset \mathscr{T}_m$ and $(\mathscr{T} \cap I)_m = \mathscr{S} \cap I \subset \mathscr{T}_m \cap I \subset (\mathscr{T} \cap I)_m$. ■

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