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Perturbation results for the local Phragmén-Lindelöf condition and stable homogeneous polynomials

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Dedicated to the memory of Klaus Floret

Abstract. The local Phragmén-Lindelöf condition for analytic varieties in complex n-space was introduced by Hörmander and plays an important role in various areas of analysis. Recently, new necessary geometric properties for a variety satisfying this condition were derived by the present authors. These results are now applied to investigate the homogeneous polynomials P with real coefficients that are stable in the following sense: Whenever f is a holomorphic function that is defined in some neighborhood of the origin, is real over real points, and has P as its localization at zero then the zero variety $V(f)$ of f satisfies the local Phragmén-Lindelöf condition at the origin. It is shown that P can only be stable if $V(P)$ satisfies the local Phragmén-Lindelöf condition at the origin and if, at each real point x in $V(P)$ of modulus 1, the localization of P at x is either linear or an indefinite quadratic form. Further, for polynomials P in three variables it is shown that these necessary conditions are also sufficient for the stability of P and therefore characterize the stable polynomials.

Resultados de perturbación para la condición local de Phragmén-Lindelöf y polinomios homogéneos estables

Resumen. La condición local de Phragmén-Lindelöf para variedades analíticas complejas n-dimensionales fue introducida por Hörmander y juega un papel importante en varias áreas del análisis. Recientemente los presentes autores han derivado nuevas propiedades geométricas que son necesarias para que la variedad satisfaga esta condición. Estos resultados se aplican ahora a investigar los polinomios homogéneos P con coeficientes reales que son estables en el siguiente sentido: Cuando f es una función holomorfa que está definida en un entorno del origen, es real en los puntos reales, y tiene a P como su localización en cero, entonces la cero-variedad $V(f)$ de f satisface la condición local de Phragmén-Lindelöf en el origen. Se prueba que P puede ser estable sólo si $V(P)$ satisface la condición local de Phragmén-Lindelöf en el origin y si, en cada punto real x en $V(P)$ de módulo 1, la localización de P en x es o bien lineal o bien una forma cuadrática indefinida. Además, para polinomios P de tres variables se muestra que estas condiciones necesarias son también suficientes para la estabilidad de P y, por tanto, caracterizan los polinomios estables.

1. Introduction

The local Phragmén-Lindelöf condition $\mathrm{PL}_{\mathrm{loc}}$ for analytic varieties in $\mathbb{C}^n$ (see Definition 3) was introduced by Hörmander [7] in his characterization of the linear partial differential operators $P(D)$ which are surjective on the space $\mathcal{A}(\mathbb{R}^n)$ of all real analytic functions on $\mathbb{R}^n$. In the intervening years $\mathrm{PL}_{\mathrm{loc}}$ has been shown

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to also be important in the study of other Phragmén-Lindelöf conditions, especially those concerned with the existence of continuous linear right inverses for linear constant coefficient partial differential operators $P(D)$. For example, it was shown in Meise, Taylor, and Vogt [9] that a homogeneous partial differential operator $P(D)$ admits a continuous linear right inverse on $\mathcal{E}(\mathbb{R}^n)$ and/or $\mathcal{D}'(\mathbb{R}^n)$ if and only if its zero variety satisfies the local Phragmén-Lindelöf condition at the origin. In other work of the present authors ([8], [5]), it was shown that $\mathrm{PL}_{\mathrm{loc}}$ at each real point of an algebraic variety V is a necessary condition in order that V satisfies (SPL), the most natural extension of the classical Phragmén-Lindelöf theorem to algebraic varieties. Moreover, Vogt [12] has recently shown that the local Phragmén-Lindelöf condition characterizes the existence of continuous linear extension operators for real analytic functions defined on compact real analytic subvarieties of $\mathbb{R}^n$.

In the recent article [4] new necessary conditions for $\mathrm{PL}_{\mathrm{loc}}$ were obtained for analytic varieties in $\mathbb{C}^n$ and it was shown that these conditions are sufficient for surfaces. In the present paper we use these results to investigate when a homogeneous polynomial $P \in \mathbb{R}[z_1, \dots, z_n]$ has the following property: Whenever f is a holomorphic function on some neighborhood of the origin in $\mathbb{C}^n$ that is real over real points and has P as the lowest degree homogeneous polynomial in its Taylor series expansion at zero, then the zero variety $V(f)$ of f satisfies the condition $\mathrm{PL}_{\mathrm{loc}}(0)$, i.e., $\mathrm{PL}_{\mathrm{loc}}$ at the origin. We call such polynomials *stable*. In [4] it was proved that each homogeneous polynomial $P \in \mathbb{R}[z_1, \dots, z_n]$ for which $V(P)$ has no real singularities outside the origin (i.e., P is of principal type) and for which $V(P)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ is stable. We show here that this condition is only sufficient and that $V(P)$ may have singular points outside the origin. Our main result is the following theorem.

Theorem 1 *For $n \geq 3$ let $P \in \mathbb{R}[z_1, \dots, z_n]$ be homogeneous of degree $m \geq 1$. If P is stable then $V(P)$ satisfies $PL_{\mathrm{loc}}(0)$ and for each $\xi \in V(P) \cap S^{n-1}$ the localization P_ξ of P at ξ (see Definition 1) satisfies the following two conditions:*

(a) $\deg P_\xi \leq 2$

(b) If $\deg P_\xi = 2$ *then* P_ξ *is an indefinite quadratic form.*

For $n = 3$ these conditions are also sufficient. □

The proof of the necessity of the conditions (a) and (b) in Theorem 1 is based on the necessary conditions for $\mathrm{PL}_{\mathrm{loc}}$ derived in [4]. To prove that they are sufficient for $n = 3$ we show that the sufficient conditions stated in [4] are fulfilled. More precisely, we show that around each real half line generated by $x \in V(P) \cap S^2$ there exists a complex truncated cone Γ so that $V(P) \cap \Gamma$ satisfies a hyperbolicity condition. To achieve this when x is a singular point of $V(P)$, we use condition (b) to find a real analytic transformation that allows the reduction of the general case to a more special one which is then handled by methods which were developed in [4].

To illustrate the main result, we provide several examples in Section 4. Example 2 shows that there are stable polynomials of any degree for which the zero variety has singularities outside the origin. Also we show in Example 3 that the polynomial $P(z_1, z_2, z_3, z_4) := z_1^2 + z_2^2 - z_3^2$ is stable. Hence, it might be that the conditions in Theorem 1 are sufficient for any dimension $n \geq 3$.

2. Preliminaries

In this preliminary section we introduce the basic definitions and notation which will be used subsequently.

Throughout the paper, $|\cdot|$ denotes the Euclidean norm on $\mathbb{C}^n$, while $B^n(\xi, r)$ denotes the ball of center ξ and radius r in $\mathbb{C}^n$. When $n = 1$ we write $B(\xi, r)$ instead of $B^1(\xi, r)$.

Definition 1 *Let $f \colon B^n(\xi, r) \to \mathbb{C}$ be a holomorphic function. Then*

$$V(f) := \{z \in B^n(\xi, r) : f(z) = 0\}$$

will be called the zero variety *of the function* f.

For $\theta \in B^n(\xi, r)$ *the* localization f_θ *of* f *at* θ *is defined as the lowest degree homogeneous term in the Taylor series expansion of* f *at* θ.

The tangent cone $T_\theta V(f)$ *of the variety* $V(f)$ *at* θ *coincides with* $V(f_\theta)$ *(see Chirka [6], 8.4 Proposition 1).*

An analytic variety V *in* $\mathbb{C}^n$ *is defined to be a closed analytic subset of some open set in* $\mathbb{C}^n$ *(see Chirka [6], 2.1). By* V_{sing} *(resp.* V_{reg}*) we denote the set of all singular (resp. regular) points in* V.

Definition 2 *Let* V *be an analytic variety in* $\mathbb{C}^n$ *and let* Ω *be an open subset of* V. *A function* $u : \Omega \to [-\infty, \infty[$ *is called* plurisubharmonic *if it is locally bounded above, plurisubharmonic in the usual sense on* Ω_{reg}*, the set of all regular points of* V *in* Ω*, and satisfies*

$$u(z) = \limsup_{\zeta \in \Omega_{\text{reg}}, \zeta \to z} u(\zeta)$$

at the singular points of V *in* Ω. *By* $\text{PSH}(\Omega)$ *we denote the set of all plurisubharmonic functions on* Ω.

Definition 3 *For* $\xi \in \mathbb{R}^n$ *and* $r_0 > 0$ *let* V *be an analytic variety in* $B^n(\xi, r_0)$ *which contains* ξ. *We say that* V *satisfies the condition* $\text{PL}_{\text{loc}}(\xi)$ *if there exist positive numbers* A *and* $r_0 \geq r_1 \geq r_2$ *such that each* $u \in \text{PSH}(V \cap B^n(\xi, r_1))$ *satisfying*

(α) $u(z) \leq 1, \ z \in V \cap B^n(\xi, r_1)$ *and*

(β) $u(z) \leq 0, \ z \in V \cap \mathbb{R}^n \cap B^n(\xi, r_1)$

also satisfies

(γ) $u(z) \leq A|\text{Im}\, z|, \ z \in V \cap B^n(\xi, r_2)$.

For various equivalent conditions for $\text{PL}_{\text{loc}}(\xi)$ we refer to [4], Lemma 3.3.

Definition 4 *A* simple curve γ *in* $\mathbb{C}^n$ *is a map* $\gamma \colon]0, \alpha[\to \mathbb{C}^n$ *which for some* $\alpha > 0$ *and some* $q \in \mathbb{N}$ *admits a convergent Puiseux series expansion*

$$\gamma(t) = \sum_{j=q}^{\infty} \xi_j t^{j/q} \text{ with } |\xi_q| = 1.$$

Then ξ_q *is the* tangent vector *to* γ *at the origin. The trace of* γ *is defined as* $\text{tr}(\gamma) := \gamma(]0, \alpha[)$. *A* real simple curve *is a simple curve* γ *satisfying* $\text{tr}(\gamma) \subset \mathbb{R}^n$.

Definition 5 *Let* $V \subset \mathbb{C}^n$ *be an analytic variety of pure dimension* $k \geq 1$ *which contains the origin, let* $\gamma :]0, \alpha[\to \mathbb{C}^n$ *be a simple curve, and let* $d \geq 1$. *Then for* $t \in]0, \alpha[$ *we define*

$$V_{\gamma,t,d} := \{w \in \mathbb{C}^n : \gamma(t) + wt^d \in V\} = \frac{1}{t^d}(V - \gamma(t))$$

and we define the limit variety $T_{\gamma,d}V$ *of* V *of order* d *along* γ *as the set*

$$T_{\gamma,d}V := \{\zeta \in \mathbb{C}^n : \zeta = \lim_{j \to \infty} z_j, \text{ where } z_j \in V_{\gamma,t_j,d} \text{ for } j \in \mathbb{N} \text{ and } (t_j)_{j \in \mathbb{N}} \text{ is a null-sequence in }]0, \alpha[\}.$$

If it is clear from the context we will sometimes write $V_{t,d}$ *or just* V_t *instead of* $V_{\gamma,t,d}$.

From [3], Theorem 3.2 and Proposition 4.1 we recall the following results.

Theorem 2 *Let V be an analytic variety of pure dimension $k \geq 1$ containing the origin, let γ be a simple curve in $\mathbb{C}^n$ with tangent vector ξ at the origin, and let $d \geq 1$ be given. Then the following assertions hold:*

(a) $T_{\gamma,d}V$ *is either empty or an algebraic variety of pure dimension k.*

(b) $T_{\gamma,1}V = T_0 V - \xi$.

(c) If $d > 1$ then $w \in T_{\gamma,d}V$ if and only if $w + \lambda\xi \in T_{\gamma,d}V$ for each $\lambda \in \mathbb{C}$.

We will also need the following definitions in Section 3.

Definition 6 *Let V be an analytic variety in $\mathbb{C}^n$ which is of pure dimension $k \geq 1$ at $\zeta \in V$. A projection $\pi : \mathbb{C}^n \to \mathbb{C}^n$ is called* noncharacteristic *for V at ζ if its rank is k, its image and its kernel are spanned by real vectors, and $T_\zeta V \cap \ker \pi = \{0\}$.*

Definition 7 *Let $\gamma\colon]0, \alpha[\to \mathbb{R}^n$ be a real simple curve, let $d \geq 1$, let U be a subset of $\mathbb{C}^n$, and let $0 < R \leq \alpha$ be given. We call*

$$\Gamma(\gamma, d, U, R) := \bigcup_{0<t<R} (\gamma(t) + t^d U)$$

the conoid *with core γ, opening exponent d, and profile U, truncated at R, provided that the origin does not belong to this set.*

Definition 8 *Let V be an analytic variety of pure dimension k in $\mathbb{C}^n$ which contains the origin, let γ be a real simple curve, let $d \geq 1$, and let $\zeta \in T_{\gamma,d}V \cap \mathbb{R}^n$. We say that V is* (γ, d)-hyperbolic at ζ *with respect to a projection $\pi\colon \mathbb{C}^n \to \mathbb{C}^n$, which is noncharacteristic for $T_{\gamma,d}V$ at ζ, if there exist a zero neighborhood U in $\mathbb{C}^n$ and $r > 0$ such that $z \in V \cap \Gamma(\gamma, d, \zeta + U, r)$ is real whenever $\pi(z)$ is real. V is called (γ, d)-hyperbolic at ζ if it is (γ, d)-hyperbolic at ζ with respect to some projection π as above.*

Definition 9 *Let V be an analytic variety of pure dimension k in $\mathbb{C}^n$ and let $\xi \in V \cap \mathbb{R}^n$. We say that V is* locally hyperbolic *at ξ if there are a neighborhood U of ξ and a projection $\pi\colon \mathbb{C}^n \to \mathbb{C}^n$ which is noncharacteristic for V at ξ such that $z \in V \cap U$ is real whenever $\pi(z)$ is real.*

By Hörmander [7], 6.5, local hyperbolicity implies $\mathrm{PL}_{\mathrm{loc}}$; however, only for $n = 2$ and $n = 3$ it is equivalent to $\mathrm{PL}_{\mathrm{loc}}$.

3. Results

In this section we state and prove the main results of this paper. Throughout this section the dimension n is assumed to satisfy $n \geq 3$.

Definition 10 *A polynomial $P \in \mathbb{R}[z_1, \ldots, z_n]$ is called* stable *if P is homogeneous of degree $m \geq 1$ and if for each holomorphic function $f\colon B^n(0, r) \to \mathbb{C}$ which is real over $B^n(0, r) \cap \mathbb{R}^n$ and for which P is the localization of f at the origin the variety $V(f)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$.*

If f is a holomorphic function on $B^n(0, r)$ for which $V(f)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ then [4], Lemma 3.18, implies that, up to a complex constant factor, f is real over real points in $B^n(0, r)$. Therefore the requirements in Definition 10 are reasonable.

To derive necessary conditions for a polynomial P be to be stable, we will use the following lemma.

Lemma 1 *Let $P \in \mathbb{R}[z_1, \ldots, z_n]$ be homogeneous of degree 3. Then there exists a real linear change of variables such that in these variables $P(z) = \sum_{|\alpha|=3} a_\alpha z^\alpha$, where $a_{(3,0,\ldots,0)} = 0$ and $a_{(0,3,0,\ldots,0)} \neq 0$.*

PROOF. Since P does not vanish identically, we can choose $\xi_1, \ldots, \xi_n \in \mathbb{R}^n$, linearly independent, so that $P(\xi_j) = 1$ for $2 \leq j \leq n$ and $P(\xi_1) = 0$. With respect to the basis $(\xi_1, \ldots, \xi_n)$ the polynomial P has the desired representation. ■

The next lemma is crucial.

Lemma 2 *If $P \in \mathbb{R}[z_1, \ldots, z_n]$ is stable, then the following conditions are satisfied for each $\xi \in V(P) \cap S^{n-1}$:*

(a) $V(P_\xi)$ *satisfies* $\mathrm{PL}_{\mathrm{loc}}(0)$

(b) $\deg P_\xi \leq 2$.

PROOF. (a) This follows from [3], Corollary 6.3 and [4], Proposition 3.5.
(b) To prove (b), we first assume that $\deg P_\xi = \nu \geq 4$. It is no restriction to assume that $\xi = (0, \ldots, 0, 1)$. Expand P in the form

$$P(z', z_n) = \sum_{k=\nu}^{m} p_k(z') z_n^{m-k}, \tag{1}$$

where the polynomials p_k are either homogeneous of degree k or identically zero and where $p_\nu \not\equiv 0$. Then $p_\nu(z') = P_\xi(z)$ (see, e.g., [1], Lemma 3.9). Now let $a := m + \nu - 3$, $b := m + \nu - 6$, and define

$$Q(z', z_n) := z_1^2 z_n^a - z_2^3 z_n^b.$$

Since $\nu \geq 4$, the degree of Q is at least $m + 1$. Hence, P is the localization at the origin of the function $f := P + Q$. Since f is real over real points and since P is stable, the variety $V(f)$ must satisfy $\mathrm{PL}_{\mathrm{loc}}(0)$. To derive a contradiction from this, note that by [4], Proposition 3.5, each limit variety $T_{\gamma,d}V(f)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\eta)$ at each real point $\eta \in T_{\gamma,d}V(f)$. Now define $\gamma(t) := t\xi$, $t > 0$, and let $d = 3$. Then it follows from a direct calculation of $V_{\gamma,t,d}$ in Definition 5 (or [3], Lemma 6.1), that

$$T_{\gamma,d}V(f) = \{z \in \mathbb{C}^n : z_1^2 - z_2^3 = 0\}.$$

Obviously, this variety is the product of $\mathbb{C}^{n-2}$ and W, where

$$W := \{(x, y) \in \mathbb{C}^2 : x^2 - y^3 = 0\}.$$

From this it follows easily that W has to satisfy $\mathrm{PL}_{\mathrm{loc}}(0)$. However, this is not the case by Hörmander [7], Theorem 6.5. In this particular case, this can also be verified directly: If W is parametrized by $x = t^3$, $y = t^2$, then the function $|\mathrm{Im}\, t| = |\mathrm{Im}\, x/y|$ vanishes at the real points, but is not $O(|\mathrm{Im}(x, y)|)$ in any neighborhood of the origin. Thus, a contradiction has been reached from the assumption $\deg P_\xi \geq 4$. Hence we have $\deg P_\xi \leq 3$.

To complete the proof of (b) we now show that also the assumption $\deg P_\xi = 3$ leads to a contradiction. To do so we argue as before and expand

$$P(z', z_n) = p_3(z') z_n^{m-3} + \ldots + p_m(z'),$$

where $p_3(z') = P_\xi(z)$. By Lemma 1 we can perform a real linear change of coordinates in $\mathbb{C}^{n-1}$ so that

$$p_3(z') = \sum_{|\alpha|=3} a_\alpha (z')^\alpha, \text{where } a_{(3,0,\ldots,0)} = 0,\ a_{(0,3,0,\ldots,0)} \neq 0.$$

Then let $Q(z', z_n) := z_1^2 z_n^{m-1}$ and define $f := P + Q$. As above, the hypothesis implies that $V(f)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Next define $\gamma(t) := t\xi$, $t > 0$, and $d := 2$. Then [3], Lemma 6.1 implies that

$$T_{\gamma,d}V(f) = \{z \in \mathbb{C}^n : p_3(z') + z_1^2 = 0\}.$$

Since $V(f)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ and since $0 \in T_{\gamma,d}V(f)$, [4], Proposition 3.5, implies that $T_{\gamma,d}V(f)$satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Since

$$T_{\gamma,d}V(f) = \{z' \in \mathbb{C}^{n-1} : p_3(z') + z_1^2 = 0\} \times \mathbb{C} =: W \times \mathbb{C},$$

W also satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Now note that T_0W is the zero set of the localization at the origin of the polynomial defining W, which shows

$$T_0W = \{z' \in \mathbb{C}^{n-1} : z_1^2 = 0\}.$$

Hence T_0W is a complex manifold. Therefore, [4], Proposition 3.12, implies the following: Whenever η is in $T_0W \cap S^{n-2}$ and $\gamma_\eta : t \mapsto t\eta$, then for each projection π in $\mathbb{C}^{n-1}$ which is noncharacteristic for $T_{\gamma_\eta,1}W = T_0W - \eta$ at zero, W is $(\gamma_\eta, 1)$-hyperbolic at 0.

To show that this hyperbolicity condition does not hold for all $\eta \in T_0W \cap S^{n-2}$, note that our assumption on p_3 implies the following expansion, where $z' = (z_1, z'')$:

$$p_3(z') = z_1^2 q_1(z'') + z_1 q_2(z'') + q_3(z'').$$

Here the polynomials q_j are either homogeneous of degree j, $1 \leq j \leq 3$, or identically zero. Since $a_{(0,3,0,\ldots,0)} \neq 0$ we must have $q_3 \not\equiv 0$. Therefore we can choose η'' so that $q_3(\eta'') > 0$. Then let $\eta := (0, \eta'')$ and note that $\pi : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$, $\pi(z') := (0, z'')$, is noncharacteristic for $T_{\gamma_\eta,1}W$ at 0 since $\ker \pi = \{(\lambda, 0, \ldots, 0) \in \mathbb{C}^{n-1} : \lambda \in \mathbb{C}\}$, while $T_0(T_{\gamma_\eta,1}W) = T_0(T_0W - \eta) = T_0W$ and hence $\ker \pi \cap T_0(T_{\gamma_\eta,1}W) = \{0\}$. Therefore our assumption implies that there exists an open zero neighborhood G such that $z' \in W \cap \Gamma(\gamma_\eta, 1, G, r)$ is real whenever $\pi(z')$ real.

To show that this does not hold, note that the points z' in W for which $\pi(z') = \gamma_\eta(t)$ satisfy the equation

$$0 = p_3(z') + z_1^2 = (1 + q_1(t\eta''))z_1^2 + q_2(t\eta'')z_1 + q_3(t\eta'').$$

The discriminant of this quadratic equation is

$$D(t) = q_2(t\eta'')^2 - 4(1 + q_1(t\eta''))q_3(t\eta'').$$

Since $q_3(t\eta'') = t^3 q_3(\eta'') > 0$ and since $q_2(t\eta'')^2$ is either identically zero or homogeneous in t of degree four, it follows that for small t the discriminant $D(t)$ is negative. Hence the two roots of the equation are not real. Since one can solve the equation explicitly one can check that the roots are inside the cone $\Gamma(\gamma_\eta, 1, G, r)$ when t is small enough. This shows that W is not $(\gamma_\eta, 1)$-hyperbolic in contradiction to our assumption. Hence we must have $\deg P_\xi \leq 2$. ■

Lemma 3 *Let $P \in \mathbb{R}[z_1, \ldots, z_n]$ be stable of degree $m \geq 2$. If $\deg P_\xi = 2$ for some $\xi \in V(P) \cap S^{n-1}$ then P_ξ is an indefinite real quadratic form.*

PROOF. After a real linear change of variables we may assume $\xi = (0, \ldots, 0, 1)$ and we can expand P as in (1), where now $p_2(z') = P_\xi(z', z_n)$. By Lemma 2 (a), $V(P_\xi)$ and hence $V(p_2)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Since p_2 is homogeneous of degree 2 by hypothesis, it follows from Meise, Taylor, and Vogt [11], Theorem 3.3, that $V(p_2)$ satisfies $\mathrm{PL}(\mathbb{R}^{n-1})$. Therefore it follows from Meise, Taylor, and Vogt [10], Lemma 3 and Proposition 2, that p_2 is either an indefinite real quadratic form or $p_2 = p_1^2$ for some real linear form p_1. To complete the proof we show that the latter case cannot occur.

Note that so far we have only used the fact that $V(P)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Now we will use the stability hypothesis. If we assume that $p_2 = p_1^2$ for some linear form p_1 then it is no restriction to assume $p_1(z') = z_1$, and hence

$$P(z', z_n) = z_1^2 z_n^{m-2} + p_3(z')z_n^{m-3} + \ldots + p_m(z').$$

Next let $Q(z', z_n) := z_n^{m+1}$ and define $f := P + Q$. Then $V(f)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ since P is stable.

To show that this is not the case assume first that $m = 2$. Then $f(z) = z_1^2 + z_n^3$ and $V(f)$ does not satisfy $\mathrm{PL}_{\mathrm{loc}}(0)$ by Hörmander [7], Theorem 6.5, or by the direct argument given in the proof of Lemma 2.

To treat the case $m \geq 3$, define the real simple curve γ by $\gamma(t) := (0, \ldots, 0, t)$ and let $d := \frac{4}{3}$. Then note that by [3], Lemma 6.1, we have

$$T_{\gamma,d}V(f) = \{z \in \mathbb{C}^n : z_1^2 = 0\}.$$

Hence $T_{\gamma,d}V(f)$ is a complex manifold. Therefore it follows from [4], Proposition 3.12, that $V(f)$ must be (γ, d)-hyperbolic at $0 \in T_{\gamma,d}V(f)$. To show that this is not the case, define $\pi : \mathbb{C}^n \to \mathbb{C}^n$, $\pi(z) := (0, z_2, \ldots, z_n)$. Then π is noncharacteristic for $T_{\gamma,d}V(f)$ at 0, since $T_{\gamma,d}V(f)$ is its own tangent cone. Hence there exist a zero neighborhood U in $\mathbb{C}^3$ and $r > 0$ such that for each $z \in V(f) \cap \Gamma(\gamma, d, U, r)$ with $\pi(z)$ real, z must be real. To show that this does not hold, consider the equation

$$0 = f(z_1, 0, \ldots, 0, t) = z_1^2 t^{m-2} + t^{m+1} + \sum_{j=3}^{m} b_j z_1^j t^{m-j},$$

where $b_j := p_j(1, 0, \ldots, 0)$, $3 \leq j \leq m$. For $0 < t < r$ let $z_1(t, \lambda) = it^{3/2} + \lambda$, where $\lambda \in \mathbb{C}$ satisfies $|\lambda| = t^{3/2}/4$. Then

$$|(z_1(t,\lambda) - t^3)t^{m-2}| \geq \frac{1}{4}t^{3/2}(2 - \frac{1}{4})t^{3/2}t^{m-2} = \frac{7}{16}t^{m+1}$$

and

$$\Big|\sum_{j=3}^{m} b_j z_1(t,\lambda)^j t^{m-j}\Big| \leq \sum_{j=3}^{m} |b_j| \frac{5}{4} t^{3j/2} t^{m-j} \leq A t^{m+1+\frac{1}{2}}$$

for a suitable constant A. Hence there exists $0 < t_0 < r$ such that $At^{1/2} < 7/16$ for $0 < t \leq t_0$. Therefore the Theorem of Rouché implies that for $0 < t \leq t_0$ there exists $z_1(t)$ satisfying $f(z_1(t), 0, \ldots, 0, t) = 0$ and $|z_1(t) - it^{3/2}| < t^{3/2}/4$. Because of this estimate we have $\operatorname{Im} z_1(t) \neq 0$. Since $d = 4/3$, it is easy to check that for sufficiently small $0 < t < t_0$ the point $w(t) := (z_1(t), 0, \ldots, 0, t)$ belongs to $V(f) \cap \Gamma(\gamma, d, U, r)$ and $\pi(w(t))$ is real, while $w(t)$ is not real, in contradiction to the (γ, d)-hyperbolicity of $V(f)$ at $0 \in T_{\gamma,d}V(f)$. ■

To derive the necessary conditions for stability in Lemma 2 and Lemma 3 we did not use the full strength of Definition 10, since the only holomorphic functions which we needed were polynomials.

The following sufficient condition for stability was proved already in [4], Proposition 7.4. Because of Euler's rule and the fact that a homogeneous polynomial P does not have any elliptic factors if $V(P)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$, we can reformulate this result as follows.

Proposition 1 *Let $P \in \mathbb{R}[z_1, \ldots, z_n]$ be homogeneous of degree $m \geq 1$. If $V(P)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ and if $\deg P_\xi = 1$ for each $\xi \in V(P) \cap \mathbb{R}^n$, then P is stable.* ■

To show that the necessary conditions for stability which we derived so far are sufficient when $n = 3$, we need the following lemmas.

Lemma 4 *Let $f \colon B^n(0, r) \to \mathbb{C}$ be a holomorphic function, and let $k \in \mathbb{N}$ satisfy $k < n$. Group the variables as $z = (s, w, \tau) \in \mathbb{C}^k \times \mathbb{C}^{n-k-1} \times \mathbb{C}$ and assume that*

$$f(z) = \sum_{l=0}^{\infty} \tau^l f_l(s, w) \quad \text{for } f_0(s, w) = \sum_{j=2}^{\infty} p_j(s, w),$$

where p_j is either zero or homogeneous of degree j. Assume furthermore that

$$p_2(s,w) = Q(s) := \sum_{i=1}^{k} \lambda_i z_i^2, \quad \lambda_i \neq 0 \text{ for } 1 \le i \le k.$$

Then there exist $\epsilon > 0$, a neighborhood U of zero in $\mathbb{C}^{n-k}$, and holomorphic maps $h\colon U \to \mathbb{C}^k$, $G\colon U \to \mathbb{C}$, and $R\colon B^k(0,\epsilon) \times U \to \mathbb{C}$ such that $h(0) = 0$, $G(0) = 0$,

$$f(s + h(w,\tau), w, \tau) = Q(s) + G(w,\tau) + R(s,w,\tau) \quad \text{for } |s| < \epsilon \text{ and } (w,\tau) \in U, \tag{2}$$

and $|R(s,w,\tau)| = O(|s|^2(|s| + |w| + |\tau|))$. Furthermore:

(a) If all Taylor coefficients of f are real, then the same holds for h and G.

(b) If Q is the localization of f at 0, then $|h(w,\tau)| = O(|(w,\tau)|^2)$ and $|G(w,\tau)| = O(|(w,\tau)|^3)$.

PROOF. Denote by $\frac{\partial f}{\partial s}$ the vector $\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_k}\right)$. We claim the existence of a function h and a neighborhood U of zero satisfying

$$\frac{\partial f}{\partial s}(h(w,\tau), w, \tau) = 0 \quad \text{for } (w,\tau) \in U. \tag{3}$$

To prove this claim, note first that

$$\frac{\partial f}{\partial s}(s,0,0) = \sum_{j=2}^{\infty} \frac{\partial p_j}{\partial s}(s,0),$$

hence $\frac{\partial f}{\partial s}(0,0,0) = 0$. To verify the remaining hypothesis of the implicit function theorem we also have to check that the Jacobi matrix of $\frac{\partial f}{\partial s}$ with respect to the s-variables is invertible at the origin. It is easy to see that this matrix is the Hessian of $f(\cdot,0,0)$ at $s = 0$ and that this Hessian is the Hessian of Q, hence invertible by hypothesis. This proves the existence of h and U as in (3).

For sufficiently small ϵ define

$$G(w,\tau) = f(h(w,\tau), w, \tau) \quad \text{for } (w,\tau) \in U,$$
$$R(s,w,\tau) = f(s + h(w,\tau), w, \tau) - Q(s) - G(w,\tau) \quad \text{for } |s| < \epsilon \text{ and } (w,\tau) \in U.$$

Then (2) is obvious.

Define $F(s,w,\tau) := f(s + h(w,\tau), w, \tau)$ and insert (3) into its Taylor series expansion with respect to s for fixed $(w,\tau) \in U$:

$$F(s,w,\tau) = F(0,w,\tau) + \frac{\partial F}{\partial s}(0,w,\tau)s + O(|s|^2) = G(w,\tau) + O(|s|^2),$$

where the O-estimates are locally uniform in (w,τ). So far, we have proved that $|R(s,w,\tau)| = O(|s|^2)$. To improve this bound, consider

$$R(s,0,0) = f(s,0,0) - Q(s) = \sum_{j=3}^{\infty} p_j(s,0).$$

Hence $|R(s,0,0)| = O(|s|^3)$. Together with the estimate $|R(s,w,\tau)| = O(|s|^2)$ we have proved the assertion concerning R.

Claim (a) follows from the real implicit function theorem. To prove (b), note first that the implicit function theorem states that the entries of $\nabla h(0,0)$ are linear combinations of elements of the form

$$\frac{\partial^2 f}{\partial z_j \partial z_l}(0,0), \quad 1 \le j \le k < l \le n.$$

Since the localization of f at 0 is Q, all of these derivatives vanish, and it follows that $|h(w,\tau)| = O(|(w,\tau)|^2)$. To prove the second assertion, recall the definition of G:

$$G(w,\tau) = Q(h(w,\tau)) + \sum_{j=3}^{\infty} p_j(h(w,\tau),w) + \sum_{l=1}^{\infty} \tau^l f_l(h(w,\tau),w).$$

The first term is $O(|(w,\tau)|^4)$ since $|h(w,\tau)| = O(|(w,\tau)|^2)$. The second one is $O(|(w,\tau)|^3)$ since the degree of p_j is j. The last term vanishes of order at least 3 since the hypotheses of (b) imply that $f_2(0,0) = 0$ and that f_1 vanishes at least of order 2 at the origin. ■

Lemma 5 *Let $\rho > 0$, let $R\colon B(0,\rho)^3 \to \mathbb{C}$ be a holomorphic function which for some $C > 0$ satisfies the estimate*

$$|R(x,y,z)| \le C|(x,y)|^2\big(|(x,y)| + |z|\big), \quad (x,y,z) \in B(0,\rho)^3,$$

and define $g\colon B(0,\rho)^3 \to \mathbb{C}$ by

$$g(x,y,z) := x^2 - y^2 + R(x,y,z).$$

Then there exists $0 < \delta < \rho$ such that the map

$$\pi\colon V(g) \cap \big(B(0,\delta) \times B(0,\tfrac{5}{4}\delta) \times B(0,\delta)\big) \to B(0,\delta)^2, \quad \pi(x,y,z) = (x,z),$$

is a two sheeted branched cover with branch locus $\{(0,0)\} \times B(0,\delta)$. If R is real for real (x,y,z) then (x,y,z) is real when $\pi(x,y,z)$ is real.

PROOF. Choose $0 < \delta < \min(\rho,1)$ so small that

$$C\left(\tfrac{9}{4}\right)^2\left(\tfrac{9}{4}\delta + \delta\right) < \tfrac{7}{16}.$$

Then fix $(x,z) \in B(0,\delta)^2$ and $\lambda \in \mathbb{C}$ with $|\lambda| = \frac{5}{4}\delta$ and note that

$$|x^2 - \lambda^2| \ge \left(\tfrac{5}{4}\delta\right)^2 - \delta^2 = \left(\tfrac{3}{4}\delta\right)^2.$$

Hence the estimate for R and the choice of δ imply

$$|R(x,\lambda,z)| \le C\left(\tfrac{9}{4}\delta\right)^2\left(\tfrac{9}{4}\delta + \delta\right) < \tfrac{7}{16}\delta^2 < \left(\tfrac{3}{4}\delta\right)^2 \le |x^2 - \lambda^2|.$$

Therefore Rouché's theorem shows that $y \mapsto g(x,y,z)$ and $y \mapsto x^2 - y^2$ have the same number of zeros in the disk $|y| < \frac{5}{4}\delta$. Hence π is a two sheeted branched cover. The estimate for R implies that $y \mapsto g(0,y,z)$ has a zero of order 2 at $y = 0$ for each $z \in B(0,\delta)$. To show that for $(x,z) \in B(0,\delta)^2$ with $x \neq 0$ the function $y \mapsto g(x,y,z)$ has two different zeros, fix λ with $|\lambda| = \frac{1}{4}|x|$ and set

$$y(x,z,\lambda) := x + \lambda.$$

Then

$$|x^2 - y(x,z,\lambda)^2| = |\lambda||2x - \lambda| \ge \tfrac{1}{4}|x|\tfrac{7}{4}|x| = \tfrac{7}{16}|x|^2.$$

Hence the estimate for R and the choice of δ imply

$$|R(x,y(x,z,\lambda),z)| \le C\left(\tfrac{9}{4}|x|\right)^2\left(\tfrac{9}{4}|x| + \delta\right) < \tfrac{7}{16}|x|^2 < |x^2 - y(x,z,\lambda)^2|.$$

Therefore Rouché's theorem implies that the function $y \mapsto g(x,y,z)$ has exactly one zero in the disk $B(x,\frac{1}{4}|x|) \subset B(0,\frac{5}{4}\delta)$. Since we can argue in the same way using $y(x,z,\lambda) := -x + \lambda$, we proved the first assertion of the lemma. The second one follows from it by the real implicit function theorem and analytic continuation. ■

Theorem 3 *Let $P \in \mathbb{R}[x,y,z]$ be homogeneous of degree $m \geq 1$. P is stable if and only if P satisfies the following three conditions*

(a) $V(P)$ *satisfies* $\mathrm{PL}_{\mathrm{loc}}(0)$.

(b) For each $\xi \in V(P) \cap S^2$, $\deg P_\xi \leq 2$.

(c) If $\deg P_\xi = 2$ *for* $\xi \in V(P) \cap S^2$ *then* P_ξ *is an indefinite quadratic form.*

PROOF. The necessity of the conditions (a)–(c) follows from the definition of stability and the Lemmas 2 and 3. To prove the sufficiency of the conditions (a)–(c), fix any holomorphic function $f\colon B^3(0,r) \to \mathbb{C}$ which is real over real points and satisfies $f_0 = P$. To show that $V := V(f)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ let

$$M := \{\xi \in V(P) \cap S^2 : \deg P_\xi = 2\}.$$

If $M = \emptyset$, the theorem follows from Proposition 1. If $M \neq \emptyset$ we want to derive the theorem from [4], Theorem 7.3. To show that its hypotheses are fulfilled, note first that $T_0 V = V(P)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ by (a). Furthermore, $T_0 V$ has multiplicity one, or equivalently, P is square-free, since P_ξ is square-free by (c) and the localization of a product is the product of the localizations of its factors.

Next we claim that all the other hypotheses of [4], Theorem 7.3, are trivially satisfied, because the set $\mathcal{C}$, defined in [4], 5.1, is empty for the variety $V(f)$. Hence the present theorem follows from [4], Theorem 7.3, once we show $\mathcal{C} = \emptyset$.

By the definition of the set $\mathcal{C}$ we have $\mathcal{C} = \emptyset$ if we prove the following assertion:

For each $\zeta \in M$ there exist $\xi \in S^2 \backslash (T_0 V \cup T_\zeta(T_0 V))$, a zero neighborhood U in $\mathbb{C}^3$ and $r > 0$ such that there is at most one branch of the set $B_\xi \cap \mathbb{R}^3$ contained in the cone $\Gamma(\gamma_\zeta, 1, U, r)$, (4)

where $\gamma_\zeta : t \mapsto t\zeta$ and where

$$B_\xi := \left\{ (x,y,z) \in V(f) : \frac{\partial f}{\partial \xi}(x,y,z) = 0 \right\}.$$

To prove (4), fix $\zeta \in M$. After a real linear change of variables, we may assume that $\zeta = (0,0,1)$ and that P is represented as in (1), namely

$$P(x,y,z) = \sum_{k=2}^{m} p_k(x,y) z^{m-k},$$

where $p_2(x,y) = P_\zeta(x,y,z)$ is an indefinite real quadratic form. Hence we may perform a further real linear change of the (x,y)-variables to obtain $p_2(x,y) = x^2 - y^2$. Then define for $z \neq 0$

$$g(x,y,z) = \frac{1}{z^m} f(xz, yz, z)$$

and note that for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \geq m$ we have

$$\frac{1}{z^m}(xz, yz, z)^\alpha = z^{|\alpha|-m} x^{\alpha_x} y^{\alpha_y}.$$

Hence the assumptions on f imply

$$g(x,y,z) = x^2 - y^2 + \sum_{k=3}^{m} p_k(x,y) + \sum_{j=1}^{\infty} z^j g_j(x,y).$$

This shows that g extends to a holomorphic function in some neighborhood of the origin in $\mathbb{C}^3$, which we also denote by g. It is easy to check that g satisfies the hypotheses of Lemma 4. Hence this lemma implies the existence of $\rho > 0$, $\epsilon > 0$, and of holomorphic functions $h\colon B(0,\rho) \to \mathbb{C}^2$, $G\colon B(0,\rho) \to \mathbb{C}$, and $R\colon B(0,\epsilon)^2 \to \mathbb{C}$ satisfying

$$g((x,y)+h(z),z) = x^2 - y^2 + G(z) + R(x,y,z), \quad (x,y,z) \in B(0,\epsilon)^2 \times B(0,\rho) \tag{5}$$

as well as

$$h(0) = 0, \quad |G(z)| = O(|z|), \quad \text{and} \quad |R(x,y,z)| = O(|(x,y)|^2(|(x,y)|+|z|)).$$

Since f has real Taylor coefficients, so do h and G by Lemma 4 (a).

Next let $h = (h_x, h_y)$ and define

$$\gamma(t) := (th_x(t), th_y(t), t), \quad 0 \le t < \rho.$$

Since $|h(t)| = O(|t|)$, the tangent vectors of γ and of $\gamma_\zeta(t) := (0,0,t)$ at zero coincide. Hence for each $0 < r_0 < 1$ and each zero neighborhood U_0 in $\mathbb{C}^3$ with $U_0 \subset B^3(0,\frac{1}{2})$ there exist a zero neighborhood U and $0 < r < 1$ so that

$$\Gamma(\gamma_\zeta, 1, U, r) \subset \Gamma(\gamma, 1, U_0, r_0).$$

Therefore it suffices to prove (4) for some cone with core γ. To do so, assume $0 < t < \rho$, $|\xi| < \epsilon t$, and $|\eta| < \epsilon t$ and note that the definition of g and (5) imply

$$\begin{aligned} f(\xi + th_x(t), \eta + th_y(t), t) &= t^m g\Big(\tfrac{\xi}{t} + h_x(t), \tfrac{\eta}{t} + h_y(t), t\Big) \\ &= t^m \left(\left(\tfrac{\xi}{t}\right)^2 - \left(\tfrac{\eta}{t}\right)^2 + G(t) + R\Big(\tfrac{\xi}{t}, \tfrac{\eta}{t}, t\Big)\right), \end{aligned} \tag{6}$$

since $|\xi/t| < \epsilon$ and $|\eta/t| < \epsilon$. Now consider two cases:

Case 1: $G \equiv 0$.
In this case Lemma 5 implies that in a suitable cone $\Gamma(\gamma, 1, U_0, \rho_0)$ we have exactly one real branch curve of V for the projection $(x,y,z) \mapsto (x,z)$. Hence (4) holds in this case.

Case 2: $G \not\equiv 0$.
In this case the properties of G imply the existence of $l \in \mathbb{N}$ such that $G(z) = \sum_{j=l}^{\infty} a_j z^j$, where $a_j \in \mathbb{R}$ and $a_l \neq 0$. In the sequel we assume $a_l > 0$. If $a_l < 0$, then x and y have to be interchanged. Since $a_l > 0$ we can choose $\sigma_0 > 0$ such that $G(t) > 0$ for $0 < t < \sigma_0$. Moreover, we can choose $A > 0$ and $0 < \rho_1 < \rho$ such that

$$|G(z)| \le A|z|, \quad |z| < \rho_1.$$

The properties of R imply the existence of $C > 0$ such that

$$|R(x,y,z)| \le C|(x,y)|^2\left(|(x,y)| + |z|\right), \quad (x,y,z) \in B(0,\rho)^3.$$

Next choose $0 < \sigma \le \sigma_0$ and $0 < \delta < \rho_1$ so small that the following conditions are fulfilled:

$$A\sigma + C(3\delta)^2(3\delta + \sigma) < 3\delta^2 \quad \text{and} \quad C(\delta + \sigma) < \tfrac{1}{2}.$$

Then we claim that for each $(\xi, t) \in \mathbb{R}^2$ satisfying $0 < t < \sigma$ and $|\xi| < \delta t$ the function

$$\eta \mapsto f(\xi + th_x(t), \eta + th_y(t), t)$$

has exactly two distinct real zeros $\eta_1(\xi,t)$ and $\eta_2(\xi,t)$ satisfying $|\eta_j(\xi,t)| < 2\delta t$. From this claim it follows that there exists a cone $\Gamma(\gamma, 1, U_0, \sigma)$ so that the projection π of $V \cap \Gamma(\gamma, 1, U_0, \sigma)$, $\pi(x,y,z) = (x,z)$, has no real branch curve, so that (4) holds also in this case.

To prove our claim, note that by (6) it is an obvious consequence of the following assertion

For each $0 < t < \sigma$ and each $-\delta < x < \delta$ the equation (7)

$$q(x,y,t) := x^2 - y^2 + G(t) + R(x,y,t) = 0$$

has exactly two real solutions y_1, y_2 satisfying $|y_j| < 2\delta$, $j = 1,2$.

To prove (7) note first that for $x \in \mathbb{C}$, $|x| < \delta$ and $0 < t < \sigma$ the choice of δ and σ together with the estimates for G and R imply that for $\lambda \in \mathbb{C}$ with $|\lambda| = 2\delta$ we have the estimate

$$\begin{aligned}|G(t) + R(x,\lambda,t)| &\leq At + C|(x,\lambda)|^2(|(x,\lambda)| + t)\\ &\leq At + C(3\delta)^2(3\delta + t) < 3\delta^2 \leq 4\delta^2 - |x|^2 \leq |x^2 - y^2|.\end{aligned}$$

Hence Rouché's theorem implies that $y \mapsto q(x,y,t)$ has exactly two zeros in the disk $B(0,2\delta)$. To show that these zeros are different and real when x is real, note that our choices of σ_0, σ, and η imply the following estimate for $0 < t < \sigma$ and $-\delta < x < \delta$:

$$\begin{aligned}q(x,0,t) &= x^2 + G(t) + R(x,0,t) \geq x^2 + G(t) - C(|x|^3 + |x|^2 t) \geq \tfrac{x^2}{2} + G(t) > 0,\\ q(x,\pm 2\delta,t) &= x^2 - (2\delta)^2 + G(t) + R(x,\pm 2\delta,t)\\ &\leq -3\delta^2 + At + C(|x| + 2\delta)^2(|x| + 2\delta + t)\\ &\leq -3\delta^2 + A\sigma + C(3\delta)^2(3\delta + \sigma) < 0.\end{aligned}$$

From these estimates it is obvious that the equation $q(x,y,t) = 0$ has at least two real solutions. This completes the proof of (7) and of the theorem. ■

Note that Theorem 1 now follows from Lemma 2, Lemma 3, and Theorem 3.

Note that the full generality of [4], Theorem 7.3, is not needed to prove Theorem 3. In fact, the present proof shows—in the notation of [4]—that for f as in the proof of Theorem 3 and each $\zeta \in V(f) \cap S^2$ the variety $V(f)$ is $(\gamma_\zeta, 1)$-hyperbolic at $0 \in T_{\gamma_\zeta,1}V(f)$, where $\gamma_\zeta(t) := t\zeta$. Hence [4], Lemma 5.7, implies that $V(f)$ satisfies $\mathrm{PL}(V(f), \Gamma(\gamma_\zeta, 1, G_\zeta, r_\zeta))$ for a suitable zero neighborhood G_ζ and $r_\zeta > 0$ and for each $\zeta \in V(f) \cap S^2$. From this and [4], Lemma 5.13, it follows that $V(f)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$, because $V(f)$ also satisfies $\mathrm{RPL}_{\mathrm{loc}}(0)$. The latter assertion follows from the $(\gamma_\zeta, 1)$-hyperbolicity stated above and [2], Theorem 10, as it was indicated at the beginning of the proof of [4], Theorem 5.3.

4. Examples

In this section we provide some examples to illustrate the results of the previous section.

Example 1 *For $n \geq 3$ and $m \geq 1$ the polynomials P_m, defined by*

$$P_m(z_1, \ldots, z_n) := \sum_{j=1}^{n-1} z_j^m - z_n^m$$

are stable.

PROOF. This follows from Proposition 1, since $\operatorname{grad} P_m(z) \neq 0$ for each $z \in \mathbb{C}^n \backslash \{0\}$ and since $V(P_m)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ by Meise, Taylor, and Vogt [9], Example 4.9, and [11], Theorem 3.3. ■

In Example 1 stable polynomials in $n \geq 3$ variables of any degree are given. However, their varieties are manifolds outside the origin. The next example shows that there are stable polynomials in three variables of any degree $m \geq 2$ for which the zero varieties have singular points outside the origin.

Example 2 *For $m \in \mathbb{N}$, $m \geq 2$, define $P_m \in \mathbb{R}[x,y,z]$ by*

$$P_2(x,y,z) := x^2 - y^2, \quad P_3(x,y,z) := (x^2-y^2)z + x(x^2+y^2)$$

$$P_m(x,y,z) := (x^2-y^2)z^{m-2} + x^m + y^{m-1}z, \quad m \geq 4.$$

Then

$$V(P_m)_{\text{sing}} \cap S^2 = \{(0,0,1),(0,0,-1)\}$$

and P_m is stable for each $m \geq 2$ and irreducible for $m \geq 3$.

PROOF. To derive this from Theorem 3, note first that the assertion obviously holds for $m = 2$. For $m \geq 3$ we claim that P_m and $\operatorname{grad} P_m$ both vanish on $\mathbb{R}^3$ exactly on the line $L := \{(0,0,t) : t \in \mathbb{R}\}$. To prove this claim, note that

$$\operatorname{grad} P_3(x,y,z) = (2xz + 3x^2 + y^2, y(-2z+2x), x^2 - y^2)$$

and for $m \geq 4$

$$\operatorname{grad} P_m(x,y,z) = \big(x(2z^{m-2} + mx^{m-2}), yz(-2z^{m-3} + (m-1)y^{m-3}), (m-2)(x^2-y^2)z^{m-3} + y^{m-1}\big).$$

From this it follows easily that P_m and $\operatorname{grad} P_m$ both vanish on L. Hence our claim is proved once we show that for $\zeta = (x,y,z) \in \mathbb{R}^3$ satisfying $P_m(\zeta) = 0$ and $\operatorname{grad} P_m(\zeta) = 0$ we have $x = 0 = y$.

To show this for $m = 3$, note first that the vanishing of the last component of $\operatorname{grad} P_3$ implies $x^2 = y^2$. Hence the vanishing of $P_3(\zeta)$ implies

$$0 = P_3(\zeta) = (x^2-y^2)z + x(x^2+y^2) = 2x^3$$

and hence $x = 0$ and $y = 0$.

Assume now that $m \geq 4$ and that $P_m(\zeta)$ and $\operatorname{grad} P_m(\zeta) = 0$, where $\zeta = (x,y,z) \in S^2$. We claim $x = y = 0$ and assume for contradiction that $x \neq 0$. Then the first component of $\operatorname{grad} P_m(\zeta) = 0$ implies

$$z^{m-2} = -\frac{m}{2}x^{m-2}, \tag{8}$$

since x and z are real. In particular, m is odd. We insert (8) into the second component of $\operatorname{grad} P_m(\zeta) = 0$ and multiply by y to get

$$my^2x^{m-2} = (1-m)zy^{m-1}. \tag{9}$$

On the other hand, we insert (8) into $P_m(\zeta) = 0$ and get

$$\left(1 - \frac{m}{2}\right)x^m + \frac{m}{2}y^2x^{m-2} = -zy^{m-1}. \tag{10}$$

Now equations (9) and (10) are combined to get a relation between x and y, namely

$$y^2 = \frac{(m-1)(m-2)}{m(m-3)}x^2. \tag{11}$$

This shows $y \neq 0$, since $x \neq 0$. The next step is to divide (9) by y^2 and to insert (11) into the result. This calculation yields

$$mx^{m-2} = (1-m)\left(\frac{(m-1)(m-2)}{m(m-3)}\right)^{(m-3)/2} zx^{m-3}.$$

Since $x \neq 0$, this equation leads to $x = \alpha z$ for a suitable α. Since m is odd, the exponent $(m-3)/2$ is an integer, and $\alpha \in \mathbb{Q}$. On the other hand, we know from (8) that $\frac{1}{\alpha} = (-m/2)^{1/(m-2)}$, which is not rational for odd m by Eisenstein's criterion. So the assumption $x \neq 0$ was false.

If $x = 0$ and $z = 0$, then the third component of $\operatorname{grad} P_m(\zeta) = 0$ implies $y = 0$. We still have to treat $x = 0$ and $z \neq 0$. In that case the second equation of $\operatorname{grad} P_m(\zeta) = 0$ and $P_m(\zeta) = 0$ imply

$$yz\left(-2z^{m-3} + (m-1)y^{m-3}\right) = 0 \quad \text{and} \quad y^2 z\left(-z^{m-3} + y^{m-3}\right) = 0,$$

respectively. So $y = 0$, which is the claim, or the terms in parentheses vanish. The latter case yields a linear system in y^{m-3} and z^{m-3} with determinant $\det\left(\begin{smallmatrix} -2 & m-1 \\ -1 & 1 \end{smallmatrix}\right) = m - 3 \neq 0$. Hence also in that case $y = 0$. The statement about the singular locus of P_m and hence the first assertion of the example is proved.

To derive the stability of P_m for $m \geq 3$ from this fact, we want to apply Theorem 3. Since the localization of P_m at the points in the set $\{(0,0,+1),(0,0,-1)\} = V(P_m)_{\text{sing}} \cap S^2$ equals $x^2 - y^2$ or $(-1)^{m-2}(x^2 - y^2)$, the conditions (b) and (c) of Theorem 3 are fulfilled. To show that also condition (a) of Theorem 3 is satisfied, namely that $V(P_m)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$, notice first that P_m is irreducible for $m \geq 3$. To see this for $m \geq 4$ write P_m as a polynomial in $\mathbb{C}[y,z][x]$, i.e.,

$$P_m(x,y,z) = x^m + z^{m-2}x^2 + (y^{m-1}z - y^2 z^{m-2}).$$

Then each term except the leading one is a multiple of z, the absolute term is not a multiple of z^2, and z is a prime element in $\mathbb{C}[y,z]$. Hence Eisenstein's criterion implies that P_m is irreducible for $m \geq 4$. Considering P_3 as an element of $\mathbb{C}[x,z][y]$, a similar argument shows that also P_3 is irreducible.

Now note that by the irreducibility of P_m it follows from Meise, Taylor, and Vogt [11], Corollary 3.14 and Theorem 3.3, that $V(P_m)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ if and only if it satisfies the condition (HPL). By Hörmander [7], Theorem 6.5, this holds if for each $\xi \in V(P_m) \cap S^2$ the variety $V(P_m)$ is locally hyperbolic at ξ or equivalently satisfies $\mathrm{PL}_{\mathrm{loc}}(\xi)$. Since $\operatorname{grad} P_m(\xi) \neq 0$ for each $\xi \in \mathbb{C}^n \backslash L$ by our claim, it suffices to show that $V(P_m)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0,0,\pm 1)$. By [4], Lemma 6.1, this holds if and only if the zero variety of the reduction of P_m at $(0,0,\pm 1)$, defined by

$$q_{\pm}(x,y) := P_m(x,y,\pm 1)$$

satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Now note that

$$q_{3\pm}(x,y) = \pm(x^2 - y^2) + x^3 + xy^2, \quad q_{m\pm}(x,y) = (\pm 1)^m (x^2 - y^2) + x^m \pm y^{m-1},\ m \geq 4.$$

From these equations it follows easily that $V(q_{m\pm}) \subset \mathbb{C}^2$ is locally hyperbolic at 0, hence satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Therefore, $V(P_m)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ for $m \geq 3$. Since we have shown that all the hypotheses of Theorem 3 are satisfied, the stability of P_m now follows from Theorem 3. ■

To provide an example of a stable polynomial P in four variables for which $V(P)_{\text{sing}} \cap S^3 \neq \emptyset$ we need some preparation. First we extend [1], Lemma 5.8.

Lemma 6 *Denote by $\mathbb{D}$ the open unit disk in $\mathbb{C}$ and assume that for $n, k \in \mathbb{N}$ and $0 \leq \epsilon \leq \frac{1}{2}$ the function $v \in \mathrm{PSH}(\mathbb{D}^n \times \mathbb{D}^k)$ satisfies the following two conditions:*

(i) $v(z,w) \leq 1$, $(z,w) \in \mathbb{D}^n \times \mathbb{D}^k$,

(ii) $v(z,w) \leq 0$ *if* (z,w) *is real and* $\|z\|_\infty \geq \epsilon$.

Then for each $\lambda < 1$ there exists a constant $C_\lambda > 0$ such that for each ν, $1 \leq \nu \leq n$, the following estimate holds on $\mathbb{D}^n \times \mathbb{D}^k$:

(iii) $v(z,w) \leq C_\lambda \left(\sum_{j=1, j\neq\nu}^n |\mathrm{Im}\, w_j| + |\mathrm{Im}\sqrt{z_\nu^2 - \epsilon^2}|\right)$, $(z,w) \in (\lambda\mathbb{D}^n) \times (\lambda\mathbb{D}^k)$.

PROOF. Denote by h the harmonic measure for the real axis in the unit disk and denote by k_ϵ the function which is harmonic in the unit disk with the real intervals $[-1,-\epsilon]$ and $[\epsilon,1]$ removed, and with boundary values 1 on $|\zeta|=1$ and 0 on $[-1,-\epsilon]\cup[\epsilon,1]$. Then fix ν, $1\le\nu\le n$, and fix

$$(z_1,\dots,z_{\nu-1},z_{\nu+1},\dots,z_n,w)\in(\mathbb{D}^{n-1}\times\mathbb{D}^k)\cap(\mathbb{R}^{n-1}\times\mathbb{R}^k)$$

and consider the function

$$\phi:\mathbb{D}\to[-\infty,\infty[,\quad \phi(z_\nu):=v(z,w).$$

Note that the hypotheses on v and the properties of k_ϵ imply

$$\phi(z_\nu)\le k_\epsilon(z_\nu),\quad z_\nu\in\mathbb{D}.$$

Next fix $z_\nu\in\mathbb{D}$ and consider the function

$$\psi:\mathbb{D}^{n-1}\times\mathbb{D}^k\to[-\infty,\infty[,\quad \psi(z',w):=v(z,w)-k_\epsilon(z_\nu),$$

where $z'=(z_1,\dots,z_{\nu-1},z_{\nu+1},\dots,z_n)$. Note that the estimates for v and for ϕ imply $\psi\le 1$ and $\psi(z',w)\le 0$ whenever (z',w) is real. Hence the maximum principle implies

$$\psi(z',w)\le\sum_{j\ne\nu}h(z_j)+\sum_{j=1}^k h(w_j).$$

By the definition of ψ, this implies

$$v(z,w)\le\sum_{j\ne\nu}h(z_j)+\sum_{j=1}^k h(w_j)+k_\epsilon(z_\nu),\quad (z,w)\in\mathbb{D}^n\times\mathbb{D}^k.$$

Now (iii) follows from known estimates for h and k_ϵ (see, e.g., [1], Lemma 5.8). ■

From [4], Definition 5.5, we recall:

Definition 11 *Let $V\subset\mathbb{C}^n$ be an analytic variety of pure dimension k which contains the origin, let γ be a real simple curve in $\mathbb{C}^n$, $d\ge 1$, $R>0$, D an open set in $\mathbb{C}^n$, and let $\Gamma:=\Gamma(\gamma,d,D,R)$ be a conoid. We say that V satisfies the condition $\mathrm{PL}(V,\Gamma)$ if the following holds: For each compact set $K\subset D$ there exist $A_0,r_0>0$ such that each $u\in\mathrm{PSH}(V\cap\Gamma)$ which satisfies*

(α) $u(z)\le|z|^d,\quad z\in V\cap\Gamma$

(β) $u(z)\le 0,\quad z\in V\cap\Gamma\cap\mathbb{R}^n$

also satisfies

(γ) $u(z)\le A_0|\operatorname{Im} z|,\quad z\in V\cap\Gamma(\gamma,d,K,R)\cap B^n(0,r_0).$

Example 3 *The polynomial $P\in\mathbb{R}[x,y,z,w]$, defined by*

$$P(x,y,z,w):=x^2+y^2-z^2,$$

is stable.

PROOF. To prove this, fix a holomorphic function F with real Taylor coefficients whose localization at the origin is P. By Lemma 4 there are $\epsilon_1,\epsilon_2>0$ and holomorphic functions $h,\xi,\eta,\zeta\colon B(0,\epsilon_1)\to\mathbb{C}$ and $g\colon B^3(0,\epsilon_2)\to\mathbb{C}$, all with real Taylor coefficients, such that for all $(x,y,z,w)\in B^3(0,\epsilon_2)\times B(0,\epsilon_1)$

$$\begin{aligned}F(x+\xi(w),y+\eta(w),z+\zeta(w),w)&=P(x,y,z)+h(w)+g(x,y,z,w),\\ \max\big(|\xi(w)|,|\eta(w)|,|\zeta(w)|\big)&=O(|w|^2),\\ |h(w)|&=O(|w|^3),\\ |g(x,y,z,w)|&=O\big(|(x,y,z)|^2|w|+|(x,y,z)|^3\big).\end{aligned}\tag{12}$$

Set $f(x, y, z, w) := F(x + \xi(w), y + \eta(w), z + \zeta(w), w)$. Since ξ, η, and ζ have real Taylor coefficients and vanish at the origin, it is easy to see that $V(F)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ if and only if $V := V(f)$ does.

To show that V satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$, note first that $V(P) = V(\tilde{P}) \times \mathbb{C}$, where $\tilde{P}$ is the polynomial P considered as an element of $\mathbb{R}[x, y, z]$. Hence it follows from Proposition 1 that $V(P)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Since $\mathrm{PL}_{\mathrm{loc}}(0)$ implies the condition $\mathrm{RPL}_{\mathrm{loc}}(0)$, defined in [4], 3.6, it follows from [4], Lemma 5.13, that V satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ if we show that for each $\xi \in V(P) \cap S^3$ there exist an open zero neighborhood G_ξ and $r_\xi > 0$ so that for $\gamma_\xi \colon t \mapsto t\xi$, the variety V satisfies $\mathrm{PL}(V, \Gamma(\gamma_\xi, 1, G_\xi, r_\xi))$. To show that this condition is fulfilled, note that $\operatorname{grad} P$ vanishes exactly on the line $L := \{(0, 0, 0, \lambda) : \lambda \in \mathbb{C}\}$. Hence it follows from [4], Lemma 7.2, that for each $\xi \in V(P) \cap S^3$, $\xi \neq \xi_\pm := (0, 0, 0, \pm 1)$, V satisfies $\mathrm{PL}(V, \Gamma)$, for a suitable cone $\Gamma = \Gamma(\gamma_\xi, 1, G_\xi, r_\xi)$. Hence it remains to show that this condition also holds for ξ_+ and ξ_-. To do so, we consider two cases.

Case 1: $h \equiv 0$

From (12) and the particular form of f it follows as in the proof of Lemma 5 that the projection π defined by $(x, y, z, w) \mapsto (x, y, 0, w)$ provides a two sheeted branched cover of $V \cap U$ with branch locus $\{(0, 0, 0)\} \times B(0, \delta)$ in a suitable neighborhood U and that (x, y, z, w) is real when $\pi(x, y, z, w)$ is real. This implies that V is $(\gamma_{\xi_\pm}, 1)$-hyperbolic at 0. Hence it follows from [4], Lemma 5.7, that there exists a cone $\Gamma = \Gamma(\gamma_{\xi_\pm}, 1, G_{\xi_\pm}, r_{\xi_\pm})$ so that V satisfies $\mathrm{PL}(V, \Gamma)$. By the preceding, this shows that V satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ in this case.

Case 2: $h \not\equiv 0$

The present hypothesis and (12) imply the existence of $k \in \mathbb{N}$, $k \geq 3$, such that $h(w) = \sum_{j=k}^{\infty} b_j w^j$, where $b_k \neq 0$. From this and [3], Lemma 6.1, it follows that for $\delta := \frac{k}{2} \geq \frac{3}{2}$ we have

$$T_{\gamma_{\xi_\pm}, d} V = \{(x, y, z, w) \in \mathbb{C}^4 : x^2 + y^2 - z^2 = 0\}, \quad 1 \leq d < \delta$$

and

$$T_{\gamma_{\xi_\pm}, \delta} V = \{(x, y, z, w) \in \mathbb{C}^4 : p_\pm(x, y, z, 1) = 0\},$$

where

$$p_\pm(x, y, z, w) := x^2 + y^2 - z^2 + b_k(\pm w)^k.$$

Since $b_k \neq 0$ by hypothesis in this case, we have

$$\operatorname{grad} p_\pm(\zeta) \neq 0 \text{ for each } \zeta \in T_{\gamma_{\xi_\pm}, \delta} V \cap \mathbb{R}^4. \tag{13}$$

Next choose $0 < \rho_1 < r$ so small that

$$|h(w)| \leq 2|b_k||w|^k \qquad \text{for } |w| \leq \rho_1. \tag{14}$$

According to (12) we may choose ρ_1 also so small that there exists $C > 0$ such that

$$|g(x, y, z, w)| \leq C|(x, y, z)|^2(|(x, y, z)| + |w|), \quad (x, y, z, w) \in B(0, \rho_1)^4. \tag{15}$$

Now choose $0 < \epsilon < \rho_1/2$ and $0 < \rho < \rho_1$ so small that

$$C\left(\frac{9}{4}\right)^2 \left(\frac{9}{4}\epsilon + 1\right)\rho < \frac{1}{4} \quad \text{and } 2|b_k|\frac{1}{\epsilon^2}\rho^{k-2} < \frac{1}{4},$$

and choose $D > 1$ so large that

$$(1 + \epsilon)^k 2|b_k| < \frac{D^2}{4}.$$

Then we claim

> Whenever $0 < t < \rho$ and $(x, y) \in \mathbb{R}^2$ satisfies $Dt^\delta < |(x, y)| < \epsilon t$ then $z \mapsto f(x, y, z, t)$ has exactly two distinct zeros in the disk $B(0, \frac{5}{\epsilon}t)$. These zeros are real and satisfy $|z| < \frac{5}{4}|(x, y)|$. (16)

To prove this claim, fix (x, y) as in (16) and assume that $\lambda \in \mathbb{C}$ satisfies $|\lambda| = \frac{5}{4}\epsilon t$. Then

$$|x^2 + y^2 - \lambda^2| \geq \left(\frac{5}{4}\epsilon t\right)^2 - (\epsilon t)^2 = \left(\frac{3}{4}\epsilon t\right)^2.$$

The estimates (14) and (15) together with the choice of ϵ and ρ imply

$$\begin{aligned} |h(t) + g(x, y, \lambda, t)| &\leq 2|b_k|t^k + C\left(\tfrac{9}{4}\epsilon t\right)^2\left(\tfrac{9}{4}\epsilon t + t\right) \\ &\leq \left[2|b_k|\epsilon^{-2}\rho^{k-2} + C(\tfrac{9}{4})^2\left(\tfrac{9}{4}\epsilon + 1\right)\rho\right]\epsilon^2 t^2 \\ &< \left(\tfrac{3}{4}\epsilon t\right)^2 \leq |x^2 + y^2 + \lambda^2|. \end{aligned}$$

Hence the first statement in (16) follows from Rouché's theorem. To prove the second one, note that for x, y real, the present choices imply

$$\begin{aligned} f(x, y, 0, t) &= x^2 + y^2 + h(t) + g(x, y, 0, t) \\ &\geq x^2 + y^2 - 2|b_k|t^k - C|(x, y)|^2(|(x, y)| + t) \\ &\geq x^2 + y^2 - \tfrac{1}{4}D^2 t^k - \tfrac{1}{4}(x^2 + y^2) \geq \tfrac{1}{2}(x^2 + y^2) > 0, \end{aligned}$$

while

$$\begin{aligned} f(x, y, \pm\tfrac{5}{4}|(x, y)|, t) &= -(\tfrac{5}{4})^2(x^2 + y^2) + x^2 + y^2 + h(t) + g(x, y, \pm\tfrac{5}{4}|(x, y)|, t) \\ &\leq -\tfrac{9}{16}(x^2 + y^2) + 2|b_k|t^k + C(\tfrac{9}{4})^2|(x, y)|^2(\tfrac{9}{4}|(x, y)| + t) \\ &\leq -\tfrac{9}{16}(x^2 + y^2) + \tfrac{1}{4}D^2 t^k + \tfrac{1}{4}(x^2 + y^2) \leq -\tfrac{1}{16}(x^2 + y^2) < 0. \end{aligned}$$

Obviously, these estimates imply the second assertion.

Next we claim that the following assertion holds:

There exist $S > 7D$ and $\rho > 0$ so that if $0 < t < \rho$, $(x, y, z, w) \in t[B(0, \epsilon)^2 \times B(0, \frac{5}{4}\epsilon) \times B(0, \epsilon)]$, $(x, y, z, t + w) \in V$, and $\|(x, y)\|_\infty \leq 3Dt^\delta$ then $|z| < St^\delta$. (17)

To prove this, let $\epsilon > 0$ and $D > 1$ be as before. Shrinking $\rho > 0$ if necessary, we may assume that

$$18C(1 + 3\epsilon)\rho < 1 \quad \text{and} \quad C[(1 + 7\epsilon)\rho + \frac{5}{4}\epsilon\rho] < \frac{1}{4}.$$

Then we choose $S > 7D$ so large that

$$6C(1 + 7\epsilon)\rho < \frac{S}{4}. \tag{18}$$

Assume now that there exists $(x, y, z, t + w) \in V$ which satisfies the conditions in (17) while $St^\delta \leq |z| < \frac{5}{4}\epsilon t$. Then our choices and (14) together with (15) and the obvious estimate $|(x, y, z)| \leq |(x, y)| + |z|$ imply

$$\begin{aligned} |z|^2 &= |x^2 + y^2 + h(t + w) + g(x, y, z, t + w)| \\ &\leq 18D^2 t^{2\delta} + 2(1 + \epsilon)^k |b_k| t^k + C(|(x, y, z)|^3 + |(x, y, z)|^2 |t + w|) \\ &\leq (18 + \tfrac{1}{4})D^2 t^{2\delta} + C[(|(x, y)| + (1 + \epsilon)t)|(x, y)|^2 \\ &\qquad + (3|(x, y)| + 2|t + w|)|(x, y)||z| + (3|(x, y)| + |t + w| + |z|)|z|^2] \\ &\leq 19D^2 t^{2\delta} + C(1 + 3\epsilon)18\rho D^2 t^{2\delta} + C(1 + 7\epsilon)6\rho D t^\delta |z| + C\left[(1 + 7\epsilon)\rho + \tfrac{5}{4}\epsilon\rho\right]|z|^2 \\ &\leq 20D^2 t^{2\delta} + \tfrac{1}{4}St^\delta|z| + \tfrac{1}{4}|z|^2 < \tfrac{3}{4}|z|^2. \end{aligned}$$

From this contradiction, it follows that no such point can exist, hence (17) holds.

To interpret (16) and (17), let $\gamma := \gamma_{\xi_+}$, $\gamma_0 : t \mapsto (0, 0, t)$, $\Gamma(\epsilon, \sigma) := \Gamma(\gamma, 1, B(0, \epsilon)^2 \times B(0, \sigma) \times B(0, \epsilon), \rho)$, and $\Gamma' := \Gamma'(\gamma_0, 1, B(0, \epsilon)^3, \rho)$. Then (16) proves the existence of $0 < \epsilon < \frac{1}{4}$, $0 < \sigma < 4\epsilon$,

$\rho > 0$, and $D > 1$ such that $(x, y, z, w) \in V \cap \Gamma(\epsilon, \sigma)$ and $(x, y, w) \in \mathbb{R}^3 \cap (\Gamma' \setminus \Gamma'(\gamma, \delta, B(0, D)^3, 1))$ implies $(x, y, z, w) \in \mathbb{R}^4$. Moreover, (17) proves that $|z| < St^\delta$ whenever $(x, y, z, w) \in V \cap \Gamma(\epsilon, \sigma)$ satisfies $(x, y, w) \in \Gamma'(\gamma_0, \delta, B(0, 2D)^3, 1)$.

Now we are going to use the assertions derived so far to show that V satisfies a weaker variant of $\mathrm{PL}(V, \Gamma)$. To do so fix $u \in \mathrm{PSH}(V \cap \Gamma)$ and assume that u satisfies the conditions (α) and (β) of Definition 11. Then define $\phi : \Gamma' \to [-\infty, \infty[$ by

$$\phi(x, y, w) := \max\{u(x, y, z, w) : (x, y, z, w) \in \Gamma \cap V\}.$$

If we choose $0 < \epsilon < \frac{1}{4}$ and $0 < \sigma < 4\epsilon$ small enough, then the projection $\pi\colon (x, y, z, w) \mapsto (x, y, 0, w)$ will be proper on $V \cap \Gamma(\epsilon, \frac{\sigma}{2})$, in particular it will be proper on $V \cap \Gamma(\epsilon, \sigma)$. Hence it follows from Hörmander, [7], Lemma 4.4, that ϕ is plurisubharmonic on $\Gamma' = \Gamma(\epsilon, \sigma)'$. Next we note that condition (α) of Definition 11, applied to u, implies the existence of a constant M, not depending on u, such that

$$\phi(x, y, w) \le M|(x, y, w)|, \quad (x, y, w) \in \Gamma'. \tag{19}$$

Furthermore, the considerations above and condition (β) of Definition 11, applied to u, imply

$$\phi(x, y, w) \le 0 \quad \text{if } (x, y, w) \in \Gamma' \cap \mathbb{R}^3 \text{ and } \|(x, y)\|_\infty \ge Dt^\delta. \tag{20}$$

Shrinking ρ if necessary , we may assume that $8D\rho^{\delta-1} \le \epsilon$. Next fix $0 < t < \rho$ and define

$$v : \mathbb{D}^3 \to [-\infty, \infty[, \quad v(\xi, \eta, w) := \frac{1}{2Mt}\phi(\epsilon\xi t, \epsilon\eta t, t + \epsilon w t).$$

Then the estimate (19) and $0 < \epsilon < \frac{1}{4}$ imply

$$v(\xi, \eta, w) \le \frac{1}{2Mt} Mt \left(\epsilon^2 + \epsilon^2 + (1 + \epsilon)^2\right)^{\frac{1}{2}} \le 1, \quad (\xi, \eta, w) \in \mathbb{D}^3,$$

while (20) implies

$$v(\xi, \eta, w) \le 0 \text{ if } (\xi, \eta, w) \text{ is real and } \|(\xi, \eta)\|_\infty \ge \frac{D}{\epsilon} t^{\delta-1}.$$

By Lemma 6 these estimates imply the existence of $C \ge 1$ so that for $\|(\xi, \eta, w)\|_\infty \le \frac{2}{3}$ we have

$$v(\xi, \eta, w) \le C \left(|\mathrm{Im}\, \eta| + |\mathrm{Im}\, w| + \left|\mathrm{Im} \sqrt{\xi^2 - \left(\tfrac{D}{\epsilon} t^{\delta-1}\right)^2}\right| \right)$$
$$v(\xi, \eta, w) \le C \left(|\mathrm{Im}\, \xi| + |\mathrm{Im}\, w| + \left|\mathrm{Im} \sqrt{\eta^2 - \left(\tfrac{D}{\epsilon} t^{\delta-1}\right)^2}\right| \right).$$

By the definition of v, these estimates imply that for $0 < t < \rho$ and $(x, y, w) \in \mathbb{C}$, $\|(x, y, w)\|_\infty < \frac{2\epsilon}{3} t$ we get

$$\begin{aligned} \phi(x, y, t + w) &\le \tfrac{CM}{\epsilon} \left(|\mathrm{Im}\, y| + |\mathrm{Im}\, w| + |\mathrm{Im} \sqrt{x^2 - (Dt^\delta)^2}| \right) \\ \phi(x, y, t + w) &\le \tfrac{CM}{\epsilon} \left(|\mathrm{Im}\, x| + |\mathrm{Im}\, w| + |\mathrm{Im} \sqrt{y^2 - (Dt^\delta)^2}| \right) \end{aligned} \tag{21}$$

Now we apply [1], Lemma 5.7, to get the existence of $C_1 > 0$, not depending on ϕ, so that

$$\phi(x, y, t + w) \le \frac{C_1 M}{\epsilon} \left(|\mathrm{Im}\, w| + |\mathrm{Im}\, x| + |\mathrm{Im}\, y| + Dt^\delta \right) \tag{22}$$

and

$$\phi(x, y, t + w) \le \frac{C_1 M}{\epsilon} |\mathrm{Im}(x, y, w)| \quad \text{if } \|(x, y)\|_\infty \ge 2Dt^\delta. \tag{23}$$

We claim that these estimates imply the existence of $C_2 > 0$ (not depending on ϕ) so that

$$\phi(\theta) \le C_2|\mathrm{Im}\,\theta|, \quad \theta \in \Gamma_1'\backslash\Gamma_2', \tag{24}$$

where for $\gamma_0(t) := (0,0,t)$, we let

$$\Gamma_1' := \Gamma'(\gamma_0, 1, B(0,\frac{\epsilon}{2})^3, \rho), \quad \Gamma_2' := \Gamma'(\gamma_0, \delta, B(0,2D)^3, \rho).$$

To show this, fix $\theta = (x, y, t + i\beta) \in \Gamma_1'$ and assume $\beta \in \mathbb{R}$. If $\|(x, y)\|_\infty \ge 2Dt^\delta$ then (24) follows from (23). If $\|(x, y)\|_\infty < 2Dt^\delta$ and $|\beta| < 2Dt^\delta$ then $\zeta \in \Gamma_2'$ and there is nothing to prove. If $\|(x, y)\|_\infty < 2Dt^\delta$ and $|\beta| > 2Dt^\delta$ then (22) implies

$$\phi(\theta) = \phi(x, y, t + i\beta) \le \frac{C_1 M}{\epsilon}(|\beta| + 5Dt^\delta) \le \frac{4C_1 M}{\epsilon}|\beta| \le \frac{4C_1 M}{\epsilon}|\mathrm{Im}\,\theta|.$$

This shows that (24) holds for an appropriate constant C_2.

Now let $\Gamma_0 := \Gamma(\gamma, 1, B(0,\frac{\epsilon}{2})^2 \times B(0,\sigma) \times B(0,\frac{\epsilon}{2}), \rho)$ and note that the definition of ϕ and the estimate (24) imply that for $\zeta \in V \cap \Gamma_0$, $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ we have

$$u(\zeta) \le \phi(\zeta_1, \zeta_2, \zeta_3) \le C_2|\mathrm{Im}(\zeta_1, \zeta_2, \zeta_3)| \le C_2|\mathrm{Im}\,\zeta|, \tag{25}$$

whenever $(\zeta_1, \zeta_2, \zeta_3) \in \Gamma_1'\backslash\Gamma_2'$. To show that (25) even holds for all $\zeta \in V \cap \Gamma_0$ and a possibly larger constant C_2, it therefore suffices to show that (25) holds whenever $\zeta \in V \cap \Gamma_0$ and $(\zeta_1, \zeta_2, \zeta_3) \in \Gamma_3' := \Gamma'(\gamma_0, \delta, B(0,\frac{5}{2}D)^3, \rho)$. To prove this note that by (17) each point $(x, y, z,$ t + w$) \in V \cap \Gamma_0$ which satisfies $\|(x, y)\|_\infty < 3Dt^\delta$ already satisfies $|z| < St^\delta$. Hence we get (25) for all $\zeta \in V \cap \Gamma_0$ if we show

$$u(\zeta) \le C_3|\mathrm{Im}\,\zeta|, \quad \zeta \in \Gamma(\gamma, \delta, B(0,\frac{5}{2}D)^2 \times B(0,\frac{3}{2}S) \times B(0,\frac{5}{2}D), \rho) \tag{26}$$

and a suitable constant C_3.

To prove (26) let $G_0 := B(0,3D)^2 \times B(0,2S) \times B(0,3D)$ and note that from (22) and the definition of ϕ we get the existence of some constant $B > 0$ so that

$$u(\zeta) \le B|\zeta|^\delta, \quad \zeta \in \Gamma(\gamma, \delta, G_0, \rho). \tag{27}$$

Then we note that (13) and [4], Lemma 7.2, imply that at each point $\kappa \in T_{\gamma,d}V \cap \mathbb{R}^4$ the variety V is (γ, d)-hyperbolic. Hence [4], Lemma 5.7, implies that for each such κ there is a zero neighborhood G_κ such that $\mathrm{PL}(V, \Gamma(\gamma, \delta, \kappa + G_\kappa, r_\kappa))$ holds. Now an application of [4], Lemma 5.6, shows that V satisfies $\mathrm{PL}(V, \Gamma(\gamma, \delta, G_0, \rho))$ since $B(0,\frac{5}{2}D)^2 \times B(0,\frac{3}{2}S) \times B(0,\frac{5}{2}D)$ is a relatively compact subset of G_0, we get (26). Altogether we proved that there exists $A \ge 1$ such that for each $u \in \mathrm{PSH}(V \cap \Gamma')$ and $\zeta \in V \cap \Gamma_0$ we have

$$u(\zeta) \le A|\mathrm{Im}\,\zeta|.$$

This is enough to apply [4], Lemma 5.10. The case $\gamma = \gamma_-$ is treated in the same way. ■

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