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Some aspects of the modern theory of Fréchet spaces

Klaus D. Bierstedt and José Bonet

Dedicated to the memory of Professor Klaus Floret, our good friend

Abstract. We survey some recent developments in the theory of Fréchet spaces and of their duals. Among other things, Section 4 contains new, direct proofs of properties of, and results on, Fréchet spaces with the density condition, and Section 5 gives an account of the modern theory of general Köthe echelon and co-echelon spaces. The final section is devoted to the developments in tensor products of Fréchet spaces since the negative solution of Grothendieck's "problème des topologies".

Algunos aspectos de la teoría moderna de espacios de Fréchet

Resumen. Discutimos progresos recientes en la teoría de espacios de Fréchet y sus duales. Entre otras cosas, la Sección 4 contiene nuevas pruebas de propiedades y resultados acerca de espacios con la condición de densidad y la Sección 5 proporciona información acerca de la teoría reciente de espacios escalonados y co-escalonados de Köthe. La sección final está dedicada a los progresos en productos tensoriales de espacios de Fréchet obtenidos desde la solución negativa del "problema de las topologías" de Grothendieck.

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Presentado por Manuel Valdivia.
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1. Introduction

Fréchet spaces have played an important role in functional analysis from its very beginning: Many vector spaces of holomorphic, differentiable or continuous functions which arise in connection with various problems in analysis and its applications are defined by (at most) countably many conditions, whence they carry a natural Fréchet topology (if they are, in addition, complete). In particular, each Banach space is a Fréchet space and so has a countable basis of absolutely convex zero neighborhoods. In the Banach case the basis can be obtained as multiples of the unit ball. Therefore the geometry of the unit ball is crucial in Banach space theory. In a Fréchet space, however, the relation between different neighborhoods of zero is, in general, more important than the local Banach spaces. This is the reason why the properties of the linking maps between the local Banach spaces are crucial in the theory of Fréchet spaces. Also, in Banach spaces the unit ball is simultaneously a typical zero neighborhood and a typical bounded set. In a Fréchet space, a zero neighborhood is bounded only when the space is normable. Accordingly, the behavior of the bounded sets also plays a significant role. Another important difference to the Banach space case is that the strong dual of a Fréchet space is not metrizable in general. Strong duals of Fréchet spaces are (DF)-spaces, a class introduced by Grothendieck [62]. Since it will be necessary to use duality theory, (DF)-spaces have to be considered here as well. Countable (locally convex) inductive limits of Banach spaces, (LB)-spaces, are also (DF)-spaces.

Köthe echelon and co-echelon spaces are among the most important examples of Fréchet and (DF)-spaces, respectively. Many spaces of functions or distributions, like

$$H(\mathbb{D}^N),\ H(\mathbb{C}^N),\ \mathcal{S},\ \mathcal{S}',\ \mathcal{D}(K),\ \mathcal{D}'(G),\ C^\infty(G)$$

for a compact subset K or an open subset G of $\mathbb{R}^N$, are topologically isomorphic to (products of) echelon or co-echelon spaces. (In the sequel, we will sometimes simply write 'isomorphic' instead of 'topologically isomorphic'.) Various classes of Fréchet spaces were defined, and they were characterized in the context of Köthe echelon spaces. In the late '80s, the authors investigated Fréchet spaces with the density condition and characterized the density condition of echelon spaces [16].

Topological invariants like (DN) and (Ω) have been essential in the structure theory of Fréchet spaces, due to Meise, Vogt and others. Moreover, they also have many interesting applications to problems arising in analysis, e.g. the existence of extension operators for C^∞-functions on compact or closed subsets of $\mathbb{R}^N$, the surjectivity or the existence of solution operators for convolution operators or for linear partial differential operators with constant coefficients on spaces of analytic or (ultra-) differentiable functions or (ultra-) distributions. We do not discuss these topics here. Short exact sequences of Fréchet spaces, the properties (DN) and (Ω), subspaces and quotients of power series spaces, the splitting theorem of Vogt and Wagner and their applications are studied in detail in the book [74] of Meise and Vogt.

We concentrate here on other recent developments in the theory of Fréchet spaces in which the authors contributed to substantial progress or which we find remarkable in connection with our own interests.

After recalling some general definitions and introducing some notation in Section 2, we report in Section 3 shortly on several lines of recent research on various classes of Fréchet spaces. Subsection (a) is devoted to the distinguishedness, subsection (b) to quasinormable Fréchet spaces and other classes of Fréchet spaces.

In Section 2.(c), approximation properties are discussed, and in (d) Moscatelli type constructions are mentioned. In the last subsection of Section 2, results of Banaszczyk [7, 8] and of Bonet, Defant [29] around the Levy Steinitz rearrangement theorem and nuclearity are surveyed.

In the rest of the article, more details are given on three important topics. Section 4 discusses the density condition for Fréchet spaces and the dual density conditions for (DF)-spaces, see [16, 17]. In particular, in subsection (a) new, short and direct proofs of some of the main results on Fréchet spaces with the density condition are presented. Section 5 surveys Köthe echelon and co-echelon spaces, in (a) and (b) beginning from scratch as in Bierstedt, Meise, Summers [28]. The relevance of the regularly decreasing condition and of condition (D) is pointed out in subsections (c) and (d). We finish Section 5 with some results on vector valued echelon and co-echelon spaces from [16, 17].

The theory of topological tensor products of locally convex spaces was started by Grothendieck [63], and it is still a broad, thriving area of present research. In particular, it took the experts in Banach space theory quite some period of time to really understand Grothendieck's metric theory of topological tensor products [61]. Then Pietsch, König and others developed the theory of operator ideals, with beautiful applications to the distribution of eigenvalues. Finally, Defant and Floret [43] gave the first treatment in book form of Grothendieck's metric theory, combined with the approach via operator ideals. This excellent book is highly recommended. However, our account of tensor products of Fréchet spaces mainly stays in the tradition of [63].

In Section 6 we present the developments in tensor products of Fréchet spaces which started with Taskinen's solution [93] of Grothendieck's famous "problème des topologies" and of some related problems. In subsection (c) we survey the counterexamples and some positive results, due to Taskinen and other authors; in particular, we mention results from the important paper [44] of Defant, Floret, Taskinen. The work was continued by Peris who in [82] developed the notion of "locally convex properties by operators" and solved a problem of Bierstedt, Meise [25] on (DFS)-spaces.

2. Fundamental definitions

A *Fréchet space* F is a complete metrizable locally convex (topological vector) space (over the field $\mathbb{R}$ or $\mathbb{C}$). Hence the topology (and the uniform structure) of F can be given by an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of *seminorms*. As in any topological vector space, the topology of F is determined by a basis of neighborhoods of 0. Since F is locally convex, one can take all the basic 0-neighborhoods to be absolutely convex, and since F is metrizable, there is a countable *basis*; viz., a decreasing sequence $\mathcal{U}_0 = \mathcal{U}_0(F) = (U_n)_{n\in\mathbb{N}}$ of (closed) absolutely convex 0-neighborhoods. It is sometimes convenient to assume that the sequence $(p_n)_n$ satisfies $2p_n \le p_{n+1}$ and that, similarly, $U_{n+1} + U_{n+1} \subset U_n$ holds. In the sequel, we will always take

$$U_n = \{f \in F;\ p_n(f) \le 1\},\ \ n = 1, 2, \ldots$$

Since Fréchet spaces are locally convex (l.c.), the Hahn-Banach Theorem and its many consequences (among them the quite useful Bipolar Theorem) hold in our context. And as in the case of Banach spaces, completeness is essential in order to obtain results following from Baire's Category Theorem, like e.g. the Open Mapping and Closed Graph Theorems. In this article, we deal with the *isomorphic* theory of Fréchet spaces, which allows us to switch from one basic sequence $(p_n)_n$ of seminorms (or from one basis $\mathcal{U}_0 = (U_n)_n$ of 0-neighborhoods) to another, equivalent one (i.e., giving the same topology), whenever convenient. – In connection with non-linear phenomena and in investigations on the inverse function theorem and Nash-Moser theory, a different category has been used. In that case, one speaks of *graded Fréchet spaces* F and fixes (natural) increasing sequences $(p_n)_n$ of seminorms. Instead of arbitrary continuous linear morphisms, the *tame category* only allows continuous linear morphisms T which, for a fixed $k \in \mathbb{N}$, satisfy estimates of the type $p_n(Tf) \le C_n p_{n+k}(f)$ for all $n \in \mathbb{N}$ and $f \in F$, where $C_n > 0$ denotes a constant. We refer the reader e.g. to [88] and to the references therein.

Each Fréchet space F is the *projective limit* $\operatorname{proj}_n F_n$ of the projective sequence of its *local Banach spaces* F_n. We will now discuss the notation which arises in this context: For each $n \in \mathbb{N}$, the quotient space $F/p_n^{-1}(0)$ is endowed with the norm induced by p_n, and F_n is the completion of this space. π_n denotes the canonical mapping $F \to F_n$, $n \in \mathbb{N}$, and for $m \geq n$ the natural continuous linear applications $\pi_{mn} : F_m \to F_n$ are said to be the *linking maps* associated with F. Switching from a fundamental sequence $(p_n)_n$ of seminorms for F to an equivalent one amounts to changing the sequence $(F_n)_n$ of local Banach spaces at the same time, but sometimes properties of the linking maps (like compact, nuclear etc.) are invariant under such a change. Thus, while it is clear that the (conveniently selected sequence of) local Banach spaces F_n can yield important information on the Fréchet space F, often new phenomena appear in Fréchet space theory which are not induced by the local Banach spaces alone, but are rather a consequence of properties of the linking maps π_{mn}.

If F is a Fréchet space and $(F_n)_n$ is the sequence of local Banach spaces associated with an increasing fundamental sequence of seminorms $(p_n)_n$ as above, then the sequence

$$0 \to F \xrightarrow{\pi} \prod_n F_n \xrightarrow{\sigma} \prod_n F_n \to 0,$$

with $\pi(x) := (\pi_n(x))_n$, $x \in F$, and $\sigma((x_n)_n) := (x_n - \pi_{n+1,n}(x_{n+1}))_n$, $x_n \in F_n$, $n \in \mathbb{N}$, is a short (topologically) exact sequence of Fréchet spaces by [74, 26.16]; i.e., both π and σ are topological homomorphisms. In the present setting this follows from the exactness of the short sequence by the Banach Schauder Open Mapping Theorem since all the spaces are Fréchet spaces and $\mathrm{Ker}\ \sigma = \mathrm{Im}\ \pi$. The short exact sequence above is called the *canonical resolution of the Fréchet space* F; it plays an important role in the characterization of certain properties of F, see [74].

As is known from the duality theory of l.c. spaces, the dual F' of the Fréchet space F; i.e., the space of all continuous linear functionals on F, can be endowed with many different, important topologies, e.g. with the weak*-topology $\sigma(F', F)$, the topology $\kappa(F', F)$ of uniform convergence on the absolutely convex compact subsets of F – F' endowed with this topology will be denoted by F'_c – and the Mackey topology $\mu(F', F)$ (the strongest admissible topology for the dual pair $< F, F' >$). In some sense, however, the most natural topology of F' is the *strong topology* $\beta(F', F)$, the topology of uniform convergence on the bounded subsets of F; F' equipped with $\beta(F', F)$ is called the *strong dual* and denoted by F'_b. We recall that $B \subset F$ is bounded if and only if each of the seminorms p_n is bounded on B or, equivalently, if for each $n \in \mathbb{N}$ there is $\lambda_n > 0$ with $B \subset \lambda_n U_n$. The system of all absolutely convex, closed and bounded subsets of F will be denoted by $\mathcal{B} = \mathcal{B}(F)$. In the case of Banach spaces, the unit ball is both a typical 0-neighborhood and a typical bounded set. But a Fréchet space with a bounded 0-neighborhood must already be a Banach space; for *proper* Fréchet spaces (i.e., Fréchet spaces which are not already Banach spaces), 0-neighborhoods and bounded sets are in fact quite different objects.

The strong dual F'_b of any Fréchet space F is complete. However, if F is a proper Fréchet space, then F'_b will no longer be metrizable; thus, F'_b has a much more complicated structure than F. In fact, the increasing sequence $(U_n^\circ)_n$ of the (absolute) polars

$$U_n^\circ = \{f' \in F'; |f'(f)| \leq 1 \text{ for all } f \in U_n\}$$

of the sets U_n (= the unit ball with respect to p_n, see above) in F' is a fundamental sequence of bounded sets; i.e., each bounded subset of F'_b is contained in some U_n°, and every countable bornivorous intersection of absolutely convex 0-neighborhoods of F'_b is also a 0-neighborhood. The locally convex spaces E which share these two properties of the strong duals of Fréchet spaces are called *(DF)-spaces*, see Grothendieck [62]. There are several good references for (DF)-spaces: [69, 74, 79]. In particular, every countable (locally convex) inductive limit of Banach spaces – the inductive limit is called *(LB)-space* – is a (DF)-space; see [13].

There is another, quite natural, "categorical" idea how the dual of F should be topologized: F is the projective limit of the local Banach spaces F_n. It looks tempting to take the transposed maps and to consider the inductive limit of the dual Banach spaces F'_n. Now, it is easy to see that the dual of the local Banach space F_n is nothing but $F'_{U_n^\circ}$, that is, the span of the polar U_n° of U_n in F', endowed with the Minkowski functional of U_n°, a norm which turns $F'_{U_n^\circ}$ into a Banach space. Each U_n° is equicontinuous and $\sigma(F', F)$-compact by the Theorem of Alaoglu-Bourbaki. The sequence $(U_n^\circ)_n$ is increasing, and its union equals

F'. $\iota(F', F)$ denotes the *inductive topology* on F'; viz., the strongest locally convex topology on F' which makes all the natural injections of the Banach spaces $F'_{U_n^\circ}$ into F' continuous. F' with this topology is called the *inductive dual*, and it is denoted by $F'_i = \mathrm{ind}_n F'_{U_n^\circ}$. The inductive dual is obtained by dualizing the projective limit $F = \mathrm{proj}_n F_n$. Grothendieck proved that $\iota(F', F)$ is the bornological topology associated with $\beta(F', F)$ and coincides with $\beta(F', F'')$, see [69].

Our notation concerning locally convex spaces is standard; e.g., see [67, 69, 74]. In particular, $\Gamma(A)$ (resp., $\overline{\Gamma}(A)$) denotes the absolutely convex hull (resp., the closed absolutely convex hull) of the subset A of a linear space (resp., locally convex space). By 'quotient' we mean a separated quotient space. To avoid trivialities, it will always be assumed that Fréchet and (DF)-spaces are different from $\{0\}$. – Clearly, it is impossible to quote all the important articles and books on Fréchet spaces in the present survey. Hence we have sometimes given references only to one article or to one book in which references to other relevant papers can then be found.

3. Classes of Fréchet spaces

(a) Distinguished Fréchet spaces

Dieudonné and Schwartz called a Fréchet space F *distinguished* if F'_b is barrelled. Grothendieck proved that, in the present setting, this is also equivalent to $F'_b = F'_i$ or, equivalently, to F'_b being bornological. Hence, exactly for the distinguished Fréchet spaces F, one obtains the strong dual F'_b by dualizing the projective sequence F_n of local Banach spaces and by taking the inductive limit of the sequence F'_n. As mentioned e.g. by Horváth in his book [66, page 288], it is important to know if the strong duals of the function spaces which appear in the theory of distributions have good locally convex properties. Indeed, if this is the case, one could apply to them the Closed Graph or the Open Mapping Theorem or the Uniform Boundedness Principle.

The first example of a non-distinguished Fréchet space was given by Grothendieck and Köthe, and it was the Köthe echelon space $\lambda_1(A)$ of order 1 for the Köthe matrix $A = (a_n)_n$ defined on $\mathbb{N} \times \mathbb{N}$ by $a_n(i,j) := j$ if $i < n$ and $a_n(i,j) = 1$ otherwise. The Köthe echelon spaces which are distinguished were characterized by the authors and Meise in the late 80's, see Section 5. One could say that, for a long time, all the examples of non-distinguished Fréchet spaces were abstract and artificial. However, Taskinen [96] showed that the Fréchet space $C(\mathbb{R}) \cap L^1(\mathbb{R})$ endowed with the natural intersection topology is not distinguished. His original proof was simplified considerably in [41]. Here is the argument, which is valid even for infinitely differentiable functions.

Theorem 1 *For every open subset G of $\mathbb{R}^N$ the intersection space $E := C^\infty(G) \cap L^1(G)$ is not distinguished.*

PROOF. Set, for $f \in E$, $p_0(f) := \int_G |f| d\mu$ and select an increasing fundamental sequence $(L_n)_n$ of compact subsets of G. The topology of E is defined by the increasing sequence of seminorms $(p_n)_n$ given by

$$p_n(f) := p_0(f) + \max_{|\alpha| \le n} \max_{x \in L_n} |f^{(\alpha)}(x)|, \; f \in E.$$

It is enough to show that for each bounded subset B of E there is $u \in B^\circ$ such that for each n there is $f_n \in E$ with $p_n(f_n) \le 1$ and $u(f_n) = 2$. Indeed, in this case the absolutely convex set

$$V := \bigcup_{n \in \mathbb{N}} \{v \in E'; \; |v(f)| \le p_n(f) \text{ for all } f \in E\}$$

is a neighborhood in E'_i, but not in E'_b. To complete the proof we fix a bounded set B in E. For each n we choose $\lambda_n > 0$ such that $p_n(f) \le \lambda_n$ for each $f \in B$. Now, for each n select a compact cube I_n

with non-empty interior contained in $L_{n+1} \setminus L_n$ and with Lebesgue measure $\mu_n < 1/(2^{n+1}\lambda_{n+1})$. Define $u(f) := 2\sum_k \int_{I_k} f d\mu$, $f \in E$. Clearly $|u| \leq 2p_0$, hence $u \in E'$. We have, for $f \in B$,

$$|u(f)| \leq 2\sum_k \mu_k \max_{x\in I_k} |f(x)| \leq 2\sum_k \mu_k \lambda_{k+1} \leq 1$$

so that $u \in B^\circ$. For each n we find $f_n \in \mathcal{D}(I_n)$ which is non-negative and satisfies $\int_{I_n} f_n d\mu = 1$. Clearly $f_n \in E$ and, since f_n vanishes on a neighborhood of L_n, $p_n(f_n) = p_0(f_n) = 1$ and $u(f_n) = 2$. ■

Grothendieck [62] proved that the non-distinguished Köthe echelon space E mentioned above even has the property that there is a discontinuous linear form on E_b' which is bounded on the bounded subsets of E_b'; i.e., $(E_b')' \neq (E_i')'$; in particular, the strong topology $\beta(E', E)$ is different from the topology $\mu(E', E'')$. This behavior is shared by all non-distinguished Köthe echelon spaces and by $E = C^\infty(G) \cap L^1(G)$. Kōmura, e.g. see [99, p. 292], gave an example of a non-distinguished Fréchet space E such that, on the other hand, $(E_b')' = (E_i')'$. More examples of Kōmura type were later given by Bonet, Dierolf, Fernández [37]. In 1993, Valdivia [101] proved that *if E is a separable Fréchet space which does not contain a copy of ℓ_1, then $(E_b')' = (E_i')'$ or, equivalently, E' endowed with the Mackey topology $\mu(E', E'')$ is bornological*. In connection with this result, Valdivia asked the following two questions:

(1) Is every separable Fréchet space not containing ℓ_1 distinguished?
(2) Does every non-separable Fréchet space E not containing ℓ_1 have the property that $(E', \mu(E', E''))$ is bornological?

All the examples of non-distinguished Fréchet spaces known at that time had many copies of ℓ_1. Both problems have a negative answer. (1) was solved by Díaz in [47], and (2) by Díaz and Miñarro in [51]. The first counterexample utilizes a variant of the James tree space, a separable Banach space which does not contain ℓ_1 and has a non-separable dual. The counterexample to question (2) requires the continuum hypothesis; the Fréchet space is constructed using weighted Banach spaces similar to the James quasireflexive Banach space defined on an uncountable index set.

(b) Quasinormable Fréchet spaces and other classes of Fréchet spaces

We recall the most important classes of Fréchet spaces. All the spaces in these classes are distinguished. A Fréchet space E is called *reflexive* if $(E_b')' = E$ algebraically via the evaluation mapping; in this case, E equals the strong dual $E'' := (E_b')_b'$ of E_b' topologically, and E_b' is barrelled. The Fréchet space E is reflexive if and only if every bounded subset of E is relatively $\sigma(E, E')$-compact. A Fréchet space E is said to be *Montel*, abbreviated by (FM), if each bounded subset of E is relatively compact. Every (FM)-space is reflexive. Köthe and Grothendieck gave examples of (FM)-spaces with a quotient topologically isomorphic to ℓ_1, hence not reflexive; see [69, 31.5]. According to Grothendieck [62], a Fréchet space E is called *totally reflexive* if every quotient of E is reflexive, as it happens in case E is a Banach space. Valdivia proved in [100] the following interesting characterization: *A Fréchet space E is totally reflexive if and only if E is the projective limit of a sequence of reflexive Banach spaces*. As a consequence, he obtained that the product of two totally reflexive Fréchet spaces is again totally reflexive, thus solving an open problem of Grothendieck [62]. These investigations of Valdivia were continued in [102].

A Fréchet space E is called *Schwartz*, abbreviated by (FS), if the linking maps are compact in the sense that, for each $n \in \mathbb{N}$ there is $m > n$ such that $\pi_{nm} : E_m \to E_n$ is compact, or equivalently, if for each $n \in \mathbb{N}$ there is $m > n$ such that for each $\varepsilon > 0$ there is a finite set F with $U_m \subset F + \varepsilon U_n$. Finally, a Fréchet space E is *nuclear*, abbreviated by (FN), if the linking maps are nuclear (or, equivalently, absolutely summable). Every (FN)-space is (FS), and every (FS)-space is (FM). The converse implications do not hold. We refer the reader to [66, 67, 74] for Schwartz and nuclear spaces.

Grothendieck [62] also proved that if the bounded subsets of the strong dual E_b' of a Fréchet space E are metrizable, then E is distinguished. The class of Fréchet spaces F for which the strong dual F_b' has metrizable bounded sets coincides with the class of Fréchet spaces which satisfy the *density condition* of Heinrich, and it contains every (FM)-space. These spaces were studied thoroughly by the authors, and they are treated in Section 4 below. An important subclass is the class of quasinormable Fréchet spaces. This class was introduced by Grothendieck [62] because it contains the most usual function spaces, and it contains every Banach space and every (FS)-space. A Fréchet space E is said to be *quasinormable* if for each $n \in \mathbb{N}$ there is $m \geq n$ such that, for each $\lambda > 0$, there exists a bounded subset B of E with $U_m \subset B + \lambda U_n$ or, equivalently, for each $n \in \mathbb{N}$ there is $m \geq n$ such that $E'_{U_m^\circ}$ and $\beta(E', E)$ induce the same topology on U_n°. A Fréchet space is Schwartz if and only if it is quasinormable and Montel. Quasinormable Fréchet spaces and their connection with the lifting of bounded sets are thoroughly investigated in [74, Chapter 26]. For example, it follows from [74, 26.17 and 26.18] that a Fréchet space F is quasinormable if and only if the transpose σ^t of the mapping σ in the canonical resolution of F is a topological homomorphism. The class of quasinormable locally convex spaces is rather large. In fact, every (DF)-space, and even every (gDF)-space, is quasinormable; see [67, 79]. – We will return to another aspect of the class of quasinormable Fréchet spaces in subsection 6.(f).

The Theorem of Josefson and Nissenzweig was proved independently by these two authors in 1975 and can be stated as follows: *A Banach space X is finite dimensional if and only if every sequence in X' which is $\sigma(X', X)$-convergent to zero also converges to zero for the norm topology of X'.* In 1980 Jarchow [67] conjectured that natural extensions of this theorem should hold for Fréchet spaces. These conjectures were proved by Bonet, Lindström, Schlumprecht, and Valdivia in the mid 90's. We summarize their results in the next theorem and refer the reader to [39] for more details, some similar results and consequences in related areas.

Theorem 2 *Let E be a Fréchet space.*

(1) E is quasinormable if and only if every null sequence in E'_b converges uniformly to zero on a 0-neighborhood in E.

(2) E is Montel (resp. Schwartz) if and only if every $\sigma(E', E)$-null sequence in E' is also strongly convergent to zero (resp., converges uniformly to zero on a 0-neighborhood in E).

(3) E does not contain a copy of ℓ_1 if and only if every null sequence in $(E', \mu(E', E))$ is also strongly convergent to zero. ■

The statement (2) in Theorem 2 is the extension of the Theorem of Josefson and Nissenzweig to Fréchet spaces. Part (1) of the theorem permits to conclude the first part of the next theorem. The second part in the theorem below can be found in [53], and it combines work by Bonet, Dierolf, Fernández [38] with [74, 26.12]. At the same time, this permitted to solve a problem of Grothendieck [62] by giving examples of distinguished Fréchet spaces E with a non-distinguished strong bidual E''.

Theorem 3 *Let E be a Fréchet space.*

(1) E is quasinormable if and only if its strong bidual E'' is quasinormable.

(2) The bidual E'' of E is distinguished if and only if (a) E and E''/E are distinguished, and (b) the quotient map $q : E'' \to E''/E$ lifts bounded sets (i.e., every bounded set in E''/E is contained in the image by the quotient map q of a bounded set in E''). ■

We refer the reader to [74, Chapter 26] for the results of Palamodov, Merzon, Bonet, Dierolf, Meise and Vogt about the lifting of bounded sets, quasinormability and the duality of exact sequences. The following recent result of Valdivia [102] is a nice complement of these results.

Theorem 4 *A Fréchet space E has the property that each quotient map $q : E \to G$ defined on E lifts bounded sets if and only if one of the following conditions holds: (a) E is a Banach space, (b) E is a Schwartz space, or (c) E is the product of a Banach space and the countable product ω of copies of the scalar field.* ■

(c) Bases and approximation properties

A sequence $(x_n)_n$ in a l.c. space E is called a *basis* if every $x \in E$ determines a unique sequence $(a_n)_n$ in the scalar field such that the series $\sum a_n x_n$ converges to x in the topology of E. Any l.c. space with a basis is separable. The basis $(x_n)_n$ is called a *Schauder basis* of E if its coefficient functionals $u_n(x) := a_n$, $n \in \mathbb{N}$, are continuous. Every basis in a Fréchet space is a Schauder basis. From this point on we will always write 'basis' and mean 'Schauder basis'.

The problem whether every separable Banach space has a basis appeared in 1931 for the first time in the Polish edition of Banach's book [6, Chapter 7, section 3]. It was clear to Banach, Mazur and Schauder that this question was related to an approximation problem mentioned by Mazur in the "Scottish Book" in 1936. This approximation problem was equivalent to the question whether every l.c. space has the approximation property, a question which was analyzed carefully by Grothendieck in his "thèse" [63]. A locally convex space E has the *approximation property (a.p.)* if the identity of E is the limit of a net of finite rank operators for the topology of uniform convergence on the absolutely convex compact subsets of E. If the net is equicontinuous, it is said that E has the *bounded a.p.* A Banach space E is said to have the *metric a.p.* if it has the bounded approximation property with a net of finite rank operators of operator norm ≤ 1.

Banach's problem was solved in the negative by Enflo in [59]: Each space ℓ_p ($1 \leq p \leq \infty$, $p \neq 2$), as well as c_0, has a closed subspace without the a.p. The case ℓ_p, $1 \leq p < 2$, is due to Szankowski in 1978. He also proved in [92] that the Banach space $\mathcal{L}(\ell_2, \ell_2)$ of all operators on the separable Hilbert space ℓ_2 does not have the a.p. This is a natural space, but it is not separable. Szankowski's result is still based essentially on the constructions of Enflo. Pisier [86], [87] constructed an infinite dimensional Banach space P such that P and P' are of cotype 2 and the injective and the projective topologies coincide on $P \otimes P$, thus solving a long standing problem of Grothendieck. The space P' does not have the approximation property [87], and it is a counterexample constructed in a completely different way.

All the usual Banach spaces (such as $C(K)$ or L_p) have the bounded approximation property (even the metric approximation property). Up to our knowledge, it is still unknown whether the (non-separable) space $H^\infty(\mathbb{D})$ of all bounded holomorphic functions on the unit disc $\mathbb{D}$ of the complex plane has the approximation property. On the other hand, the disc algebra $A(\mathbb{D})$ consisting of all the elements in $H^\infty(\mathbb{D})$ with continuous boundary values even has a basis.

All implications between the various approximation properties and the property of having a basis are either false or trivially true: In 1973, Figiel and Johnson constructed an example of a Banach space with separable dual and the approximation property, but without the bounded approximation property; Szarek in 1984 showed the existence of a reflexive, separable Banach space with the bounded (even the metric) approximation property, but without a basis. – For a more detailed account on approximation properties in Banach spaces see Casazza [42].

Every nuclear space has the approximation property. In 1960, Dynin and Mitjagin proved that every equicontinuous basis in a nuclear space is absolute. For a long time it was an open problem whether there exists a nuclear Fréchet space without a basis. The first example of such a space was given by Mitjagin and Zobin; we refer the reader to [67]. It was an open problem of Grothendieck since 1955 if every nuclear Fréchet space had the bounded approximation property. This was solved in the negative by Dubinsky in 1981; the example was simplified considerably by Vogt in [103].

It is a classical problem, but still open, whether every complemented subspace of an (FN)-space with a basis must itself have a basis. For more information, see the article 'Structure theory of power series spaces of infinite type' by Dietmar Vogt in this special issue of *Rev. R. Acad. Cien. Serie A. Mat.* Using methods due to Mitjagin, Zobin and Pełczyński, Taskinen constructed an (FS)-space with a basis and with a complemented subspace which is (FN) and does not have a basis, cf. [97].

Every nuclear space is Schwartz. In 1973, Hogbe-Nlend used Enflo's example to construct a Fréchet Schwartz space without the approximation property, see [67]. If an (FS)-space E has approximable linking maps (in the sense that they are limits in the operator norm of sequences of finite rank operators), then E has the approximation property. Nelimarkka proved in 1982 that every (FS)-space with the bounded approximation property must have approximable linking maps. The converse does not hold due to the examples of Dubinsky and Vogt mentioned above. Peris [83] gave an example of a Fréchet Schwartz space with the approximation property, but without approximable linking maps, thus answering a problem of Ramanujan in the negative.

In their important paper [58], Domański and Vogt show that if G is an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$, then the separable, complete and nuclear space $A(G)$ of all real analytic functions on G, endowed with its natural topology (which, however, has a rather complicated structure), does not have a Schauder basis. This is the first example of a separable complete function space without a basis which is natural in the sense that it had been in existence in analysis for many years and was not constructed on purpose.

(d) Moscatelli type constructions

A different way to construct examples of nuclear Fréchet spaces without basis was presented by Moscatelli in 1980. His approach is based on the following result due to Floret and Moscatelli [79, 8.4.38]: *Every Fréchet space with an unconditional basis is topologically isomorphic to a countable product of Fréchet spaces with a continuous norm and unconditional basis*. Moscatelli's idea is to use a "shifting" device which is implicit in the example of Grothendieck and Köthe of a non-distinguished echelon space. Moscatelli also utilized his method to construct a Fréchet space which is the projective limit of a sequence of Banach spaces with surjective linking maps and which is not isomorphic to a complemented subspace of a countable product of Banach spaces. A Fréchet space E is called a *quojection* if it is the projective limit of a sequence of Banach spaces with surjective linking maps or, equivalently, if every quotient with a continuous norm is a Banach space for the quotient topology. Every Fréchet space $C(X)$ of continuous functions, endowed with the compact open topology, is a quojection. Several authors proved that every quojection is a quotient of a countable product of Banach spaces; in particular, it is quasinormable. We refer the reader to the survey article [75] on quojections. Constructions of Moscatelli type with a shifting device have recently been used several times to construct various counterexamples. Bonet and Dierolf studied this type of constructions in a series of articles; e.g. see [35].

It is well-known that every non-normable Fréchet space admits a quotient isomophic to ω, and that it has a subspace topologically isomorphic to ω if and only if it does not admit a continuous norm; e.g. see [79]. In 1961, Bessaga, Pełczyński and Rolewicz showed that *a Fréchet space contains a subspace which is topologically isomorphic to an infinite dimensional nuclear Fréchet space with basis and a continuous norm if and only if it is not isomorphic to the product of a Banach space and ω*. As a consequence of the results mentioned above, *every non-normable Fréchet space always contains a subspace which can be written in the form $F \oplus G$ with F and G infinite dimensional spaces*. Miñarro [77] even proved that *a Fréchet space which is neither normable nor nuclear contains a closed subspace $F \oplus G$ with F and G infinite dimensional and such that F is not nuclear*. These results should be compared with the existence of *hereditarily indecomposable* Banach spaces established by Gowers and Maurey [60].

The situation for quotients is more complicated. Bellenot and Dubinsky in the separable case in 1982, and Önal and Terzioğlu in general in 1990 proved the following result: *A Fréchet space E does not have a quotient which is nuclear with a basis and a continuous norm if and only if the bidual E'' of E is a quojection*. Fréchet spaces satisfying this condition were introduced with another definition. Vogt showed that the original definition was equivalent to the one mentioned above. Dierolf, Moscatelli, Behrends and Harmand constructed Fréchet spaces E such that E'' is a quojection, but E is not a quojection. Fréchet spaces E such that E'' is a quojection are called *prequojections*; they are also quasinormable by Theorem 3. More information about prequojections can be seen in [75].

(e) Nuclearity and the Levy Steinitz rearrangement theorem

In this subsection we assume that all the vector spaces are real. For a convergent series $\sum u_k$ in a locally convex space E the *domain of sums* $S(\sum u_k)$ is the set of all $x \in E$ which can be obtained as the sum of a convergent rearrangement of the series. In terms of this notion, Riemann's famous rearrangement theorem from 1867 states that on the real line the set of sums of a convergent series is either a single point or the whole line. Later on, Levy and Steinitz extended Riemann's result to finite dimensional spaces by describing the sets of sums of each convergent series in $\mathbb{R}^N$, $N \in \mathbb{N}$. To state their result we introduce the following notation. The set of summing functionals of a convergent series $\sum u_k$ in a locally convex space E is defined by $G(\sum u_k) := \{f \in E';\ \sum_k |f(u_k)| < \infty\}$. The result is as follows: *If $\sum u_k$ is a convergent series in $\mathbb{R}^N$, then $S(\sum u_k) = \sum_k u_k + G(\sum u_k)^\circ$, which is a closed affine subspace of $\mathbb{R}^N$.* The result fails for infinite dimensional Banach spaces. In fact, Kadets, Enflo and others proved that every infinite dimensional Banach space contains a convergent series whose set of sums consists exactly of two different points. We refer the interested reader to the nice book of Kadets and Kadets [68].

In his two papers [7, 8], Banaszczyk proved the following extension of the theorem of Levy and Steinitz. The result gives a new characterization of nuclear Fréchet spaces.

Theorem 5 *A Fréchet space E is nuclear if and only if the domain of sums of each convergent series $\sum u_k$ in E is given by the formula $S(\sum u_k) = \sum_k u_k + G(\sum u_k)^\circ$. In particular, in each (FN)-space E the domain of sums is a closed affine subspace of E.* ■

In the article [29], Bonet and Defant investigated the domains of sums of convergent series in (DF)-spaces. Their description of the set of sums requires some notation. Let E be a complete (DF)-space with a fundamental sequence $(B_n)_n$ of bounded sets. We denote by E_n the Banach space E_{B_n} spanned by all positive multiples of B_n, with its Minkowski functional as norm. Assume that every convergent sequence in E is contained and converges in some E_n. Then, given a convergent series $\sum u_k$ in E, there is $n(0)$ such that $\sum_k u_k$ converges in $E_{n(0)}$. We denote by $G^\circ_{loc}(\sum u_k)$ the subspace of E which is the union for $n \geq n(0)$ of all the elements $x \in E_n$ such that $f(x) = 0$ for each $f \in (E_n)'$ with $\sum_k |f(u_k)| < \infty$.

Theorem 6 *Let E be the strong dual of a nuclear Fréchet space; i.e., a (DFN)-space.*

(1) The domain of sums of every convergent series $\sum u_k$ in E is the affine subspace

$$S(\sum u_k) = \sum_k u_k + G^\circ_{loc}(\sum u_k).$$

(2) If E is not isomorphic to the space φ of all finite sequences, then there is a convergent series in E such that its domain of sums is not closed. ■

Theorem 7 *Let E be a complete (DF)-space in which every convergent sequence converges locally. If for each convergent series $\sum u_k$ in E we have*

$$S(\sum u_k) = \sum_k u_k + G^\circ_{loc}(\sum u_k),$$

then E is nuclear. ■

The approach of [29], using local convergence and bounded sets, permits to show that in a large class of nonmetrizable spaces the domains of sums of all convergent series are affine subspaces. This class includes the space of test functions, the space of distributions and the space of real analytic functions.

4. The density condition and the dual density conditions

(a) The density condition: definition and direct proofs of some theorems

A Fréchet space E with a basis $(U_n)_n$ of closed absolutely convex 0-neighborhoods (of which we assume in the sequel that $U_{n+1} + U_{n+1} \subset U_n$ holds) is said to have the *density condition* if for every sequence

$(\lambda_j)_j$ of strictly positive numbers and for each $n \in \mathbb{N}$ there is a set $B \in \mathcal{B}(E)$ and there is $m(n) \geq n$ such that

$$\bigcap_{j=1}^{m(n)} \lambda_j U_j \subset B + U_n.$$

It is easy to see that the definition does not depend on the basis $(U_n)_n$ of 0-neighborhoods. The density condition was introduced by Heinrich [65] in connection with his investigations on ultrapowers of locally convex spaces. The density condition for Fréchet spaces was thoroughly investigated by the authors in [16]. The dual density conditions for (DF)-spaces were treated later in [17, 18]. The density condition turned out to be connected to vector valued sequence spaces, the abstract theory of (DF)-spaces, projective description of weighted inductive limits, tensor products of Fréchet or (DF)-spaces, operator ideals, infinite dimensional holomorphy, and unbounded operator algebras. Below we give direct proofs of some of the main results and properties of Fréchet spaces with the density condition.

Theorem 8 *The Fréchet space E satisfies the density condition if and only if*

$$\forall (\lambda_j)_j \subset (0, \infty)\ \exists B \in \mathcal{B}(E)\ \forall n\ \exists m \geq n :\ \bigcap_{j=1}^{m} \lambda_j U_j \subset B + U_n.$$

PROOF. Only the necessity requires a proof. We fix $(\lambda_j)_j \subset (0, \infty)$. Set $V_m := \bigcap_{j=1}^m \lambda_j U_j$, $m \in \mathbb{N}$. Since E satisfies the density condition, for each n there are $B_n \in \mathcal{B}(E)$ and $m(n)$ with $V_{m(n)} \subset B_n + U_n$. Define

$$B := \bigcup_{n\in\mathbb{N}} (U_n + V_{m(n)}) \cap B_n.$$

We show that B is bounded in E: Fix $k \in \mathbb{N}$. For $n \geq k$, we have $U_n + V_{m(n)} \subset (1 + \lambda_k)U_k$. On the other hand, the set $(U_n + V_{m(n)}) \cap B_n$ is bounded for $1 \leq n < k$, from where the boundedness of B follows. It remains to prove that $V_{m(n)}$ is contained in $B + U_n$. This is easy: If $x \in V_{m(n)}$, then $x = y + z$ for some $y \in B_n$ and $z \in U_n$, hence $y = x - z \in (U_n + V_{m(n)}) \cap B_n \subset B$. Thus $x \in B + U_n$. ■

Corollary 1 *Every quasinormable Fréchet space E satisfies the density condition, and every Fréchet space E with the density condition is distinguished.*

PROOF. First we suppose that E is quasinormable. To show that E satisfies the density condition, we fix $(\lambda_j)_j$ and n. Since E is quasinormable, there is m such that for each $\varepsilon > 0$ there is $B_\varepsilon \in \mathcal{B}(E)$ with $U_m \subset B_\varepsilon + \varepsilon U_n$. We put $C := B_{1/\lambda_m} \in \mathcal{B}(E)$ to conclude

$$\bigcap_{j=1}^{m} \lambda_j U_j \subset \lambda_m U_m \subset \lambda_m(C + (1/\lambda_m)U_n) \subset \lambda_m C + U_n.$$

Now assume that E satisfies the density condition. Let W be a 0-neighborhood in E'_i. Find a sequence $(\mu_j)_j \in (0, \infty)$ such that $\Gamma\left(\bigcup_{j\in\mathbb{N}} \mu_j U_j^\circ\right) \subset W$. We apply Theorem 8 to the sequence $(2\mu_j^{-1})_j$ to find $B \in \mathcal{B}(E)$ satisfying the equivalent condition. If $u \in B^\circ$, there is n with $u \in U_n^\circ$. For this n, select m as in the condition in Theorem 8, and apply the bipolar theorem (as in [67, 8.2.4]; also note that the sets U_j° are $\sigma(E', E)$-compact) to conclude

$$u \in 2(B + U_n)^\circ \subset 2\left(\bigcap_{j=1}^{m} 2\mu_j^{-1} U_j\right)^\circ = \Gamma\left(\bigcup_{j=1}^{m} \mu_j U_j^\circ\right) \subset W.$$

Hence $B^\circ \subset W$, and W is a 0-neighborhood in E'_b. ■

Theorem 9 *The Fréchet space E has the density condition if and only if there is a double sequence $(B_{n,k})_{n,k} \subset \mathcal{B}(E)$ such that for each n and each $C \in \mathcal{B}(E)$ there is k with $C \subset B_{n,k} + U_n$.*

PROOF. Suppose first that the condition is satisfied. Fix $(\lambda_j)_j$ and n. Suppose that for each m the set $V_m := \bigcap_{j=1}^m \lambda_j U_j$ is not contained in $B_{n,m} + U_n$. For each m, select $x_m \in V_m \setminus (B_{n,m} + U_n)$. Clearly, the set $C := \{x_m;\ m \in \mathbb{N}\}$ is bounded in E. By assumption it is contained in $B_{n,l} + U_n$ for some l, which is a contradiction to the choice of x_l.

To prove the converse, fix n, and set

$$\mathcal{W}_n := \{\bigcap_{j=1}^{m} \lambda_j U_j;\ m \in \mathbb{N},\ \lambda_j \in \mathbb{N} \text{ such that } \bigcap_{j=1}^{m} \lambda_j U_j \subset B + U_n \text{ for some } B \in \mathcal{B}(E)\}.$$

The family $\mathcal{W}_n$ is at most countable, and for each $V \in \mathcal{W}_n$ there is $B_V \in \mathcal{B}(E)$ with $V \subset B_V + U_n$. We denote by $(B_{n,k})_k$ the sequence of bounded sets obtained in this way. Letting n again be arbitrary, we have found a double sequence $(B_{n,k})_{n,k} \subset \mathcal{B}(E)$. Given $C \in \mathcal{B}(E)$, we determine a sequence $(\lambda_j)_j \subset \mathbb{N}$ with $C \subset \bigcap_{j\in\mathbb{N}} \lambda_j U_j$. By the definition of the density condition, for this sequence and $n \in \mathbb{N}$, we find $V \in \mathcal{W}_n$ with $C \subset V \subset B_{n,k} + U_n$ for some k. ■

Corollary 2 *Every Fréchet Montel space has the density condition.*

PROOF. Since E is an (FM)-space, E is separable (cf. [67, 11.6.2]). Let $\{x_k;\ k \in \mathbb{N}\}$ be a dense subset of E. For each n, k we put $B_{n,k} := \Gamma(x_1, ..., x_k)$. Fix n. If C is a bounded, hence relatively compact, subset of E, there is a finite subset F of E such that $C \subset F + U_{n+1}$. By density of $(x_k)_k$, we find k such that $F \subset B_{n,k} + U_{n+1}$. This yields $C \subset B_{n,k} + U_n$. The conclusion follows from Theorem 9. ■

Corollary 3 *A Fréchet space has the density condition if and only if the bounded subsets of E'_b are metrizable.*

PROOF. The bounded sets of E'_b are metrizable if and only if for each n the origin has a countable basis of closed absolutely convex neighborhoods in U_n° for the topology $\beta(E', E)$ (cf. [67, 9.2.4]). This is equivalent to the existence of a double sequence $(B_{n,k})_{n,k} \subset \mathcal{B}(E)$ such that for each n and each $C \in \mathcal{B}(E)$ of E there is k with $B_{n,k}^\circ \cap U_n^\circ \subset C^\circ$. By the bipolar theorem and simple properties of polars (see [67, 8.2.1, 8.2.4]), this condition is equivalent to the condition in Theorem 9:

$$B_{n,k}^\circ \cap U_n^\circ \subset C^\circ \quad\Rightarrow\quad C \subset (B_{n,k}^\circ \cap U_n^\circ)^\circ = \overline{\Gamma}(B_{n,k} \cup U_n) \subset B_{n,k} + U_n + U_n \subset B_{n,k} + U_{n-1},$$
$$C \subset B_{n,k} + U_n \quad\Rightarrow\quad B_{n,k}^\circ \cap U_n^\circ \subset 2(B_{n,k} + U_n)^\circ \subset 2C^\circ. \quad ■$$

Corollary 4 *A Fréchet space has the density condition if and only if there is $B \in \mathcal{B}(E)$ such that for each n and for each $C \in \mathcal{B}(E)$ there is $\lambda > 0$ with $C \subset \lambda B + U_n$.*

PROOF. Suppose first that E has the density condition, and select the double sequence $(B_{n,k})_{n,k}$ as in Theorem 9. Since E is metrizable, there are $\rho_{n,k} > 0$, n, $k \in \mathbb{N}$, such that $B := \Gamma\left(\bigcup_{n,k} \rho_{n,k} B_{n,k}\right)$ is bounded (cf. [74, 26.6.(a)]). It is easy to see that $\overline{B}$ satisfies the desired condition. The converse follows by applying Theorem 9 to the double sequence $B_{n,k} := kB$, $n, k \in \mathbb{N}$. ■

Corollary 5 *Every Fréchet space with the density condition has a total bounded set, or equivalently, E'_b admits a continuous norm.*

PROOF. If B is the bounded set whose existence is ensured by Corollary 4, it is easy to see that the linear span of B is dense in E. ■

Amemiya constructed reflexive Fréchet spaces with no total bounded sets [69, 29.6]. They are examples of distinguished Fréchet spaces which do not satisfy the density condition.

For a locally convex space E, we denote by $\ell_1(E)$ the space of all absolutely summable sequences in E. If E is complete, then $\ell_1(E)$ is topologically isomorphic to the complete projective tensor product $\ell_1 \hat{\otimes}_\pi E$; see [69].

Theorem 10 *A Fréchet space E has the density condition if and only if the Fréchet space $\ell_1(E)$ is distinguished.*

PROOF. The strong dual $(\ell_1(E))'_b$ of $\ell_1(E)$ is topologically isomorphic to the space $\ell_\infty(E'_b)$ of all bounded sequences in E'_b, endowed with the topology of uniform convergence. The duality is given by

$$< x, u >:= \sum_i < x(i), u(i) >,\ x = (x(i))_i \in \ell_1(E),\ \ u = (u(i))_i \in \ell_\infty(E'_b).$$

This follows from a direct argument using the special form of the bounded sets in $\ell_1(E)$ which is ensured by [85, 1.5.8]; compare with the property (BB) of the pair (ℓ_1, E) mentioned in Section 6. Accordingly, it is enough to show that E has the density condition if and only if $\ell_\infty(E'_b)$ is bornological. We denote by $(B_n)_n$ the fundamental sequence of bounded sets in E'_b given by the polars of the basis $(U_n)_n$ of 0-neighborhoods in E. The sets

$$C_n := \{u \in \ell_\infty(E'_b);\ u(i) \in B_n \text{ for each } i \in \mathbb{N}\},\ \ n \in \mathbb{N},$$

form a fundamental sequence of bounded sets in the (DF)-space $\ell_\infty(E'_b)$.

Suppose first that E satisfies the density condition. We let W' denote an absolutely convex bornivorous subset of $\ell_\infty(E'_b)$ and choose a sequence $(\lambda_j)_j$ of positive numbers such that $W := \bigcup_m \sum_{j=1}^m \lambda_j C_j \subset W'$. Since the topology of the (DF)-space $\ell_\infty(E'_b)$ is localized to its bounded subsets ([69, 29.3.(2)]), it is enough to show that $W \cap C_n$ is a 0-neighborhood in C_n, $n = 1, 2, ...$ Since E has the density condition, we can apply the bipolar theorem to get $m > n$ and a 0-neighborhood B° in E'_b such that $B_n \cap B^\circ \subset \sum_{j=1}^m \lambda_j B_j$. Let V denote the 0-neighborhood in $\ell_\infty(E'_b)$ defined by $V := \{u \in \ell_\infty(E'_b);\ u(i) \in B^\circ \text{ for each } i \in \mathbb{N}\}$. It is easy to see that $C_n \cap V \subset \sum_{j=1}^m \lambda_j C_j \subset W$, from which the conclusion follows.

Suppose now that E does not satisfy the density condition. By the bipolar theorem, we find a sequence $(\lambda_j)_j$ and $n \in \mathbb{N}$ such that for each m and each bounded set B in E the set $B_n \cap B^\circ$ is not contained in $D_m := \Gamma(\bigcup_{j=1}^m \lambda_j B_j)$. We define $A_m := \{u \in \ell_\infty(E'_b);\ u(i) \in D_m \text{ for each } i \in \mathbb{N}\}$ for every $m = 1, 2, ...$ Clearly the set $A := \bigcup_m A_m$ is absolutely convex and bornivorous in $\ell_\infty(E'_b)$. Since we assume that this space is bornological, there is a bounded set B in E such that

$$U := \{u \in \ell_\infty(E'_b);\ u(i) \in B^\circ \text{ for each } i \in \mathbb{N}\} \subset A.$$

However, given B, for each m we can find $x(m) \in (B_n \cap B^\circ) \setminus D_m$. We clearly have $x = (x(m))_m \in U$. Thus, there must be k with $x \in A_k$, hence $x(k) \in D_k$, which yields the desired contradiction. ■

The density condition for Köthe echelon spaces will be considered in Section 5.(d). The density condition for Fréchet spaces is stable under the formation of complemented subspaces and of countable products. It is not stable under the formation of closed subspaces or quotients. In fact, every Fréchet space is topologically isomorphic to a closed subspace of a countable product of Banach spaces, and every separable Fréchet space is topologically isomorphic to a quotient of an (FM)-space (see [99, page 221]). Bonet, Dierolf, Fernández showed that neither the density condition nor distinguishedness is a three-space property; see [53]. Peris [80] showed that a Fréchet space E satisfies the density condition if and only if E'' does. This should be compared with Theorem 3.

(b) Additional results

The following examples related to Theorem 1 were given in [41]:

Theorem 11 *For every $1 < p < \infty$ and for every open subset G of $\mathbb{R}^N$, $E := C^\infty(G) \cap L^p(G)$ is reflexive, but it does not satisfy the density condition.* ■

Note that the isomorphic classification and sequence space representations of such intersection function spaces were treated by Albanese, Metafune, Moscatelli [4, 5].

Önal and Terzioğlu [78], confirming a conjecture in [33], proved the following result.

Theorem 12 *Every closed subspace of a Fréchet space E has the density condition if and only if E is Montel or E is topologically isomorphic to the product of a Banach space and a (finite or infinite) countable product of copies of $\mathbb{K}$.* ■

The corresponding result for quotients confirmed a conjecture in [33] and was proved by Albanese [1, 2].

Theorem 13 *A Fréchet space E is quasinormable if and only if every quotient of E has the density condition.* ■

This deep result permits to obtain as a corollary a theorem of Bellenot [11] which was proved using non-standard analysis. A different proof was included in [100].

Corollary 6 *A Fréchet space E is Schwartz if and only if every quotient is Montel.*

PROOF. Every quotient of an (FS)-space is also Schwartz. We assume conversely that every quotient of E is Montel. In particular, E is Montel. To conclude it is enough to show that E is quasinormable. By assumption every quotient of E has the density condition, hence the conclusion follows from Theorem 13. ■

The stability of the density condition under the formation of projective tensor products is analyzed in Section 6.(f).

(c) The dual density conditions

A (DF)-space E with a fundamental sequence $(B_n)_n$ of absolutely convex bounded sets is said to satisfy the *strong dual density condition* (resp. *dual density condition*) if for each decreasing sequence $(\lambda_j)_j$ of strictly positive numbers and for each n there exist a neighborhood U in E and $m \geq n$ such that

$$B_n \cap U \subset \Gamma\left(\bigcup_{j=1}^m \lambda_j B_j\right) \quad (\text{resp., } B_n \cap U \subset \overline{\Gamma}\left(\bigcup_{j=1}^m \lambda_j B_j\right)).$$

We abbreviate these conditions as (SDDC) and (DDC), respectively. Every (DF)-space E with the (DDC) (resp. (SDDC)) must be quasibarrelled (resp., bornological). If the fundamental sequence $(B_n)_n$ can be selected in such a way that each B_n is compact for a coarser Hausdorff locally convex topology on E, then (DDC) and (SDDC) are equivalent for E. In particular, it follows from the bipolar theorem that a Fréchet space F satisfies the density condition if and only if F'_b satisfies (DDC) or (SDDC). However, in general these two conditions are not equivalent, even in the framework of (DF)-spaces, which is the "good" setting for (DDC) and (SDDC). We refer the reader to [17, 18] for details and applications to spaces of vector valued co-echelon spaces (also see Section 5.(f)) and to weighted inductive limits of spaces of continuous functions. The main characterization of these two properties is the following theorem.

Theorem 14 *Let E denote a (DF)-space.*

(a) The following conditions are equivalent:

(1) E satisfies the dual density condition (DDC),

(2) each bounded subset of E is metrizable,

(3) $\ell_\infty(E)$ is quasibarrelled,

(4) $\ell_\infty(E)$ satisfies (DDC).

(b) The following conditions are equivalent:

(1) E satisfies the strong dual density condition (SDDC),

(2) $\ell_\infty(E)$ is bornological,

(3) $\ell_\infty(E)$ satisfies (SDDC). ■

Corollary 7 *Let E again denote a (DF)-space.*

(a) The following assertions are equivalent:

(1) (i) E is barrelled, and (ii) each bounded subset of E is metrizable (or equivalently, E satisfies (DDC)),

(2) $\ell_\infty(E)$ is barrelled.

(b) Similarly, the following assertions are equivalent:

(1) (i) E is ultrabornological, and (ii) E satisfies the strong dual density condition (SDDC),

(2) $\ell_\infty(E)$ is ultrabornological. ■

5. Köthe echelon and co-echelon spaces

(a) Definition and first properties of $\lambda_p(A)$, $k_p(V)$, and $K_p(\overline{V})$

In this section, I will always denote an arbitrary (infinite) index set and $A = (a_n)_{n\in\mathbb{N}}$ an increasing sequence of strictly positive functions, which will also be called a *Köthe matrix* on I. (One usually thinks of the case $I = \mathbb{N}$.) Corresponding to each Köthe matrix $A = (a_n)_n$ and $1 \le p < \infty$, we associate the spaces

$$\lambda_p(I,A) = \{x = (x(i))_{i\in I} \in \mathbb{C}^I \text{ (or } \mathbb{R}^I);\ \forall n \in \mathbb{N}:\ q_n^p(x) = (\textstyle\sum_{i\in I}(a_n(i)|x(i)|)^p)^{1/p} < \infty\},$$
$$\lambda_\infty(I,A) = \{x = (x(i))_{i\in I} \in \mathbb{C}^I \text{ (or } \mathbb{R}^I);\ \forall n \in \mathbb{N}:\ q_n^\infty(x) = \sup_{i\in I} a_n(i)|x(i)| < \infty\},$$
$$\lambda_0(I,A) = \{x = (x(i))_{i\in I} \in \mathbb{C}^I \text{ (or } \mathbb{R}^I);\ \forall n \in \mathbb{N}:\ (a_n(i)x(i))_i \text{ converges to } 0\},$$

the last space endowed with the topology induced by $\lambda_\infty(I,A)$. Very often the index set I is omitted from the notation; we will follow this tradition from now on. The spaces $\lambda_p(A)$ are called (Köthe) *echelon spaces* of order p, $1 \le p \le \infty$ or $p = 0$; they are Fréchet spaces with the sequence of norms $p_n = q_n^p$, $n = 1, 2, ...$ If A consists of a single function $a = (a(i))_i$, we sometimes write $\ell_p(a)$ instead of $\lambda_p(A)$, $1 \le p \le \infty$, and $c_0(a)$ instead of $\lambda_0(A)$. The elements of the echelon spaces are considered as generalized sequences, and $\ell_p(a)$ is a diagonal transform (via a) of the space $\ell_p(I) = \ell_p(I,1)$, $1 \le p \le \infty$.

For a Köthe matrix $A = (a_n)_n$, let $V = (v_n)_n$ denote the associated decreasing sequence of functions $v_n = 1/a_n$, and put

$$k_p(V) = k_p(I,V) = \operatorname*{ind}_n \ell_p(v_n),\ 1 \le p \le \infty, \text{ and } k_0(V) = \operatorname*{ind}_n c_0(v_n).$$

That is, $k_p(V)$ is the increasing union of the Banach spaces $\ell_p(v_n)$ resp. $c_0(v_n)$, endowed with the strongest locally convex topology under which the injection of each of these Banach spaces is continuous. The spaces $k_p(V)$ are called *co-echelon spaces* of order p; as (LB)-spaces, they are ultrabornological (DF)-spaces. The mapping $k_0(V) \to k_\infty(V)$ is obviously continuous, but it turns out that it is even a topological isomorphism into $k_\infty(V)$. For a systematic treatment of echelon and co-echelon spaces see Bierstedt, Meise, Summers [28], from which the next definitions and results are also taken.

It is helpful in the treatment of echelon and co-echelon spaces to introduce, for a given decreasing sequence $V = (v_n)_n$ of strictly positive functions on I or for the corresponding Köthe matrix $A = (a_n)_n$, the system

$$\lambda_\infty(A)_+ = \{\overline{v} = (\overline{v}(i))_{i\in I} \in \mathbb{R}^I_+;\ \forall n \in \mathbb{N}:\ \sup_{i\in I} \frac{\overline{v}(i)}{v_n(i)} = \sup_{i\in I} a_n(i)\overline{v}(i) < \infty\},$$

which in the sequel will be denoted by $\overline{V} = \overline{V}(V)$. If I is countable, this system always contains strictly positive functions. $\overline{V}$ can be used to characterize the bounded subsets of $\lambda_p(A)$, as follows.

Proposition 1 *Let A be a Köthe matrix on I. Then a subset B of $\lambda_p(A)$, $1 \le p \le \infty$, is bounded if and only if there exists $\overline{v} \in \overline{V}$ so that*

$$B \subset \overline{v}B(\ell_p) = \{y \in \mathbb{C}^I \text{ (or } \mathbb{R}^I);\ \exists z \in B(\ell_p):\ y(i) = \overline{v}(i)z(i)\ \forall i \in I\},$$

where $B(\ell_p)$ denotes the closed unit ball of the Banach space $\ell_p = \ell_p(I, 1)$. ■

Next, with $\overline{V}$ we associate the following spaces

$$K_p(\overline{V}) = K_p(I, \overline{V}) = \operatorname*{proj}_{\overline{v}\in\overline{V}}\ \ell_p(\overline{v}),\ 1 \le p \le \infty, \text{ and } = \operatorname*{proj}_{\overline{v}\in\overline{V}}\ c_0(\overline{v}),\ p = 0.$$

These spaces are equipped with the complete locally convex topology given by the seminorms $q^p_{\overline{v}}$, $\overline{v} \in \overline{V}$, where $q^p_{\overline{v}}(x) = (\sum_{i\in I}(\overline{v}(i)|x(i)|)^p)^{1/p}$, $1 \le p < \infty$, and $q^\infty_{\overline{v}}(x) = \sup_{i\in I} \overline{v}(i)|x(i)|$. The notation suggests that $K_p(\overline{V})$ is, in some sense, related to $k_p(V)$. In fact, it is easily seen that $k_p(V)$ is continuously embedded in $K_p(\overline{V})$, $p = 0$ or $1 \le p \le \infty$, and that $k_p(V) = K_p(\overline{V})$ algebraically for $1 \le p \le \infty$. More exactly, one proves:

Theorem 15 *Let A be a Köthe matrix on I and take V and $\overline{V} = \overline{V}(V)$ as above.*

(a) For $1 \le p < \infty$, $k_p(V)$ equals $K_p(\overline{V})$ algebraically and topologically. In particular, the inductive limit topology is given by the system $(q^p_{\overline{v}})_{\overline{v}\in\overline{V}}$ of seminorms, and $k_p(V)$ is always complete.

(b) $K_0(\overline{V})$ is the completion of $k_0(V)$. The inductive limit topology of $k_0(V)$ is given by the seminorms $q^\infty_{\overline{v}}$. However, $k_0(V)$ can be a proper subspace of $K_0(\overline{V})$.

(c) $k_\infty(V)$ equals $K_\infty(\overline{V})$ algebraically, and the two spaces have the same bounded sets. $k_\infty(V)$ is the bornological space associated with $K_\infty(\overline{V})$, but, in general, the inductive limit topology is strictly stronger than the topology of $K_\infty(\overline{V})$. ■

(b) Duality of echelon and co-echelon spaces

At this point, we are able to state the duality of the echelon and co-echelon spaces.

Theorem 16 *For $1 \le p < \infty$ or $p = 0$, if $\frac{1}{p} + \frac{1}{q} = 1$ (where we take $q = \infty$ for $p = 1$ and $q = 1$ for $p = 0$), then $(\lambda_p(A))'_b = K_q(\overline{V})$ and $(k_p(V))'_b = \lambda_q(A)$.* ■

Corollary 8 *(a) For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ or for $p = 0$ and $q = 1$, we have $(\lambda_p(A))'_b = k_q(V)$.*

(b) In case $1 < p < \infty$, the spaces $\lambda_p(A)$ and $k_p(V)$ are reflexive.

(c) $\lambda_0(A)$ is always distinguished, and $((\lambda_0(A))'_b)'_b = (k_1(V))'_b = \lambda_\infty(A)$.

(d) $K_0(\overline{V})$ is a barrelled (DF)-space with $(K_0(\overline{V}))'_b = (k_0(V))'_b = \lambda_1(A)$, and hence there is the biduality $((k_0(V))'_b)'_b = ((K_0(\overline{V}))'_b)'_b = K_\infty(\overline{V})$.

(e) $k_\infty(V) = (\lambda_1(A))'_i$, and this space is always complete.

(f) The following assertions are equivalent:

(i) $(\lambda_1(A))'_b = k_\infty(V)$, *(i')* $K_\infty(\overline{V}) = k_\infty(V)$,

(ii) $\lambda_1(A)$ *is distinguished.* ■

(c) The regularly decreasing condition

It remains to discuss when $k_0(V)$ is complete and when $\lambda_1(A)$ is distinguished. The first question was solved in [27, 28]; part of this was also found by Valdivia, independently. The sequence $V = (v_n)_n$ associated with the Köthe matrix $A = (a_n)_n$ is said to be *regularly decreasing* if

$$\forall n \, \exists m \geq n \, \forall I_0 \subset I \ \ \inf_{i \in I_0} \frac{v_m(i)}{v_n(i)} > 0 \implies \inf_{i \in I_0} \frac{v_k(i)}{v_n(i)} > 0 \ \ \forall k \geq m.$$

Theorem 17 *(a) For* $1 \leq p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *or for* $p = 0$ *and* $q = 1$*, the following assertions are equivalent:*

(1) V *is regularly decreasing.*

(2) $\lambda_p(A)$ *is quasinormable.*

(3) $K_q(\overline{V})$ *satisfies the strict Mackey convergence condition.*

(4) $k_q(V) = \mathrm{ind}_n \, \ell_q(v_n)$ *is boundedly retractive; i.e., for each bounded set* B *in the inductive limit there exists* $n \in \mathbb{N}$ *such that* B *is a bounded subset of* $\ell_q(v_n)$ *and such that the norm topology of* $\ell_q(v_n)$ *induces the inductive limit topology on* B*.*

(b) $\lambda_\infty(A)$ *is quasinormable if and only if* V *is regularly decreasing.*

(c) The following assertions are equivalent:

(1) V *is regularly decreasing.*

(2) $k_0(V) = \mathrm{ind}_n \, c_0(v_n)$ *is a regular inductive limit; i.e., every bounded set* $B \subset k_0(V)$ *must be contained and bounded in some* $c_0(v_n)$*.*

(3) $k_0(V)$ *is complete, or equivalently,* $k_0(V) = K_0(\overline{V})$*.*

(4) $k_0(V)$ *is closed in* $k_\infty(V)$*.*

(5) $k_0(V)$ *is boundedly retractive.* ■

Every compact subset of a Fréchet space is contained in the closed absolutely convex hull of a null sequence. (This fact is an easy consequence of the Banach-Dieudonné theorem, cf. [74, 26.21] or [69, 21.10].) If an (LB)-space $E = \mathrm{ind}_n \, E_n$ is boundedly retractive, then every compact subset $C \subset E$ is contained and compact in a step E_n; thus, C is a subset of the closed absolutely convex hull of some null sequence in E_n, hence in E. However, this assertion may fail in arbitrary (LB)-spaces. Using an idea of Frerick and Wengenroth, Albanese [3] proved that *the sequence* V *is regularly decreasing if and only if every compact subset of the complete (LB)-space* $k_\infty(V)$ *is contained in the closed absolutely convex hull of a null sequence*.

(d) Condition (D)

To treat the distinguishedness of $\lambda_1(A)$, the following *condition (D)* was introduced in Bierstedt, Meise [26]. The decreasing sequence $V = (v_n)_n$ on I is said to satisfy (D) if there exists an increasing sequence $\mathcal{I} = (I_m)_{m \in \mathbb{N}}$ of subsets of I such that

$$\forall m \, \exists n_m \, \forall k > n_m : \ \inf_{i \in I_m} \frac{v_k(i)}{v_{n_m}(i)} > 0,$$

$$\forall n \, \forall I_0 \subset I \ \text{ with } \ I_0 \cap (I \setminus I_m) \neq \emptyset \ \text{ for each } \ m \in \mathbb{N} \, \exists n' = n'(n, I_0) > n : \ \inf_{i \in I_0} \frac{v_{n'}(i)}{v_n(i)} = 0.$$

Theorem 18 *Let $A = (a_n)_n$ be a Köthe matrix on I and let V and $\overline{V}$ be as before.*

(1) Then the following assertions are equivalent:

(i) V satisfies the condition (D).

(ii) $\lambda_1(A)$ is distinguished.

(iii) $\lambda_p(A)$, $1 \le p \le \infty$ or $p = 0$, satisfies the density condition.

(iv) $\lambda_\infty(A)$ is distinguished.

(2) The following assertions are equivalent:

(i) A satisfies the condition (ND): There is n and there is a decreasing sequence $(J_k)_k$ of subsets of I such that

$$\forall k \ge n : \inf_{i \in J_k} \frac{a_n(i)}{a_k(i)} > 0 \text{ and } \forall k \ge n\ \exists \ell(k) > k : \inf_{i \in J_k} \frac{a_n(i)}{a_{\ell(k)}(i)} = 0.$$

(ii) $\lambda_1(A)$ is not distinguished.

(iii) The duals of $(\lambda_1(A))'_b$ and $(\lambda_1(A))'_i$ do not coincide.

(iv) There is a sectional subspace of $\lambda_1(A)$ isomorphic to an echelon space $\lambda_1(\mathbb{N}\times\mathbb{N}, B)$ for a Köthe matrix $B = (b_n)_n$ on $\mathbb{N}\times\mathbb{N}$ which satisfies $b_n(i,j) = b_1(i,j)$ if $n \le i$ and $\lim_{j\to\infty} (b_n(n,j)/b_{n+1}(n,j)) = 0$. ■

Bierstedt, Meise [26] proved (1)(i)⇒(ii). The equivalences of (i), (ii) and (iii) in part (1) were completed by the authors in [16]. The equivalence of (iv) with the other conditions in (1) was proved by Bastin [9]. The condition (ND) in Theorem 18 was introduced by Bierstedt, Meise [26] who proved (2)(i)⇒(ii). Vogt [108] introduced the condition on the matrix B in (2)(iv) and showed that, if B satisfies this assumption, then $\lambda_1(\mathbb{N} \times \mathbb{N}, B)$ is not distinguished. The other implications in part (2) are due to Bastin, Bonet [10]. Bastin and Vogt also discussed other, equivalent formulations of condition (D).

It should be noted that some of the results presented here for co-echelon spaces of order ∞ have generalizations in the context of *weighted inductive limits of spaces of continuous functions and projective description*, see [25, 27, 26, 18]. Moreover, weighted inductive limits of spaces of *holomorphic* functions – for a recent survey on this topic see [15] – sometimes follow more closely the pattern set by the behavior of co-echelon spaces than the weighted inductive limits of spaces of *continuous* functions; e.g., concerning certain biduality results, cf. [21]. For properties of weighted inductive limits and projective description, the regularly decreasing condition and condition (D) remain very important.

(e) Subspaces and quotients of echelon spaces

The next two theorems were proved by Bonet, Díaz [31, 33].

Theorem 19 *Let A be a Köthe matrix on $I = \mathbb{N}$.*

(1) If $1 < p < \infty$ or $p = 0$, then every closed subspace and every quotient of $\lambda_p(A)$ is distinguished.

(2) Every quotient of $\lambda_1(A)$ is distinguished if and only if $\lambda_1(A)$ is quasinormable.

(3) Every closed subspace of $\lambda_1(A)$ is distinguished if and only if $\lambda_1(A)$ is normable or Montel or isomorphic to $\ell_1 \times \omega$.

(4) Every quotient of $\lambda_p(A), 1 < p < \infty$ or $p = 0$, has the density condition if and only if $\lambda_p(A)$ is quasinormable. ■

The following theorem should be compared with these two well-known facts: (1) Every Banach space is isomorphic to a quotient of the space $\ell_1(I)$ for a suitable index set I; e.g. see [69]. (2) Every separable Fréchet space is topologically isomorphic to a quotient of a Montel echelon space $\lambda_1(\mathbb{N}, A)$; e.g. see [99, page 221].

Theorem 20 *A Fréchet space F is the quotient of a suitable Köthe echelon space $\lambda_1(A) = \lambda_1(I, A)$ satisfying the density condition if and only if F has a total bounded set.* ■

(f) Some results on vector valued echelon and co-echelon spaces

It should be clear how echelon spaces $\lambda_p(A, E)$ and co-echelon spaces $k_p(V, E)$, $K_p(\overline{V}, E)$ with values in a locally convex space E are defined. See [16, 2, Lemma 1] for a generalization of the proposition in subsection (a) in this context when E is a Fréchet space. If E is a Fréchet space, then one has the duality $(\lambda_p(A, E))'_b = K_q(\overline{V}, E'_b)$, where $1 \leq p < \infty$ or $p = 0$, $p^{-1} + q^{-1} = 1$ with $q = 1$ for $p = 0$ and $q = \infty$ for $p = 1$. Similarly to the case $p = 1$ of this duality, one can show also that for any locally complete locally convex space E, there is a topological isomorphism

$$K_\infty(\overline{V}, E) = \mathcal{L}_b(\lambda_1(A), E).$$

In this subsection we concentrate on some results on vector valued echelon and co-echelon spaces which are related to the (DF)-property, to distinguishedness and the density condition. The following two theorems are taken from [16].

Theorem 21 *For a Fréchet space E, the following assertions are equivalent:*

(1) E satisfies the density condition.

(2) For each distinguished echelon space $\lambda_1(A)$, $\lambda_1(A, E)$ is distinguished, or equivalently, $\lambda_1(A, E)$ satisfies the density condition.

(3) The topological equality $k_\infty(V) = K_\infty(\overline{V})$ always implies that $K_\infty(\overline{V}, E'_b)$ is bornological, or equivalently, that the bounded subsets of $K_\infty(\overline{V}, E'_b)$ are metrizable. ■

Theorem 22 *(a) If $\lambda_1(A)$ is an echelon space and E is a Fréchet space, then $\lambda_1(A, E)$ is distinguished if and only if (i) $\lambda_1(A)$ is Montel and E is distinguished, or (ii) $\lambda_1(A)$ is distinguished and E satisfies the density condition.*

(b) If $\lambda_1(A)$ is an echelon space and E is a Fréchet space, then $\lambda_1(A, E)$ satisfies the density condition if and only if both $\lambda_1(A)$ and E have the density condition. ■

We finish this section with two theorems on vector valued co-echelon spaces, mainly from [17]. Note that (c) in the next theorem generalizes part (a) of the previous theorem.

Theorem 23 *(a) $K_\infty(\overline{V}, E)$ is a (DF)-space if and only if E has the (DF)-property.*

(b) For an arbitrary (DF)-space E, the following assertions are equivalent:

(i) E satisfies (DDC).

(ii) The topological equality $k_\infty(V) = K_\infty(\overline{V})$ always implies that the bounded subsets of $K_\infty(\overline{V}, E)$ are metrizable.

(iii) $K_\infty(\overline{V})$ quasibarrelled always implies $K_\infty(\overline{V}, E)$ quasibarrelled.

(iv) For each distinguished echelon space $\lambda_1(A)$, $K_\infty(\overline{V}, E)$ satisfies (DDC).

(c) Let E be a (DF)-space, A a Köthe matrix. Then $K_\infty(\overline{V}, E)$ is quasibarrelled if and only if (i) $\lambda_1(A)$ is Montel and E is quasibarrelled, or (ii) $\lambda_1(A)$ is distinguished and E satisfies (DDC).

(d) Let E be a locally convex space with the countable neighborhood property and $\lambda_1(A)$ be a distinguished echelon space. Then one has, algebraically and topologically, $k_\infty(V, E) = K_\infty(\overline{V}, E)$. ■

Theorem 24 *(a) For any Köthe matrix A and any locally convex space E with the countable neighborhood property, one has $k_p(V, E) = K_p(\overline{V}, E)$ algebraically and topologically, $1 \leq p < \infty$.*

(b) Let I denote a countable index set and fix p with $1 \leq p < \infty$. For an arbitrary (DF)-space E, $K_p(\overline{V}, E) = k_p(V, E)$ is again a (DF)-space, and if the (DF)-space E is quasibarrelled, barrelled or bornological, then this property is inherited by $K_p(\overline{V}, E)$. ■

When co-echelon spaces with values in a Fréchet space are investigated more thoroughly, conditions of a completely different type come into the play, see [20, Section 3].

6. Tensor products of Fréchet spaces, the problem of topologies of Grothendieck

(a) ε-product, injective and projective tensor products: a short reminder

If E and F are locally convex spaces, we denote by $\mathcal{L}_b(E, F)$ the space of all continuous linear maps from E to F, endowed with the topology of uniform convergence on the bounded subsets of E. A basis of absolutely convex 0-neighborhoods of $\mathcal{L}_b(E, F)$ is given by the sets of the form

$$W(B, V) := \{f \in \mathcal{L}(E, F);\ f(B) \subset V\},$$

as B runs through $\mathcal{B}(E)$ and V varies in a basis of absolutely convex 0-neighborhoods in F. It is well known that E'_b and F are topologically isomorphic to complemented subspaces of $\mathcal{L}_b(E, F)$. The tensor product $E' \otimes F$ can be identified canonically with the subspace of $\mathcal{L}(E, F)$ of all finite rank operators, via $u \otimes y \to (x \to u(x)y)$ for $x \in E$, $u \in E'$ and $y \in F$.

The ε-product of Schwartz [91] of E and F is defined as $E\varepsilon F := \mathcal{L}_e(E'_c, F)$; also see [67, 69]. Here the index e stands for the family of the E-equicontinuous sets in E' and c for the topology of uniform convergence on the absolutely convex compact subsets of E. Using transposed maps it is easy to see that $E\varepsilon F$ is topologically isomorphic to $F\varepsilon E$. If E and F are complete, $E\varepsilon F$ is also complete; if E and F are Fréchet spaces, $E\varepsilon F$ is again a Fréchet space. If E is a Fréchet Montel space, then $E\varepsilon F = \mathcal{L}_b(E'_b, F)$ and $\mathcal{L}_b(E, F) = E'_b\varepsilon F$.

The *injective* (or ε-) tensor product $E \otimes_\varepsilon F$ is defined by inducing on the tensor product the topology given by the canonical inclusion $E \otimes F \to E\varepsilon F$. If E and F are complete and one of these spaces has the approximation property, then $E\varepsilon F = E\check{\otimes}_\varepsilon F$; i.e., it equals the completion of $E \otimes_\varepsilon F$. Conversely, a locally convex space E for which $E \otimes F$ is dense in $E\varepsilon F$ for each locally convex space F (or only for each Banach space F) must have the approximation property. If E is a Fréchet space, the inclusion $E'_b \otimes_\varepsilon F \to \mathcal{L}_b(E, F)$ is a topological isomorphism into.

The *projective* (or π-) tensor product $E \otimes_\pi F$ is endowed with the locally convex topology with a basis of absolutely convex 0-neighborhoods of the form $\Gamma(U \otimes V)$ as U and V run through bases of closed absolutely convex neighborhoods of zero in E and F, respectively. If E and F are Fréchet spaces, then $E\hat{\otimes}_\pi F$, the completion of $E \otimes_\pi F$, is again a Fréchet space.

These two topologies on tensor products of locally convex spaces were defined by Grothendieck [63]. For more details and for permanence properties of the complete tensor products, we also refer the reader e.g. to [67, 69]. Tensor products have been utilized widely to represent spaces of vector valued functions or of vector valued sequences and spaces of functions of several variables on product sets; see [90, 63, 91, 67, 69]. Tensor product representations of weighted spaces of vector valued sequences were obtained in [18], and representations of weighted spaces of vector valued continuous or holomorphic functions as topological tensor products or as spaces of operators were given in [12, 22, 24].

We will not mention many of the "classical" results on topological tensor products of Fréchet or (DF)-spaces which one can find in the books quoted above. Instead, it is our main aim in the next subsections to report on the new developments which started when Taskinen solved Grothendieck's problem of topologies.

(b) Grothendieck's "problème des topologies"

Proposition 2 *Let E and F be Fréchet spaces. Then the map*

$$\Psi : \mathcal{L}(E, F_b') \to (E \otimes_\pi F)',\ \Psi(u)(x \otimes y) := (u(x))(y),\ x \in E,\ y \in F,\ u \in \mathcal{L}(E, F_b'),$$

is a linear isomorphism. Moreover, it yields the topological isomorphism $\mathcal{L}_b(E, F_b') = (E\hat{\otimes}_\pi F)'_{b\otimes b}$, where the index $b \otimes b$ stands for the topology of uniform convergence on the bounded sets of the form $\overline{\Gamma}(C \otimes D)$, C bounded in E and D bounded in F.

PROOF. To see that Ψ is well defined and continuous, we fix $u \in \mathcal{L}(E, F_b')$. Since E is a Fréchet space and F_b' is a (DF)-space, there is a 0-neighborhood U in E such that $u(U)$ is bounded in F_b' (see [64, Cor. 2, page 168]), hence equicontinuous. Thus, we can find a 0-neighborhood V in F with $u(U) \subset V^\circ$. Now it is easy to see that $|\Psi(u)(z)| \leq 1$ for each $z \in \Gamma(U \otimes V)$. On the other hand, for $u \in (E \otimes_\pi F)'$ define $\Phi(u) \in \mathcal{L}(E, F_b')$ by $(\Phi(u)(x))(y) := u(x \otimes y)$ for $x \in E$, $y \in F$. It is easily seen that Φ is well defined and linear; for $u \in (\Gamma(U \otimes V))^\circ$ we get $\Phi(u)(U) \subset V^\circ$. Since $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the respective identities, it follows that Ψ is a linear isomorphism. For the second assertion of the proposition, it is enough to observe that for C bounded in E and D bounded in F, we have

$$\Phi((\overline{\Gamma}(C \otimes D))^\circ) = \Phi((\Gamma(C \otimes D))^\circ) = W(C, D^\circ) = \{f \in \mathcal{L}(E, F_b')\ ;\ f(C) \subset D^\circ\}. \quad \blacksquare$$

The *problem of topologies* of Grothendieck [63] asked whether, for every pair (E, F) of Fréchet spaces every bounded subset B of $E\hat{\otimes}_\pi F$ is contained in a bounded set of the form $\overline{\Gamma}(C \otimes D)$, for some bounded set C in E and some bounded set D in F. Grothendieck showed that the answer is positive if E is nuclear, or if $E = \lambda_1(A)$ is a Köthe echelon space of order 1. He also claimed a positive answer if both E and F are hilbertisable; i.e., projective limits of sequences of Hilbert spaces. A proof of this fact was later published by Kürsten [70]. The problem of topologies remained open for more than 30 years and was finally solved in the negative by Taskinen [93]. In this article, Taskinen introduced the following terminology which we will use in the rest of the section: A pair of Fréchet spaces (E, F) satisfies the *property (BB)* (for bi-bounded) if the problem of topologies of Grothendieck has a positive answer for the pair (E, F). A Fréchet space is said to be an *(FBa)-space* if (E, X) satisfies the property (BB) for every Banach space X.

Closely related to the problem of topologies are Grothendieck's problems on (DF)-spaces:

(1) Suppose that E is a Fréchet space and G is a (DF)-space. Must then $\mathcal{L}_b(E, G)$ be a (DF)-space?

(2) Suppose that G and H are (DF)-spaces. Must then the injective tensor product $G \otimes_\varepsilon H$ also be a (DF)-space?

In a certain sense these two problems are dual to the problem of topologies. Indeed, if the pair (E, F) of Fréchet spaces satisfies the property (BB), then $\mathcal{L}_b(E, F_b') = (E\hat{\otimes}_\pi F)_b'$ is the strong dual of the Fréchet space $E\hat{\otimes}_\pi F$, hence a (DF)-space. On the other hand, if for Fréchet spaces E and F, $\mathcal{L}_b(E, F_b')$ is a (DF)-space, then the pair (E, F) satisfies the property (BB) if one of the following conditions holds (see [34]): (a) both E and F have the density condition, (b) both E and F are separable, or (c) E or F is separable and satisfies the bounded approximation property. However, it is not known whether this implication holds in general. On the other hand, for the relation between the problems (1) and (2) above, see Defant and Peris [45]: The spaces $\mathcal{L}_b(E, G)$ and $E_b' \otimes_\varepsilon G$ "have the same local structure".

(c) Counterexamples and positive results

Our next theorem summarizes the main counterexamples of Taskinen [93, 94, 96].

Theorem 25 *(1) There is a separable reflexive Fréchet space E such that the pair (E, ℓ_2) does not satisfy the property (BB).*

(2) There is an (FM)-space E such that both (E, E) and (E, ℓ_2) do not satisfy (BB). Moreover, none of the

spaces $\mathcal{L}_b(E, E'_b)$, $\mathcal{L}_b(E, \ell_2)$, $E'_b \otimes_\varepsilon E'_b$, $E'_b \otimes_\varepsilon \ell_2$ *is a (DF)-space.*

(3) For each Köthe echelon space $\lambda_1(A)$ *there is an (FM)-space* E *such that* $\lambda_1(A)$ *is topologically isomorphic to a complemented subspace of* $E\hat{\otimes}_\pi E$. ■

The work of Taskinen showed that the questions of Grothendieck were related to the geometry of finite dimensional Banach spaces. This led to a large amount of research on Grothendieck's problems, on the stability of various properties of Fréchet spaces under the formation of projective or injective tensor products, and to applications in infinite holomorphy, by Defant, Díaz, Domański, Floret, Galbis, Metafune, Peris, Taskinen, the authors, and many others. Below we collect a number of results which we find interesting or illustrative.

In [93], Taskinen exhibited pairs (E, F) of Fréchet spaces which satisfy property (BB) when one of the spaces admits a certain type of decomposition and the other one is a Banach space, or when both spaces admit such a decomposition. For example, he proved that *every Köthe echelon space* $\lambda_p(A)$ *is an (FBa)-space*, $1 \leq p < \infty$. This line of research led Bonet, Díaz, Taskinen [32, 34] to the introduction of a general class of Fréchet resp. (DF)-spaces, called (FG)- resp. (DFG)-spaces, such that the three problems of Grothendieck have a positive answer for spaces within these classes. The approach was continued by Peris, Rivera [84] and had applications in infinite dimensional holomorphy; see e.g. [56, 57].

Here are some results concerning the (DF)-problems and the tensor product of quasibarrelled (DF)-spaces [17, 18, 34].

Theorem 26 *(1) If* $\lambda_1(A)$ *is a Köthe echelon space and* G *is a (DF)-space, then* $(\lambda_1(A))'_b \otimes_\varepsilon G$ *and* $\mathcal{L}_b(\lambda_1(A), G)$ *are (DF)-spaces.*

(2) $\mathcal{L}_b(\lambda_1(A), G)$ *is quasibarrelled for every quasibarrelled (DF)-space* G *if and only if* $\lambda_1(A)$ *is distinguished.*

(3) If $1 < p < \infty$ *and* X *is a normed space, then* $(\lambda_p(A))'_b \otimes_\varepsilon X$ *is a bornological (DF)-space, and* $\mathcal{L}_b(\lambda_p(A), G)$ *is a (DF)-space.*

(4) Assume that $1 < p,\ q < \infty$.

(i) If $\lambda_p(A)$ *is Montel, then* $\mathcal{L}_b(\lambda_p(A), \ell_q)$ *is reflexive, hence quasibarrelled.*

(ii) If $\lambda_p(A)$ *is not Montel, then* $\mathcal{L}_b(\lambda_p(A), \ell_q)$ *is reflexive if and only if* $p > q$.

(iii) If $\lambda_p(A)$ *is not Montel and* $p \leq q$*, then* $\mathcal{L}_b(\lambda_p(A), \ell_q)$ *is quasibarrelled if and only if* $\lambda_p(A)$ *has the density condition.* ■

The following important results due to Taskinen [95] and to Defant, Floret, Taskinen [44] emphasize the "local character" of the problems of Grothendieck; proofs can be found in [43, Section 35].

Theorem 27 *(a) The following conditions are equivalent for a Banach space* X*:*

(1) X *is a* $\mathcal{L}_1$*-space in the sense of Lindenstrauss and Pełczyński, e.g. see [43].*

(2) The pair (X, F) *satisfies the property (BB) for every Fréchet space* F.

(3) The space $\mathcal{L}_b(X, G)$ *is (DF) for every reflexive (DF)-space* G.

(b) The following conditions are equivalent for a Banach space X*:*

(1) X *is a* $\mathcal{L}_\infty$*-space in the sense of Lindenstrauss and Pełczyński.*

(2) $X \otimes_\varepsilon G$ *is (DF) for every (DF)-space* G.

(c) ([30]) Let $1 \leq p \leq \infty$.

(1) (ℓ_p, F) *has the property (BB) if and only if* (Y, F) *has (BB) for every* $\mathcal{L}_p$*-space* Y.

(2) $\ell_p \otimes_\varepsilon G$ *is a (DF)-space if and only if* $Y \otimes_\varepsilon G$ *is (DF) for every* $\mathcal{L}_p$*-space* Y. ■

These results have the following interesting consequence for weighted Banach spaces $Hv(G)$ and $Hv_0(G)$ of holomorphic functions, see [22]. For an open subset G of $\mathbb{C}^N$ and a strictly positive continuous function v on G, put $Hv(G) := \{f \text{ holomorphic on } G;\ ||f||_v = \sup_G v|f| < \infty\}$, and let $Hv_0(G)$ be the closed subspace of $Hv(G)$ consisting of all those functions f for which $v|f|$ vanishes at infinity on G. It is then clear how the vector valued analog $Hv(G, E)$ of $Hv(G)$ is defined when E is a (locally complete) locally convex space.

Theorem 28 *Let v be a radial (i.e., $v(\lambda z) = v(z)$ for all $z \in G$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$), stricly positive, continuous function on a balanced open subset $G \subset \mathbb{C}^N$ such that $Hv_0(G)$ contains all polynomials. Then the following conditions are equivalent:*

(1) $Hv_0(G)$ is a $\mathcal{L}_\infty$-space.

(2) $Hv(G)$ is a $\mathcal{L}_\infty$-space.

(3) The pair $((Hv_0(G))'_b, F)$ has the property (BB) for every Fréchet space F.

(4) The space $\mathcal{L}_b((Hv_0(G))'_b, F'_b)$, topologically isomorphic to $Hv(G, F'_b)$, is a (DF)-space for every Fréchet space F. ■

Lusky [71] showed that, if $G = \mathbb{D}$ is the unit disc and if the radial weight v is of moderate growth, then the conditions (1)–(4) in Theorem 28 are equivalent to $Hv_0(\mathbb{D}) = c_0$ (in the sense of a topological isomorphism) and can be characterized in terms of the weight.

Peris [82] gave the first example of a Fréchet space which is not an (FBa)-space and which is "natural"; i.e., not constructed on purpose: *The Fréchet space $\ell_{p+} = \cap \ell_{q>p}$ is quasinormable [76], and it is not an (FBa)-space*. In fact, the pair (ℓ_{p+}, C_2) fails to have the property (BB), where C_2 is the so-called Johnson space (i.e., the ℓ_2-sum of a sequence of finite dimensional Banach spaces which is dense in the Banach Mazur compactum of all finite dimensional spaces). There was some hope that the pair (ℓ_{p+}, ℓ_2) would not have the property (BB). However, this is false: Defant and Peris [46] proved that *if $2 \le p < \infty$, then (ℓ_{p+}, X) has the property (BB) for every normed space X whose dual X' has cotype 2.*

(d) (FS)-, (DFS)-spaces, and interchanging inductive limits with the ε-product

It was an open problem of Taskinen [95] whether every (FS)-space is an (FBa)-space. This question turned out to be related with the problem whether the countable inductive limit in an (LB)-space $E = \text{ind}_n E_n$ with compact linking maps (that is, E is a (DFS)-space, the strong dual of an (FS)-space) interchanges with the ε-(tensor) product with arbitrary Banach spaces [25]. The important work of Peris [81, 82, 83] solved this problem and clarified the situation completely. The survey article [14] of Bierstedt contains a detailed report of Peris' work and of its applications to weighted inductive limits of spaces of vector valued holomorphic functions [22] and to vector valued holomorphic germs on (FS)-spaces [23]. We refer the reader to this survey and to the original articles of Peris and mention only a few results here.

Theorem 29 *(a) There is an (FS)-space E with the approximation property which is not an (FBa)-space and such that, for the (DFS)-space E'_b and some Banach space X, $E'_b \otimes_\varepsilon X$ is not bornological.*

(b) For a (DFS)-space $E = \text{ind}_n E_n$ the following conditions are equivalent:

(1) E'_b is an (FBa)-space.

(2) For each Banach space X, $\mathcal{L}_b(E'_b, X')$ is a bornological (DF)-space.

(3) For each Banach space X, $E\varepsilon X = \text{ind}_n(E_n \varepsilon X)$ holds topologically.

(4) For each Banach space X, $E \otimes_\varepsilon X = \text{ind}_n(E_n \otimes_\varepsilon X)$ holds topologically.

(5) The (FS)-space $F = E'_b$ is quasinormable by operators; i.e., for each n there is $m > n$ such that, for each $\varepsilon > 0$, there is a continuous linear operator P from F into F with $P(U_m)$ bounded in F and $(I - P)(U_m) \subset \varepsilon U_n$, where $(U_n)_n$ denotes a basis of absolutely convex 0-neighborhoods in F.

(c) Every (FS)-space with approximable linking maps is quasinormable by operators. And, conversely,

if an (FS)-space has the approximation property and is quasinormable by operators, then it must have approximable linking maps. ■

If one leaves the scope of (DFS)-spaces and, in particular, considers nonreflexive (LB)-spaces, then it is still open under which conditions the interchangeability of the inductive limit with the ε-product with every Banach space, as in (b)(3) of the preceding theorem, holds. However, Mangino [72] discussed this question in the framework of (totally reflexive) (LF)-spaces. The relevance of the condition of quasinormability by operators and of its dual formulation (strict Mackey condition by operators), in connection with the stability of the class of quasinormable spaces under the formation of tensor products and with the problems of Grothendieck, is discussed in Peris' article [82].

(e) Two related open problems

The interchangeability of inductive limits and tensor products is related to another direction of research which recently had some progress. In 1977, Bierstedt and Schmets asked whether $C(K, G)$ must be bornological if K is a compact Hausdorff space and $G = \mathrm{ind}_n\, G_n$ is an (LB)-space. The results known in 1983 in connection with this problem were collected in the excellent monograph of Schmets [89]. Since then, the work of Dierolf and Domański [54, 55] made new contributions and showed that this problem is connected with other open questions on (LB)-spaces. For example, they proved that *the space $c_0(G)$ is bornological if G is the strong dual of an (FM)-space or is the inductive dual of a Köthe echelon space.* However, the problem remains open.

After Taskinen solved Grothendieck's problem of topologies, only one of Grothendieck's questions has remained unanswered, the question of the completeness of regular (LF)-spaces. Even the particular case whether every regular (LB)-space must be complete is unsolved; e.g. see [13, p. 78] or [79, Problem 13.8.6]. We recall that an (LB)-space $E = \mathrm{ind}_n\, E_n$ is said to be *regular* if every bounded set $B \subset E$ is contained and bounded in one of the steps E_n. Grothendieck's factorization theorem implies that an (LB)-space is regular if and only if it is locally complete. A positive solution of Grothendieck's problem would imply that the completion of every (LB)-space is also an (LB)-space [79, Problem 13.8.1], and this would in turn imply a positive solution to the above problem of Bierstedt and Schmets on $C(K, E)$ spaces. – It is worth mentioning at this point that Bonet and Dierolf showed that every regular (LB)-space of Moscatelli type is complete, hence that no counterexample to Grothendieck's problem can be constructed with the shifting method of Moscatelli.

Since usually it is easy to show the regularity of an (LB)-space (if it is indeed regular), but sometimes quite hard to show its completeness, many authors have felt that the solution of Grothendieck's question should be in the negative. But, so far, also all attempts to give counterexamples have failed (although it was claimed erroneously in several published articles that the author had found a counterexample or had found a proof). Thus, at this time, the last problem of Grothendieck is still wide open.

In this survey we have mainly concentrated on recent results in the theory of Fréchet spaces and their duals, rather than pointing out all the open problems that have remained despite intensive research. E.g. in the area of weighted inductive limits of spaces of holomorphic functions and projective description there are quite a number of important unsolved questions for which we refer to [15].

(f) Some stability results for the projective tensor product

Some of the problems discussed so far are related to the stability of properties of Fréchet spaces under the formation of tensor products. If E is a nuclear Fréchet space and F is a Fréchet space which is nuclear, Schwartz, Montel, reflexive, quasinormable, distinguished, or has the density condition, then the complete projective tensor product $E\hat{\otimes}_\pi F$ satisfies the same property; see [63]. The following result is also due to Grothendieck [63]: *If both Fréchet spaces E and F are Schwartz (resp. quasinormable), then $E\hat{\otimes}_\pi F$ is again Schwartz (resp. quasinormable).*

The class of quasinormable Fréchet spaces is stable under the formation of quotients and complete projective tensor products. Meise and Vogt [73] proved that *it is the smallest class of Fréchet spaces, stable under quotients and complete projective tensor products, which contains both the nuclear Köthe echelon* *spaces and the Banach spaces*. The equivalence of the first two conditions in the next theorem can be seen in [74, 26.14].

Theorem 30 *For a Fréchet space E, the following conditions are equivalent:*

(1) E is quasinormable.

(2) $\forall n\, \exists m > n\, \forall k > m\, \forall \varepsilon > 0\, \exists S > 0 : \ U_m \subset SU_k + \varepsilon U_n$.

(3) There is an index set I and there is a nuclear Köthe space $\lambda_1(A)$ with a continuous norm such that E is topologically isomorphic to a quotient of $\ell_1(I)\hat{\otimes}_\pi \lambda_1(A)$. ■

It is worth pointing out that the equivalence between (1) and (2) in Theorem 30 need not hold for metrizable locally convex spaces E. Indeed, it is not difficult to see that every dense subspace of a quasinormable Fréchet space satisfies the condition (2). On the other hand, Bonet and Dierolf [36] constructed a non-separable quasinormable Fréchet space E with a dense subspace F which is not quasinormable. Thus, the metrizable space F satisfies condition (2), but not condition (1). The construction is based on a classical example of Amemiya.

The characterization of the smallest class of Fréchet spaces, stable under closed subspaces and complete projective tensor products, which contains the nuclear Köthe echelon spaces with a continuous norm and the Banach spaces, is due to Terzioğlu and Vogt [98]. The two conditions below are equivalent to saying that E satisfies the topological invariant (DN_φ) of Vogt for some φ; see [98].

Theorem 31 *For a Fréchet space E with basis $(p_n)_n$ of seminorms, the following conditions are equivalent:*

(1) E is asymptotically normable; i.e., $\exists n(0)\, \forall n\, \exists k > n\, \forall \varepsilon > 0\, \exists M > 0 : \ p_n \leq M p_{n(0)} + \varepsilon p_k$.

(2) There is an index set I and there is a nuclear Köthe space $\lambda_1(A)$ with a continuous norm such that E is isomorphic to a subspace of $\ell_\infty(I)\hat{\otimes}_\pi \lambda_1(A)$. ■

The characterizations in Theorems 30 and 31 are closely related to Vogt's treatment of the properties (DN) and (Ω) in [105]. Note that since the echelon space in (2) above is nuclear, one could have written $\lambda_\infty(A)$ instead of $\lambda_1(A)$ (compare with (2) in the next theorem) and $\check{\otimes}_\varepsilon$ instead of $\hat{\otimes}_\pi$. – In case one restricts the attention to (FS)-spaces, it is possible to give other characterizations, which are due to Vogt and Waldorf [110]. Their theorem below should be compared with Theorem 20 and the comments before it. The corresponding results for nuclear spaces were obtained earlier by Apiola and Wagner; see [110] for precise references.

Theorem 32 *(1) Every (FS)-space is topologically isomorphic to a quotient of a Köthe echelon space $\lambda_1(A)$ which is Schwartz.*

(2) An (FS)-space is asymptotically normable if and only if it is topologically isomorphic to a subspace of a Köthe echelon space $\lambda_\infty(A)$ which is Schwartz and has a continuous norm. ■

By Theorem 25.(2), there are pairs (E, F) of Fréchet spaces with the density condition which do not satisfy (BB). Bonet and Taskinen constructed quojections E such that (E, ℓ_2) does not satisfy the property (BB), although $E\hat{\otimes}_\pi \ell_2$ is clearly quasinormable as complete projective tensor product of two quasinormable spaces. Díaz and Metafune [49] characterized the quojections E of Moscatelli type which are (FBa)-spaces and obtained many interesting examples. Our next result, proved in [19], shows that the failure of the property (BB) is the only obstruction for the stability of the density condition in complete projective tensor products of Fréchet spaces.

Theorem 33 *If E and F are Fréchet spaces with the density condition such that (E, F) satisfies (BB), then $E\hat{\otimes}_\pi F$ has the density condition, too.* ■

Recalling our characterization of distinguished echelon spaces of order 1 with values in a Fréchet space in Theorem 22, we note that this result can of course be interpreted as a stability theorem for distinguishedness under complete projective tensor products. The research about distinguished projective tensor products of Fréchet spaces was continued by Díaz and Miñarro in [50]. We mention the following example: Theorem 22 completely describes when the space $\lambda_p(A)\hat{\otimes}_\pi\lambda_q(B)$, $1 \leq p,\ q < \infty$ or $p = 0$ or $q = 0$, is distinguished in case $p = 1$. It follows from results of Díaz and Miñarro that *(i)* $\lambda_0(A)\hat{\otimes}_\pi\lambda_q(B)$ *is distinguished for each* q *as above, and that (ii)* $\lambda_p(A)\hat{\otimes}_\pi\lambda_q(B)$, $1 < p,\ q < \infty$, *is distinguished if and only if one of the spaces is Montel or both spaces satisfy the density condition or* $p > q/(q-1)$.

Distinguished complete injective tensor products of Fréchet spaces have recently been investigated by Díaz and Domański [48]. They constructed a quasinormable Fréchet space E such that $E\check{\otimes}_\varepsilon\ell_1$ is not distinguished. On the other hand, if E is a reflexive Fréchet space and K is a compact Hausdorff space, then $E\check{\otimes}_\varepsilon C(K) = C(K,E)$ is distinguished. In [40], examples are given of quojections E of Moscatelli type (hence quasinormable Fréchet spaces) and Banach spaces Z such that $E\check{\otimes}_\varepsilon Z$ does not satisfy the density condition. More results about the stability of quasinormability are due to Peris [82].

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K. D. Bierstedt
Mathematik
Universität Paderborn
D-33095 Paderborn, Germany
`klausd@upb.de`

J. Bonet
Depto. de Matemática Aplicada, E.T.S. Arq.
Universidad Politécnica de Valencia
E-46071 Valencia, Spain
`jbonet@mat.upv.es`