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Relative rearrangement and interpolation inequalities

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Abstract. We prove here that the Poincaré-Sobolev pointwise inequalities for the relative rearrangement can be considered as the root of a great number of inequalities in various sets not necessarily vector spaces. In particular, new interpolation inequalities can be derived.

Reordenamiento relativo y desigualdades de interpolación

Resumen. Mostramos que las desigualdades puntuales de Poincaré-Sobolev para el reordenamiento relativo pueden ser consideradas como el origen de bastantes desigualdades sobre varios conjuntos que que no necesitan ser espacios vectoriales. En concreto, es posible obtener nuevas desigualdades de interpolación.

1. Introduction

The Poincaré-Sobolev inequalities for the relative rearrangement called PSR property are revealed to be a common root for a large class of Sobolev embeddings of normed spaces (see [17][15] [16]). These inequalities can be summarized in the following definition for an open set Ω of $\mathbb{R}^N$

A subset V of $\bigcup\limits_{1\leqslant p\leqslant+\infty} W^{1,p}(\Omega)$ satisfies the *PSR property* if

1. $\forall u \in V, u_* \in W^{1,1}_{loc}(\Omega_*),\ \Omega_* = (0, \mathrm{measure}(\Omega))$
2. There is a measurable map $K : \Omega_* \to [0, +\infty[$ such that :

$$-u'_*(s) \leqslant K(s) \left|\nabla u\right|_{*u}(s), \text{ for a.e.} s,\ \forall u \in V.$$

Here, u_* is the monotone decreasing rearrangement of u, $|\nabla u|_{*u}$ is the relative rearrangement of the gradient $|\nabla u|$ with respect to u. The map K can depend on $\Omega,\ V$.

It often happens that u_* is replaced by the average quantity

$$u_{**}(s) = \frac{1}{s}\int_0^s u_*(t)dt$$

(see for instance the Lorentz spaces). Thus, we will show the following *new result* : If V satisfies the (PSR) property then

$$-u'_{**}(s) \leqslant \frac{1}{s} \operatorname*{ess\,sup}_{0\leqslant\sigma\leqslant s} [\sigma K(\sigma)] \cdot \left(|\nabla u|_{*u}\right)_{**}(s), \quad \text{for } a.e.\ s.$$

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Since $\big(|\nabla u|_{*u}\big)_{**}(s) \leqslant |\nabla u|_{**}(s)$, the above inequality can be replaced by the following stronger inequality

$$-u'_{**}(s) \leqslant \widetilde{K}(s)\,|\nabla u|_{**}(s)$$

with $\widetilde{K}(s) = \dfrac{1}{s} \operatorname*{ess\,sup}_{0\leqslant\sigma\leqslant s} [\sigma K(\sigma)]$.

For instance, if $V = W^{1,p}_{0+}(\Omega)$, then $\widetilde{K}(s) = K(s) = \dfrac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}$.

We have shown that if ρ is a norm on the set of all measurable functions on Ω_* then the set $V^1(\Omega,\rho) = \Big\{u \in V : \rho\big(|\nabla u|_{*u}\big) < +\infty\Big\}$ is included in $L^\infty(\Omega)$ provided that the constant $\rho'(K)$ is finite, we call $\rho'(K)$ the index of inclusion (see [17]). Here, ρ' denotes the associate norm of ρ (see definition below). More results can be found in the previous papers. As for the applications, we have used different norms, as the Birnbaum-Orlicz norms that can be found in [10], or the norm in the *small Lebesgue spaces* (see [8], [9]).

The definition of a norm, we use is the one in [4], but since we shall use some maps which are not norms, we recall for convenience the :

Definition 1 *A map $\rho : L^0(\Omega_*) \to \mathbb{R}_+$ is said to be*

1. *definite if* $\rho(f) = 0 \Longleftrightarrow f = 0,$
2. *homogeneous if* $\forall\lambda \in \mathbb{R},\ \rho(\lambda f) = |\lambda|\,\rho(f),$
3. *monotone if* $0 \leqslant f \leqslant g \Longrightarrow \rho(f) \leqslant \rho(g).$

We always assume that $\rho(f) = \rho\big(|f|\big)$ and ρ is non trivial in the sense that there is $f_0 \in L^0(\Omega_*)$: $0 < \rho(f_0) < +\infty$. Here, $L^0(\Omega_*)$ denotes the set of all measurable functions on Ω_*.
If ρ satisfies 1,2,3 and the triangular inequality, then we say that ρ is a norm. We shall use the associate norm of ρ defined

$$\rho'(f) = \sup\Big\{ \int_{\Omega_*} |fg|\,(t)dt,\ \rho(g) \leqslant 1\Big\}.$$

We also have exhibited examples of sets V satisfying the PSR property and to complete those previous results, we shall give a new proof of the following theorem shown in [14]:

If Ω is a connected set, $u \in W^{1,1}_{loc}(\Omega)$ then $u_* \in W^{1,1}_{loc}(\Omega_*)$.

Nevertheless, the (PSR) is true even if the open set Ω is not connected. We shall provide an example of such set Ω in the first paragraph. Moreover, it is possible to derive a (PSR) property for functions vanishing partly on the boundary that is if Ω is connected Lipschitz open bounded set $\Gamma_0 \subset \partial\Omega$ with $H_{N-1}(\Gamma_0) > 0$ then the set

$$W^{1,1}_{\Gamma_{0+}}(\Omega) = \Big\{v \in W^{1,1}(\Omega),\ \gamma_0 v = 0 \ on\ \Gamma_0,\ v \geqslant 0\Big\}$$

satisfies the (PSR) property with $K(s) = \dfrac{s^{\frac{1}{N}-1}}{N\sigma_N^{\frac{1}{N}}},\ s \in \Omega_*$.

Thus, the C_α-rearrangement (see [12]) associated to a function $u \in W^{1,1}_{\Gamma_{0+}}(\Omega)$ satisfies the Polyà-Szëgo type pointwise inequalities :

$$\int_0^s |\nabla C_\alpha u|_*(\sigma)d\sigma \leqslant \int_0^s \Big(|\nabla u|_{*u}\Big)_*(\sigma)d\sigma \leqslant \int_0^s |\nabla u|_*(\sigma)d\sigma \qquad \forall s \in \Omega_*.$$

These results are detailed in [17] and the arguments follows the one given in [15].

To complete the Sobolev inclusions, we will show the following general Poincaré-Sobolev inequality for bounded domain Ω:

If ρ_0 is a convenient map on $L^0(\Omega_*)$ (set of all measurable functions on Ω_*), then there is an number $\rho_0(b)$ such that if $\rho_0(1)\rho_0(b)$ is finite then

$$\forall u \in V^1(\Omega, \rho) \qquad \rho_0\Big(u_* - u_*\left(\frac{|\Omega|}{2}\right)\Big) \leqslant \rho_0(b)\rho\big(\,|\nabla u|_{*u}\,\big).$$

In particular, $V^1(\Omega,\rho) \subset L(\Omega,\rho_0) = \Big\{v \ : \ \text{measurable}, \ \rho_0(v_*) < +\infty\Big\}$ and if ρ is a Fatou norm invariant under rearrangement then

$$\rho_0(u_*) \leqslant \rho_0(b)\rho(|\nabla u|_*) + \frac{2}{|\Omega|}\rho_0(1)\,|u|_1$$

for all $u \in W^1(\Omega,\rho) = \Big\{v \in V, \ \rho(|\nabla v|_*) < +\infty\Big\}$. We shall call $\rho_0(b)$ an *index number* as in [17].

To illustrate the above result, we will estimate the index number for Lorentz norms.

In the last paragraph, we show that one can obtain general interpolations inequalities for normed spaces (or not) leading to some new interpolations in our knowledge, with explicit formulas on the interpolation constants. For instance, we have : $\forall u \in W_0^{1,N}(\Omega), \ N' = \dfrac{N}{N-1} \leqslant p < +\infty$

$$|u|_p \leqslant \left(\frac{1}{aN\alpha_N^{\frac{1}{N}}}\right)^a \Big||\nabla u|_{*u}\Big|_N^a \, |u|_{p-N'}^{1-a}\,, \ with \ a = \frac{N'}{p}.$$

Note that $\Big||\nabla u|_{*u}\Big|_N \leqslant |\nabla u|_N$ and $p - N'$ might be less than 1, thus the quantity $|u|_{p-N'}$ is not a norm but is finite. We thus recover the following interpolation frequently used in $\Omega \subset \mathbb{R}^2$,

$$|u|_4 \leqslant \left(\frac{1}{\pi}\right)^{\frac{1}{4}} |\nabla u|_2^{\frac{1}{2}}\,|u|_2^{\frac{1}{2}}\,, \quad \forall u \in H_0^1(\Omega).$$

If we replace $H_0^1(\Omega)$ by $W_{\Gamma_0}^{1,2}(\Omega) = \Big\{v \in W^{1,2}(\Omega) : \gamma_0 v = 0 \ on \ \Gamma_0\Big\}$ the the inequality reads:

$$|u|_4 \leqslant \left(\frac{1}{\sigma_2}\right)^{\frac{1}{4}} |\nabla u|_2^{\frac{1}{2}}\,|u|_2^{\frac{1}{2}}\,, \quad \forall u \in W_{\Gamma_0}^{1,2}(\Omega).$$

σ_2 can be computed for sectorial sets (see [12]).

2. Notations and preliminary results

Let Ω be an open set of $\mathbb{R}^N$. For a measurable set $E \subset \Omega$, we shall denote by $|E|$ its Lebesgue measure. We set $\Omega_* =]0, |\Omega|\,[$. The distribution of a measurable function

$$u : \Omega \to I(u) = \left(\operatorname{ess}\inf_{\Omega}\,(u), \operatorname{ess}\sup_{\Omega}\,(u)\right) \subset \mathbb{R},$$

is the map $m_u : I(u) \to \overline{\mathbb{R}}_+$ defined by $m_u(t) = |u > t|\,, \ t \in I(u)$. We always assume that $u \geqslant 0$ if Ω is unbounded. The function u_* called the monotone decreasing rearrangement of u is the generalized inverse of m_u. To introduce the definition of the relative rearrangement of a function $v \in L^1(\Omega)$ with respect to a function $u \in L^1(\Omega)$, we shall define on Ω_*:

$$w(s) = \int_{u>u_*(s)} v(x)dx \ + \int_0^{s-|u>u_*(s)|} \Big(v\big|_{\{u=u_*(s)\}}\Big)_*(\sigma)d\sigma$$

where $v\big|_{\{u=u_*(s)\}}$ is the restriction of v to the set $\{u = u_*(s)\}$. This is a general definition of the relative rearrangement.

Property 1 *Let Ω be a measurable subset of $\mathbb{R}^N$.*
(a) If Ω is bounded then $w \in W^{1,p}(\Omega_*)$:

$$\frac{(u+\lambda v)_* - u_*}{\lambda} \underset{\lambda\to 0}{\rightharpoonup} \frac{dw}{ds} \text{ in } L^p(\Omega_*)\text{-weak if } 1 \leqslant p < +\infty \text{ and in } L^\infty(\Omega_*)\text{-weak-star if } p = +\infty$$

.
(b) If Ω is unbounded then one has :
i) $w \in W^{1,p}_{loc}([0, +\infty)$,
ii) $\dfrac{dw}{ds} \in L^p(0, +\infty)$,
iii) $\dfrac{(u+\lambda v)_{+*} - u_*}{\lambda} \underset{\lambda\to 0}{\rightharpoonup} \dfrac{dw}{ds}$ *in* $L^p(0, +\infty)$*-weak if* $1 < p < +\infty$
(weak-star for $p = +\infty$) and in $L^1(0, M)$-weak, $\forall M$ finite.
Here, $v \in L^p(\Omega),\ u \geqslant 0$ being in $L^1(\Omega)$. In any case, $\left|\dfrac{dw}{ds}\right|_{L^p(\Omega_*)} \leqslant |v|_{L^p(\Omega)}$.

For the proof of property 1, one can consult [13], [5], [18].

Definition 2
The function $\dfrac{dw}{ds}$ is called the relative rearrangement of v with respect to u and is denoted by v_{*u}.

This notion was introduced first by Mossino and Temam [13], similar notion was used by Alvino and Trombetti for bounded domains and the Naples school (see [1], [2], [6], [7] and references therein). The part (b) of the above property comes from [5].

3. An alternative proof for the local regularity of the monotone rearrangement and more results on the PSR property

We have shown the following local regularity in [14] :

Theorem 1 *If Ω is an open connected set and $u \in W^{1,1}_{loc}(\Omega)$ then $u_* \in W^{1,1}_{loc}(\Omega_*)$.*

Remark 1 In [14], it was assumed that Ω can be decomposed as $\Omega = \bigcup_{j\geqslant 0} \Omega_j,\ \overline{\Omega}_j \subset \Omega_{j+1},\ \Omega_j$ being connected open bounded Lipschitz set. But it can be shown that any open connected set can be decomposed as above (see [11]). ■

Here, we shall present a slightly different proof using a dyadic decomposition of Ω.
Proof of theorem 1 **1^{st} step.** $u_* \in C(\Omega_*)$.
Let $s \in \Omega_*$. Since u_* is monotone and continuous from the right, then the following quantities are finite

$$u_*(s) = \lim_{h>0,\ h\to 0} u_*(s+h),\ u_*(s_-) = \lim_{h>0,\ h\to 0} u_*(s-h).$$

If u_* was not continuous at the point s then $u_*(s) < u_*(s_-)$ and $|u_*(s) < u < u_*(s_-)| = 0$. But $|u > u_*(s_-)| > 0$ and $|u < u_*(s)| > 0$. So as in [3], let us consider the function

$$v(x) = \max\left(u_*(s); \min\left(u(x), u_*(s_-)\right)\right) = \begin{cases} u_*(s_-) & if\ u(x) \geqslant u_*(s_-) \\ u_*(s) & if\ u(x) \leqslant u_*(s)\ a.e\ in \Omega \end{cases}$$

Since $u \in W^{1,1}_{loc}(\Omega)$ then $v \in W^{1,1}_{loc}(\Omega)$ and $\nabla v = 0$ a.e in Ω. So, on any ball B contained in Ω, v is constant. Since Ω is an open connected set,this implies that $u_*(s_-) = u_*(s)$ which is a contradiction.

2^{nd}step. u_* maps null set of $[a, b],\ 0 < a < b < |\Omega|$ into a null set.

We argue as in [14]. Let $E \subset [a, b]$ with $|E| = 0$. Since the set $E_d = \Big\{s \in E : u_*'(s)$ exists and is finite $\Big\}$ is of a measure zero, we know (see [21], [20]) that $|u_*(E_d)| = 0$. It remains to show that $|u_*(E\backslash E_d)| = 0$. Let D_u be the set of all $t \in \mathbb{R}$ such that $|u = t| > 0$ thus D_u is at most countable, and let us set $I(u) = \Big(\operatorname*{ess\,inf}_{\Omega}\ u, \operatorname*{ess\,sup}_{\Omega}\ u\Big)$. We have to show that $I_1(u) = \Big\{t \in I(u) : m_u'(t) < 0\Big\}$ has the same measure as $I(u)$ and $u_*(E\backslash E_d) - D_u$ is contained in $I(u) - I_1(u)$. To show the first statement, we decompose Ω into an union of countable cubes $(Q_j)_{j\geqslant 0}$ with disjoint interior i.e $\mathring{Q}_j \cap \mathring{Q}_k = \emptyset$ if $k \neq j$. We set $u_j = u|_{Q_j}$ restriction to Q_j, $m_j = m_{u_j}$ and $I_j = \Big(\operatorname*{ess\,inf}_{Q_j}\ u, \operatorname*{ess\,inf}_{Q_j}\ u\Big)$. Then, $I(u) = \bigcup_j I_j$ and $m_u'(t) = \sum_{j=0}^{+\infty} m_j'(t)$ a.e in $\mathbb{R}$. But $u_{j*} \in W^{1,1}_{loc}(Q_{j*})$ (see [21], [20]) thus $\Big\{t \in I_j : m_j'(t) < 0\Big\}$ has the same measure as I_j so $\Big|\Big\{t \in I(u) : m_u'(t) < 0\Big\}\Big| = |I(u)|$. If $t \in u_*(E\backslash E_d) - D_u$ by the chain rule, we necessarily have $t \in I(u) - \Big\{t \in I(u) : m_u'(t) < 0\Big\}$:

$$|u_*(E)| \leqslant |u_*(E_d)| + |u_*(E\backslash E_d)| = 0. \quad \blacksquare$$

But to have the local regularity of u_* and the (PSR) property, the domain do not need to be connected. Here is an example of such a set V.

Theorem 2 *Let $\Omega = \Omega_1 \cup \Omega_2$ with for $i = 1, 2$, Ω_i a bounded connected open set with Lipschitz boundary, $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$.*
Let

$$W^1(\Omega, |\cdot|_{N,1}) = \Big\{v \in W^{1,1}(\Omega) : \int_{\Omega_*} t^{\frac{1}{N}}\, |\nabla v|_*\,(t)\frac{dt}{t} < +\infty\Big\}$$
$$V = \Big\{v \in W^1(\Omega, |\cdot|_{N,1}), \text{ess inf }_{\Omega_1} v = \text{ess sup }_{\Omega_2} v \text{ or } \text{ess sup }_{\Omega_1} v = \text{ess sup }_{\Omega_1} v\Big\}.$$

Then V satisfies the (PSR) property.

PROOF. Let $v \in V,\ v_i = v|_{\Omega_i}\ i = 1, 2$. Assume that $\operatorname*{ess\,inf}_{\Omega_1}\ v = \operatorname*{ess\,sup}_{\Omega_2}\ v$ (the proof is the same for the other case). We have for $s \in \overline{\Omega}_*$

$$v_*(s) = \begin{cases} v_{1*}(s) & \text{if } 0 \leqslant s \leqslant |\Omega_1|\,, \\ v_{2*}(s - |\Omega_1|) & \text{if } |\Omega_1| \leqslant s \leqslant |\Omega|\,. \end{cases}$$

Since $v_i \in W^1(\Omega_i, |\cdot|_{N,1})$ thus $v_{i*} \in W^{1,1}(\Omega_{i*}),\ i = 1, 2$ and with the condition $v_{1*}(|\Omega_1|) = v_{2*}(0)$, we deduce that $v_* \in W^{1,1}(\Omega_*)$. According to the existence of PSR property given in [17], there are two constants $Q_i > 0$

$$K_i(s) = Q_i \max(s, |\Omega_i| - s)^{\frac{1}{N}-1}\,,\ i = 1, 2,\ s \in \Omega_{i*}$$

and

$$-v_{i*}'(s) \in K_i(s)\, |\nabla v_i|_{*v_i}\,(s),\ s \in \Omega_{i*}.$$

Since

$$\int_{v>v_*(s)} |\nabla v|\, dx \ = \ \int_{v_1>v_*(s)} |\nabla v_1|\, dx \ + \ \int_{v_2>v_*(s)} |\nabla v_2|\, dx \ ,\ s \in \Omega_*,$$

we then have if $s \leqslant |\Omega|_1$, $|\nabla v|_{*v}(s) = |\nabla v_1|_{*v_1}(s)$ and if $|\Omega_1| < s < |\Omega|$, $|\nabla v|_{*v}(s) = |\nabla v_2|_{*v_2}(s- |\Omega_1|)$. Combining the above relation, we then have

$$-v_*'(s) \leqslant \left[\widetilde{K}_1(s) + \widetilde{K}_2(s - |\Omega_1|)\right] |\nabla v|_{*v}(s),\ s \in \Omega_*,$$

with

$$\widetilde{K}_1(s) = \begin{cases} K_1(s) & \text{for } 0 \leqslant s \leqslant |\Omega_1,| \\ 0 & \text{otherwise.} \end{cases}$$

$$\widetilde{K}_2(s) = \begin{cases} K_2(s) & \text{if } 0 \leqslant s \leqslant |\Omega_2| \\ 0 & \text{otherwise.} \quad \blacksquare \end{cases}$$

Let us now set $u_{**}(s) = \dfrac{1}{s}\displaystyle\int_0^s u_*(t)dt,\ s \in \Omega_*$, $u : \Omega \to \mathbb{R}$ measurable.

Theorem 3 *Let V be a subset of $\bigcup_{1\leqslant p\leqslant +\infty} W^{1,p}(\Omega)$ satisfying the (PSR) property. Then for all $u \in V$,*

$$-u_{**}'(s) \leqslant \frac{1}{s} \operatorname*{ess\,sup}_{0\leqslant\sigma\leqslant s} [\sigma K(\sigma)] \left(|\nabla u|_{*u} \right)_{**}(s) \text{ a.e } s.$$

In particular, $-u_{**}'(s) \leqslant \dfrac{1}{s} \operatorname*{ess\,sup}\limits_{0\leqslant\sigma\leqslant s} [\sigma K(\sigma)] |\nabla u|_{**}(s)$. We set $\widetilde{K}(s) = \dfrac{1}{s} \operatorname*{ess\,sup}\limits_{0\leqslant\sigma\leqslant s} [\sigma K(\sigma)]$.

PROOF. Let $u \in V$. By integration by parts, we have for all $s \in \Omega_*$

$$\frac{1}{s}\int_0^s [u_*(t) - u_*(s)]\, dt = \frac{1}{s}\int_0^s t\, |u_*'(t)|\, dt.$$

Since V satisfies the (PSR) property, one deduces, using the Hardy-Littlewood inequality

$$u_{**}(s) - u_*(s) \leqslant \frac{1}{s}\int_0^s tK(t)\, |\nabla u|_{*u}(t) \leqslant s\widetilde{K}(s)\Big(|\nabla u|_{*u} \Big)_{**}(s).$$

But,

$$-\frac{d}{ds} u_{**}(s) = \frac{1}{s}\left[u_{**}(s) - u_*(s)\right],$$

thus the two last relations give the result. $\blacksquare$

4. Index of inclusion and generalized Poincaré-Sobolev inequalities for normed spaces

We assume in this paragraph that Ω is bounded.

Theorem 4 *For $s \in \Omega_*$, we set $I(s) = \left[\min\left(s, \dfrac{|\Omega|}{2}\right), \max\left(s, \dfrac{|\Omega|}{2}\right)\right]$ and $\chi_{I(s)}$ its characteristic function. Let V be a subset of $W^{1,1}(\Omega)$ satisfying the (PSR) property associated to a function K. For a nontrivial norm ρ on $L^0(\Omega_*)$ if ρ' is its associate norm, we define $b(s) = \rho'\big(\chi_{I(s)}K\big),\ s \in \Omega_*$. Then, for all homogeneous, monotone map ρ_0 on $L^0(\Omega_*)$ satisfying $0 < \rho_0(1)\cdot\rho_0(b) < +\infty$ we have $V^1(\Omega, \rho) \subset L(\Omega, \rho_0)$. Furthermore, for all $u \in V^1(\Omega, \rho)$:*

$$\inf_{c\in\mathbb{R}} \rho_0(u_* - c) \leqslant \rho_0\left(u_* - u_*\left(\frac{|\Omega|}{2}\right)\right) \leqslant \rho_0(b)\rho\big(|\nabla u|_{*u} \big).$$

The most usual inequalities are for those norms that are invariant under rearrangement.

Corollary 1 *Under the conditions of theorem 4 if ρ is a Fatou norm invariant under rearrangement and $\rho_0(f) \leqslant \rho_0(f-\lambda) + \lambda\rho_0(1)$, $\forall\lambda \in \mathbb{R}$, $\forall f \in L^0(\Omega_*)$, then*

$$\rho_0(u_*) \leqslant \rho_0(b)\rho\big(\,|\nabla u|_*\big) + \frac{2}{|\Omega|}\rho_0(1)\,|u|_1 \qquad \forall u \in W^1(\Omega,\rho). \quad \square$$

We first prove those theorems and then will give some examples of usual norms ρ and ρ_0.
PROOF OF THEOREM 4 Let u be in $V^1(\Omega,\rho) \subset V$. By the (PSR) property we deduce that for all $s \in \Omega_*$

$$\left| u_*(s) - u_*\left(\frac{|\Omega|}{2}\right)\right| \leqslant \int_{\Omega_*} \chi_{I(s)}(\sigma)K(\sigma)\,|\nabla u|_{*u}(\sigma)d\sigma \leqslant \rho'\big(\chi_{I(s)}K\big)\rho\big(\,|\nabla u|_{*u}\big).$$

If ρ_0 is a monotone homogeneous map, we deduce :

$$\rho_0\left(u_* - u_*\left(\frac{|\Omega|}{2}\right)\right) \leqslant \rho_0(b)\rho\big(\,|\nabla u|_{*u}\big).$$

$\diamondsuit$

To eliminate the term $u_*\left(\frac{|\Omega|}{2}\right)$, one can observe the following inequality

$$\frac{2}{|\Omega|}\int_{\frac{|\Omega|}{2}}^{|\Omega|} u_*(t)dt \leqslant u_*\left(\frac{|\Omega|}{2}\right) \leqslant \frac{2}{|\Omega|}\int_0^{\frac{|\Omega|}{2}} u_*(t)dt.$$

Thus,

$$\left|u_*\left(\frac{|\Omega|}{2}\right)\right| \leqslant \frac{2}{|\Omega|}\int_\Omega |u|\,dx. \quad \blacksquare \tag{1}$$

PROOF OF COROLLARY 1 Since ρ is a Fatou norm invariant under rearrangement, we know that $\rho\big(\,|\nabla u|_{*u}\big) \leqslant \rho\big(\,|\nabla u|_*\big)$. From theorem 4, we obtain by monotonicity, homogeneity of ρ_0 and the $\rho_0(u_* - \lambda) \geqslant \rho_0(u_*) - \lambda\rho_0(1)$, $\rho_0(u_*) \leqslant \rho_0(b)\rho\big(\,|\nabla u|_*\big) + \left|u_*\left(\frac{|\Omega|}{2}\right)\right|\rho_0(1)$, which gives the result, with the help of relation (1). $\blacksquare$

Definition 3 *We shall call $\rho_0(b)$ the index of inclusion associated to $V^1(\Omega,\rho)$.*

Sometimes, one has information on the average $\overline{u} = \frac{1}{|\Omega|}\int_\Omega u(x)dx$. One may replace $u_*\left(\frac{|\Omega|}{2}\right)$ by $\overline{u}$. That is,

Theorem 5 *Under the same conditions as for the theorem 4, we have $\forall u \in V^1(\Omega,\rho)$*

$$\rho_0\left(u_* - \frac{1}{|\Omega|}\int_\Omega u(x)dx\right) \leqslant \left[\rho_0(b) + c_u\rho'(1)\rho_0(1)\right]\rho\big(\,|\nabla u|_{*u}\big),$$

where $c_u = \mathrm{Max}\left\{K(\sigma), \sigma \in \left[\min\left(|u>\overline{u}|\,, \frac{|\Omega|}{2}\right), \max\left(|u>\overline{u}|\,, \frac{|\Omega|}{2}\right)\right]\right\}$, $\overline{u} = \frac{1}{|\Omega|}\int_\Omega u(x)dx.$

PROOF. Introducing $I(u,s) = \big[\min\big(s, |u>\overline{u}|\,\big), \max\big(s, |u>\overline{u}|\,\big)\big]$ and $b_u(s) = \rho'\big(\chi_{I(u,s)}K\big),\ s \in \Omega_*$ the same argument as in Theorem 4 leads to

$$\rho_0\Big(u_* - u_*\big(\,|u>\overline{u}|\,\big)\Big) \leqslant \rho_0(b_u)\rho\big(\,|\nabla u|_{*u}\big).$$

Since $u_*\big(\,|u>\overline{u}|\,\big)=\overline{u}$ and $\rho(b_u)\leqslant\rho_0(b)+c_u\rho'(1)\rho_0(1)$, we deduce the result. ■

Let us give a direct application of Theorem 4. For this, we consider an open bounded set with a Lipschitz boundary. We recall that in that case, the set $V=W^{1,1}(\Omega)$ satisfies the (PSR) property with $K(s)=Q\max(s,|\Omega|-s)^{\frac{1}{N}-1}$, Q is a constant depending only on N and Ω (see [15], [16]). We shall denote by $L^{p,q}(\Omega),\ 1\leqslant p\leqslant+\infty,\ 1\leqslant q\leqslant+\infty$ the usual Lorentz space endowed with the following norm :

$$|f|_{(p,q)}=\begin{cases}\left[\displaystyle\int_{\Omega_*}\left[t^{\frac{1}{p}}\,|f|_{**}(t)\right]^q\frac{dt}{t}\right]^{\frac{1}{q}} & 1\leqslant q<+\infty,\\ \displaystyle\sup_t t^{\frac{1}{p}}\,|f|_{**}(t) & if\ q=+\infty,\end{cases}$$

with $|f|_{**}(t)=\dfrac{1}{t}\displaystyle\int_0^t|f|_*(\sigma)d\sigma$.

For the computation, we shall use also an equivalent *quantity* (not always a norm), which is

$$|f|_{p,q}=\begin{cases}\left[\displaystyle\int_{\Omega_*}\left[t^{\frac{1}{p}}\,|f|_{*}(t)\right]^q\frac{dt}{t}\right]^{\frac{1}{q}} & 1\leqslant q<+\infty,\\ \displaystyle\sup_t t^{\frac{1}{p}} & if\ q=+\infty.\end{cases}$$

If $1\leqslant q\leqslant p<+\infty$, the map $f\to|f|_{p,q}$ is a norm. Otherwise, it is a definite, monotone, homogeneous map on $L^0(\Omega_*)$.

We denote by $W^1(\Omega,|\cdot|_{p,q})=\Big\{v\in L^1(\Omega):|\nabla v|\in L^{p,q}(\Omega)\Big\}$. We then have

Theorem 6 *Let* $1\leqslant p<N,\ 1\leqslant q\leqslant+\infty$. *Then*

$$W^1(\Omega,|\cdot|_{p,q})\subset L^{p^*,+\infty}(\Omega)\ with\quad\frac{1}{p^*}=\frac{1}{p}-\frac{1}{N}.$$

Furthermore, $\forall u\in W^1(\Omega,|\cdot|_{p,q})$ *one has:*

i) If $q\neq1$ *then*

$$\left|u-u_*\left(\frac{|\Omega|}{2}\right)\right|_{p^*,+\infty}\leqslant2^{\frac{1}{p^*}}Q\left(\int_1^{+\infty}(\theta-1)^\gamma\theta^{-\nu}d\theta\right)^{1-\frac{1}{q}}\Big|\,|\nabla u|_{*u}\Big|_{(p,q)}$$

with $\gamma+1=\dfrac{q}{p}\dfrac{p-1}{q-1},\ \nu=\dfrac{q}{N}\dfrac{N-1}{q-1}$.

ii) If $q=1$ *then*

$$\left|u-u_*\left(\frac{|\Omega|}{2}\right)\right|_{p^*,+\infty}\leqslant Q\gamma(N,p)\Big|\,|\nabla u|_{*u}\Big|_{(p,1)}$$

where $\gamma(N,p)=\dfrac{(N')^{\frac{1}{p'}}}{(p')^{\frac{1}{N'}}}(p'-N')^{\frac{1}{p^*}},\ p'=\dfrac{p}{p-1},\ N'=\dfrac{N}{N-1}$.

PROOF. We apply Theorem 4 with $\rho_0=|\cdot|_{p^*,+\infty}$, $\rho(\cdot)=|\cdot|_{(p,q)}$. Then $\rho'(\cdot)\leqslant p\,|\cdot|_{p',q'}$, $\dfrac{1}{p}+\dfrac{1}{p'}=1$, $\dfrac{1}{q}+\dfrac{1}{q'}=1$. Since $|\cdot|_{(p',q')}\leqslant p\,|\cdot|_{p',q'}$, it suffices to compute the quantity $\rho_0\big(\,|\chi_{I(\cdot)}K|_{p',q'}\,\big)$. By symmetry, we have to compute for $s < \dfrac{|\Omega|}{2}$, $\left|\left(\chi_{I(s)}K\right)_*\right|_{p',q'} = b(s)$. But

$$\left(\chi_{I(s)}K\right)_*(\sigma) = Q(s+\sigma)^{\frac{1}{N}-1}\chi_{[0,\frac{|\Omega|}{2}-s]}(s)$$

for $s \in \overline{\Omega}_*$ if $0 < s < \dfrac{|\Omega|}{2}$. Thus we have for $0 < s < \dfrac{|\Omega|}{2}$

$$b(s) = Qs^{\frac{\gamma+1-\nu}{q'}}\left(\int_1^{\frac{|\Omega|}{2s}}(\theta-1)^\gamma\theta^{-\nu}d\theta\right)^{\frac{1}{q'}},$$

with $\gamma,\ \nu$ as in the theorem. We have $\dfrac{\gamma+1-\nu}{q'} = -\dfrac{1}{p^*}$. Setting $J = Q\left(\displaystyle\int_1^{+\infty}(\theta-1)^\gamma\theta^{-\nu}d\theta\right)^{\frac{1}{q'}}$, we then have

$$b_*(s) \leqslant 2^{\frac{1}{p^*}}Js^{-\frac{1}{p^*}} : |b|_{p^*,+\infty} \leqslant 2^{\frac{1}{p^*}}J.$$

Applying Theorem 4, we get

$$\left|u - u_*\left(\frac{|\Omega|}{2}\right)\right|_{p^*,+\infty} \leqslant |b|_{p^*,+\infty}\left|\,|\nabla u|_{*u}\right|_{(p,q)}$$

We get the result. ■

Since $\left|\,|\nabla u|_{*u}\right|_{(p,q)} \leqslant |\nabla u|_{(p,q)}$, we also have :

$$\left|u - u_*\left(\frac{|\Omega|}{2}\right)\right|_{p^*,+\infty} \leqslant 2^{\frac{1}{p^*}}J\,|\nabla u|_{(p,q)},$$

which leads to the following result according to Corollary 1 of Theorem 4: if $q = 1$ then $b_*(s) \leqslant Q\gamma(N;p)s^{-\frac{1}{p^*}}$ with $\gamma(N,p)$ as in the theorem. The same argument as for $q \neq 1$ leads to the following result.

Corollary 2 *We have :*

$$|u|_{p^*,+\infty} \leqslant 2^{\frac{1}{p^*}}J\,|\nabla u|_{(p,q)} + 2\,|\Omega|^{\frac{1}{p^*}-1}\,|u|_1 \qquad \forall u \in W^1(\Omega,|\cdot|_{p,q}).$$

Remark 2 The choice of $\rho_0 = |\cdot|_{p^*,+\infty}$ is just for computational case. In fact, if $1 \leqslant q < +\infty,\ 1 \leqslant p < N$, one can show using (PSR) property the following theorem (see [1] for an alternative proof). ■

Theorem 7 *If $1 \leqslant p < N,\ 1 \leqslant q \leqslant +\infty$ then,*

$$W^1(\Omega,|\cdot|_{p,q}) \subsetneq L^{p^*,q}(\Omega), \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

Moreover, for all $v \in W^1(\Omega,|\cdot|_{p,q})$, we have:

1. *If $\gamma_0 v = 0$ on $\partial\Omega$ then*

$$|v|_{p^*,q} \leqslant \frac{p^*}{N\alpha_N^{\frac{1}{N}}}\left|\,|\nabla v|_{*|v|}\right|_{p,q} \leqslant \frac{p^*}{N\alpha_N^{\frac{1}{N}}}\,|\nabla v|_{p,q}\ \ if\ 1 \leqslant q \leqslant p$$

and

$$|v|_{(p^*,q)} \leqslant \frac{p^*}{N\alpha_N^{\frac{1}{N}}}\left|\,|\nabla v|_{*|v|}\right|_{(p,q)} + (p^*)^{\frac{1}{q}}\,|\Omega|^{\frac{1}{p^*-1}}\,|v|_1\ \ if\ p < q \leqslant +\infty$$

2. *If* $\gamma_0 v \not\equiv 0$ *on* $\partial\Omega$ *then there exists* $c_2 > 0$ *depending on* $p,\ q,\ \Omega,\ N$ *and* Q *such that*

$$\left| v - v_*\left(\frac{|\Omega|}{2}\right)\right|_{(p^*,q)} \leqslant c_2 \Big|\, |\nabla v|_{*|v|}\, \Big|_{(p,q)}.$$

We prove only part 1 of the Theorem 7 since the second follows the same idea.
PROOF OF PART 1 OF THEOREM 7 Let $1 \leqslant q \leqslant p$. By integration by parts and using the (PSR) property, we have, for $u = |v|$ with $\gamma_0 u = 0$:

$$\int_{\Omega_*} t^{\frac{q}{p^*}-1} u_*(t)^q dt = p^* \int_{\Omega_*} t^{\frac{q}{p^*}} u_*(t)^{q-1} \left|u'_*\right|(t) \leqslant$$

$$\leqslant \frac{p^*}{N\alpha_N^{\frac{1}{N}}} \int_{\Omega_*} t^{\frac{1}{N}} \left|\nabla u\right|_{*u}(t) u_*(t)^{q-1} t^{\frac{q}{p^*}-1} dt.$$

Applying the Hölder inequality to the last integral, we get after simplification and the utilization of the Hardy-Littlewood inequality :

$$|v|_{p^*,q} \leqslant \frac{p^*}{N\alpha_N^{\frac{1}{N}}} \left(\int_{\Omega_*} t^{\frac{q}{p}-1} \left[\left|\nabla u\right|_{*u}(t)\right]^q dt\right)^{\frac{1}{q}} \leqslant \frac{p^*}{N\alpha_N^{\frac{1}{N}}} \Big|\, |\nabla u|_{*u}\, \Big|_{p,q}.$$

Since the map $f \to |f|_{p,q}$ is a Fatou norm invariant under rearrangement, one deduces $\Big|\, |\nabla u|_{*u}\, \Big|_{p,q} \leqslant |\nabla u|_{p,q}$. Since this last property is not true for $p < q$, we replace u_* by u_{**}. Let $p < q < +\infty$. By integration by parts and theorem 3, one has as before

$$\int_{\Omega_*} t^{\frac{q}{p^*}-1} u_{**}(t)^q dt \leqslant \frac{p^*}{q} |\Omega|^{\frac{p^*}{q}} u_{**}\big(|\Omega|\big)^q +$$

$$+ \frac{p^*}{N\alpha_N^{\frac{1}{N}}} \int_{\Omega_*} t^{\frac{1}{N}} \Big(|\nabla u|_{*u}\Big)_{**}(t) u_{**}(t)^{q-1} t^{\frac{q}{p^*}-1} dt.$$

Using Hölder inequality and the Young inequality, we have

$$\int_{\Omega*} t^{\frac{q}{p^*}-1} u_{**}(t)^q dt \leqslant p^* |\Omega|^{\frac{q}{p^*}} u_{**}\big(|\Omega|\big)^q + \left(\frac{p^*}{N\alpha_N^{\frac{1}{N}}}\right)^q \Big|\, |\nabla u|_{*u}\, \Big|^q_{(q,p)}.$$

From which we get the result. Notice that, we always have $\Big|\, |\nabla u|_{*u}\, \Big|_{(p,q)} \leqslant |\nabla v|_{(p,q)}$.
The proof of part 2 is similar. ■

The case $p \geqslant N$ can be deduced from Theorem 4 with $\rho_0(\cdot) = |\cdot|_\infty$. Applying the Corollary 1 of Theorem 4, we have

$$|v|_\infty \leqslant pQ\left(\frac{|\Omega|}{2}\right)^{\frac{1}{N}-\frac{1}{p}} |\nabla v|_{(p,1)} + \frac{2}{|\Omega|}|v|_1 \qquad \forall v \in W^1(\Omega, |\cdot|_{p,1}),\ p \geqslant N.$$

Moreover, using the (PSR) property for $W^{1,1}\big(B(x,r)\big),\ B(x,r) \subset \Omega$, one has

$$\operatorname{osc}_{B(x,r)} v \leqslant \frac{\alpha_N^{1-\frac{1}{p}}}{\alpha_{N-1}} |\nabla v|_{L^{p,1}(B(x,r))} \cdot r^{1-\frac{N}{p}}.$$

5. General interpolations of Gagliardo-Nirenberg type

We have the following interpolation theorems:

Theorem 8 *Let ρ be a non trivial norm on $L^0(\Omega_*)$. Let $V_+ \subset L^0_+(\Omega_*)$ satisfying the (PSR) property associated to K. Then, $\forall u \in V^1_+(\Omega,\rho) = \Big\{v \in V_+,\ \rho(|\nabla v|_{*v}) < +\infty\Big\}$ and for all $q \in]0,\infty[$,*

$$\rho_0(u_*) \leqslant q^{\frac{1}{q}}\rho\big(|\nabla u|_{*u}\big)^{\frac{1}{q}}\rho_0\Big[\Big(\rho'\ \big(Ku_*^{q-1}\chi_{I(q,s)}\big)\ \Big)^{\frac{1}{q}}\Big].$$

Whenever ρ_0 is a monotone, homogeneous map from $L^0(\Omega_*)$ into R_+ and $I(q,s) = [s, |\Omega_q|]$ with

$$|\Omega_q| = \begin{cases} |\Omega| & \text{if } q \geqslant 1, \\ |u>0| & if\ 0<q<1. \end{cases}$$

Corollary 3 *Assume that Ω is an open bounded set of $\mathbb{R}^N$, $N \geqslant 2$. If $N' \leqslant p < \infty$ then for all $u \in W^{1,N}_0(\Omega)$*

$$|u|_p \leqslant \left(\frac{1}{aN\alpha_N^{\frac{1}{N}}}\right)^a \Big|\,|\nabla u|_{*u}\,\Big|_N^a\,|u|_{p-N'}^{1-a}\,.$$

If furthermore Ω is connected and Lipschtiz then for all $u \in W^{1,N}_{\Gamma_0}(\Omega) = \Big\{v \in W^{1,1}(\Omega),\ v = 0\ on\ \Gamma_0$ with $\Gamma_0 \subset \Omega$, $H_{N-1}(\Gamma_0) > 0\Big\}$ then

$$|u|_p \leqslant \left(\frac{1}{aN\sigma_N^{\frac{1}{N}}}\right)^a \Big|\,|\nabla u|_{*u}\,\Big|_N^a\,|u|_{p-N'}^{1-a}\,, \qquad with \quad a = \frac{N'}{p}.$$

Remark 3 Here $p - N'$ may be less than 1. ■

PROOF OF THEOREM 8 Let $u \in V^1_+(\Omega,\rho)$. Then,

$$-u_*'(s) \leqslant K(s)|\nabla u|_{*u}(s) \qquad a.e.$$

From which we derive that $\forall s \in \Omega_*$,

$$u_*^q(s) \leqslant q\int_{\Omega_*} |\nabla u|_{*u}\,(t)K(t)u_*^{q-1}(t)\chi_{I(q,s)}(t)dt.$$

Then, using ρ and ρ', one deduces

$$u_*(s) \leqslant q^{\frac{1}{q}}\rho\big(|\nabla u|_{*u}\big)^{\frac{1}{q}}\Big(\rho'\ \big(Ku_*^{q-1}\chi_{I(q,s)}(\cdot)\big)\Big)^{\frac{1}{q}}.$$

Since ρ_0 is a monotone, homogeneous map then

$$\rho_0(u_*) \leqslant q^{\frac{1}{q}}\rho\big(|\nabla u|_{*u}\big)^{\frac{1}{q}}\rho_0\left[\Big(\rho'\big(Ku_*^{q-1}\chi_{I(q,s)}(\cdot)\big)\Big)^{\frac{1}{q}}\right]. \quad ■$$

PROOF OF THE COROLLARY 3 We may assume that $u \geqslant 0$, we choose $\rho_0(\cdot) = |\cdot|_p$, $qN' = p$. Since $K(s) = \dfrac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}$, for $W^{1,N}_{0+}(\Omega)$, $s \in \Omega_*$, we deduce from Theorem 8

$$|u|_p \leqslant \left(\frac{q}{N\alpha_N^{\frac{1}{N}}}\right)^{\frac{1}{q}} \Big|\,|\nabla u|_{*u}\,\Big|_N \left(\int_{\Omega_*} ds \int_s^{|\Omega|} t^{-1}u_*^{p-N'}(t)dt\right)^{\frac{1}{p}}.$$

By Fubini theorem and setting $a = \dfrac{N'}{p}$, we deduce the result. ■

For $W^{1,N}_{\Gamma_{0+}}(\Omega)$, $K(s) = \frac{s^{\frac{1}{N}-1}}{N\sigma_N^{\frac{1}{N}}}$ the result follows arguing as above.

Remark 4 If $\Omega \subset \mathbb{R}^2$, $p = 4$ then

$$|u|_4 \leqslant \left(\frac{1}{\pi}\right)^{\frac{1}{4}} |\nabla u|_2^{\frac{1}{2}}\, |u|_2^{\frac{1}{2}}, \qquad \forall u \in H_0^1(\Omega),$$

$$|u|_4 \leqslant \left(\frac{1}{\sigma_2}\right)^{\frac{1}{4}} |\nabla u|_2^{\frac{1}{2}}\, |u|_2^{\frac{1}{2}}, \qquad \forall u \in W^{1,2}_{\Gamma_0}(\Omega).$$

Other interpolation inequalities can also be derived.

Using [18], [19], we have the PSR property for weighted Poincare Sobolev sets, or for measures other than Lebesgue measure. Thus, interpolation inequalities associated to those spaces can be obtained (see [17]). ■

References

[1] Alvino, A., Lions, P. L. and Trombetti, G. (1998). On Optimization Problems with Prescribed Rearrangements, *Nonlinear Anal.*, **13** (2), 185–220.

[2] Betta, F. and Mercaldo, A. (1991). Existence and Regularity results for a nonlinear Elliptic Equations, *Rend. Mat.*, **VII** (11), 737–759.

[3] Chabi, A. and Haraux, A. (1985). Un théorème de valeurs intermédiaires dans les espaces de Sobolev et applications, *Ann. Fac. Sci. Toulouse*, **VII**, 87–100.

[4] Chong, K. M. and Rice, N. M. (1971). *Equimesurable rearrangements of functions.* Queen's University.

[5] Díaz, J. I., Nagai, T. and Rakotoson, J. M. Symmetrization Techniques on Unbounded Domains: Application to a Chemotaxis System on $\mathbb{R}^{\mathrm{N}}$, *J. Differential Equations*, **145** (1), 156–183.

[6] Ferone, A. and Volpicelli, R. (2000). Some relations between pseudo-rearrangement and relative rearrangement, *Nonlinear Anal.*, **41** (7-8), 855–869.

[7] Ferone, V. and Posteraro, M. R. (1991). A remark on a Comparison theorm, *Comm. Partial Differential Equations*, **16**, 1255–1262.

[8] Fiorenza, A. (2000). Duality and reflexivity in Grand Lebesgue spaces, *Collect. Math.*, **51** (2), 131–148.

[9] Fiorenza, A. and Rakotoson, J. M. (2002). Petits espaces de Lebesgue et quelques applications, *C. R. Acad. Sci. Paris, Sér. I*, **334**, 23–26.

[10] Hewitt, E. and Stromberg, K. (1965). *Real and Abstract Analysis*, Springer-Verlag, New-York, Hidelburg, Berlin.

[11] Hnid, M., *personnal communication*.

[12] Lions, P. L., Pacella, F. and Tricarico, M. (1988). Best constants on Sobolev inequalities for functios vanishing on some part of the boundary and related question, *Indiana Univ. Math. J.*, **37**, 301–324.

[13] Mossino, J. and Temam, R. (1981). Directional Derivative of the Increasing Rearrangement Mapping and Application to a Queer Differential Equation in Plasma Physics, *Duke Math. J.*, **48** (3), 475–495.

[14] Rakotoson, J. E. and Rakotoson, J.M. (2003). Local regularity of the monotone rearrangement, *to appear*.

[15] Rakotoson, J. M. (2001). General pointwise relations for the relative rearrangement and applications, *Appl. Anal.*, **80**, 201–232.

[16] Rakotoson, J. M. (2002). Some new applications of the pointwise relations for the relative rearrangement. *Adv. Differential Equations*, **7** (5), 617–640.

[17] Rakotoson, J. M. (2003). *Réarrangement Relatif*, book in preparation.

[18] Rakotoson, J.M. and Simon, B. (1993). Relative rearrangement on a measure space. Application to the regularity of weighted monotone rearrangement, Part I-II, *Appl. Math. Lett.*, **6** (1), 75–78, 79–92.

[19] Rakotoson, J. M. and Simon, B. (1997). Relative rearrangement on a finite measure space. Application to weighted spaces and to P.D.E., *Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.)*, **91** (1), 33–45.

[20] Rakotoson, J. M. and Temam, R. (1987). Une formule intégrale du type Federer et Applications, *C. R. Acad. Sci. Paris, Sér. I*, **304**, 443–446.

[21] Rakotoson, J. M. and Temam, R. (1990). A co–area formula with applications to monotone rearrangement and to regularity, *Arch. Rational Mech. Anal.*, **109**, 213–238.

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