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On the structure of symmetric 2×2 gradients

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Abstract. A way of geometrically representing symmetric 2×2-gradients is proposed, and a general theorem characterizing sets of gradients is proved. We believe this perspective may help in understanding the structure of gradients and visualizing it. Several non-trivial examples are discussed.

Sobre la estructura de gradientes simétricos 2×2

Resumen. Se propone una manera de representar geométricamente gradientes simétricos 2×2 y se prueba un teorema que caracteriza tales conjuntos de gradientes. Creemos que esta perspectiva puede ayudar a comprender la estructura de tales gradientes y visualizarla geométricamente. Se ilustran los resultados con varios ejemplos interesantes.

1. Introduction

In this paper we provide some insight on some geometric properties of gradients, with the aim of better understanding their structure as it relates to important issues for vector problems in the Calculus of Variations. Specifically, it is well-known that quasiconvexity ([2], [7]) is the key constitutive assumption for an existence theory based on the direct method and weak lower semicontinuity. But it is also relevant for problems where nonconvexity is the central ingredient (see for instance [1]) and the oscillatory behavior recorded in the underlying Young measure minimizer is the true question, since there exists a duality ([3], [4]) between quasiconvexity and gradient Young measures through Jensen's inequality (see also [5] and [15]). In particular, Morrey's conjecture about the equivalence of quasiconvexity and rank-one convexity has turned out to be false ([14]) for matrices $\mathbf{M}^{m\times N}$ when $m \geq 3$ but it remains open when $m = 2$. The main motivation for the analysis that follows was to further pursue this question for 2×2 matrices. Since the answer does not seem to be a direct generalization of Sverak's counterexample ([12], [13]), it requires a better understanding of the structure and geometry of gradients. Some relevant references from this point of view are [9], [10].

To be able to rely on our intuition it is extremely convenient to "visualize" gradients. Since for 2×2 matrices, we already have to deal with a four-dimensional space, we will restrict attention to symmetric 2×2 matrices which is a three-dimensional subspace. In fact, the analysis of variational problems for functionals of the type

$$I(u) = \int_\Omega \varphi(x, u(x), \nabla u(x), \nabla^2 u(x))\, dx$$

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where $\Omega \subset \mathbb{R}^2$ is a regular domain, competing functions $u : \Omega \to \mathbb{R}$ belong to appropriate spaces of weakly differentiable functions up to second derivatives, and $\nabla^2 u$ indicates the hessian matrix of u, need to address the issue of the appropriate convexity properties ensuring the applicability of the direct method. Since $\nabla^2 u = \nabla(\nabla u)$ is a symmetric gradient, the question we would like to analyze is relevant for this type of variational problems. It is a particular case for two-dimensional, vector variational problems.

A very convenient identification of 2×2-symmetric matrices with $\mathbb{R}^3$ was already used in [11] and [13], namely

$$(x, y, z) \quad \text{is identified with} \quad \begin{pmatrix} x+z & z \\ z & y+z \end{pmatrix}. \tag{1}$$

Due to the close relationship between rank-one convexity and quasiconvexity, identifying rank-one directions in our model is important. In our situation, rank-one directions correspond to vectors (x, y, z) such that

$$xy + xz + yz = 0.$$

In particular, the three basis vectors in $\mathbb{R}^3$ are rank-one directions. We would like to distinguish these three directions given by the three basis vectors of $\mathbb{R}^3$. In our identification, these correspond to the rank-one matrices

$$(1,0,0) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1,0) \otimes (1,0),$$
$$(0,1,0) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (0,1) \otimes (0,1),$$
$$(0,0,1) \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = (1,1) \otimes (1,1).$$

Notice that the “normals” associated to these three matrices are $(1,0)$, $(0,1)$ and $(1,1)$.

We would like to consider discrete periodic gradients with period the unit square $T = (0,1) \times (0,1)$ taking on symmetric values. In this way, after our identification the matrices involved in such gradients can be properly identified with points in $\mathbb{R}^3$. Given such a periodic gradient, the set of matrices taken on throughout T will be a discrete set of points in $\mathbb{R}^3$. Moreover, we would like to use only normals $(1,0)$, $(0,1)$ and $(1,1)$ in our discretizations of periodic symmetric gradients as these are the normals corresponding to the coordinate axes in our model, as remarked earlier. It is the structure of these finite sets of points in $\mathbb{R}^3$ that we would like to better understand. Therefore for a given positive integer m we consider piecewise affine, periodic, symmetric gradients where the gradient is constant on each of the triangles of the discretization of T coming from subdividing T in $m \times m$ equal subsquares, and considering on each such subsquare the two triangles along the diagonal with normal $(1,1)$. See Figure 1.

Figure 1. The mesh for a periodic gradient for $m = 4$.

Some clever, particular cases of these piece-wise affine gradients have been used throughtout the literature (see [6] and references therein, [11], [13]).

Altogether we have, at most, $2m^2$ symmetric matrices which will correspond to a set of $2m^2$ points in $\mathbb{R}^3$. Let Γ_m designate the set of all such symmetric, piecewise-affine gradients. We would like to exactly describe the structure of sets of $2m^2$ points in $\mathbb{R}^3$ corresponding, through our identification, to gradients in Γ_m. The main result follows.

Theorem 1 *A set $\mathcal{A}_m$ of at most $2m^2$ points of $\mathbb{R}^3$ corresponds to a gradient in Γ_m if and only if there exist at most $3m$ numbers*

$$r_j^{(i)}, \quad i=1,2,3, \quad j=1,2,\dots m,$$

such that
1.

$$r_1^{(3)}=0;$$

2.

$$\sum_j r_j^{(1)}=\sum_j r_j^{(2)}=-\sum_j r_j^{(3)};$$

3.

$$\mathcal{A}_m=\left\{\left(r_i^{(1)},r_j^{(2)},r_{i+j-1}^{(3)}\right),\left(r_i^{(1)},r_j^{(2)},r_{i+j}^{(3)}\right):1\le i,j\le m\right\},$$

where it is assumed that

$$r_{m+j}^{(3)}=r_j^{(3)}. \quad \square$$

If we insist in having the three families of numbers ordered in an increasing fashion then we have to ask for the existence of suitable permutations of indeces $\sigma^{(i)}$, $i=1,2,3$, such that
1.

$$r_{\sigma^{(3)}(1)}^{(3)}=0;$$

2.

$$\sum_j r_j^{(1)}=\sum_j r_j^{(2)}=-\sum_j r_j^{(3)};$$

3.

$$\begin{aligned}\mathcal{A}_m= \ &\left\{\left(r_{\sigma^{(1)}(i)}^{(1)},r_{\sigma^{(2)}(j)}^{(2)},r_{\sigma^{(3)}(\sigma^{(1)}(i)+\sigma^{(2)}(j)-1)}^{(3)}\right),\right.\\ &\left.\left(r_{\sigma^{(1)}(i)}^{(1)},r_{\sigma^{(2)}(j)}^{(2)},r_{\sigma^{(3)}(\sigma^{(1)}(i)+\sigma^{(2)}(j))}^{(3)}\right):1\le i,j\le m\right\}.\end{aligned}$$

The structure of such sets of points in $\mathbb{R}^3$ can be made even more transparent by a closer examination of the structure of gradients.

Theorem 2 *A set $\mathcal{A}_m$ of at most $2m^2$ points of $\mathbb{R}^3$ corresponds to a gradient in Γ_m if and only if*
1. there exist at most $3m$ numbers

$$r_j^{(i)}, \quad i=1,2,3, j=1,2,\dots m, \quad r_1^{(3)}=0,$$

with

$$r_j^{(i)}\le r_{j+1}^{(i)},$$

for all i and j, such that

$$\mathcal{A}_m\subset\left\{\left(r_{j_1}^{(1)},r_{j_2}^{(2)},r_{j_3}^{(3)}\right):1\le j_1,j_2,j_3\le m\right\};$$

2. if we put

$$x=\left(x^{(1)},x^{(2)},x^{(3)}\right)$$

for vectors in $\mathbb{R}^3$*, then for each possible pair* $i_1 \neq i_2$ *and every* i*,* j*, we must have*

$$\left|\mathcal{A}_m \cap \left\{x \in \mathbb{R}^3 : x^{(i_1)} = r_i^{(i_1)}, x^{(i_2)} = r_j^{(i_2)}\right\}\right| = 2,$$

counting appropriately multiplicity when some of the $r_j^{(i)}$*'s are repeated;*
3. because of the previous statement, we can define maps

$$\Psi^{(i)} : \mathcal{A}_m \mapsto \mathcal{A}_m, \quad i = 1, 2, 3,$$

by putting

$$\mathcal{A}_m \cap \left\{x \in \mathbb{R}^3 : x^{(i_1)} = r_i^{(i_1)}, x^{(i_2)} = r_j^{(i_2)}\right\} = \left\{x, \Psi^{(i_3)}(x)\right\},$$

if $\{i_1, i_2, i_3\} = \{1, 2, 3\}$*; then for* $i_1 \neq i_2$*, we must have*

$$\left(\Psi^{(i_1)} \cdot \Psi^{(i_2)}\right)^j \equiv \textit{identity}$$

if and only if j *is a multiple of* m*;*
4. the baricenter of $\mathcal{A}_m$*, counting multiplicity of points,*

$$\sum_{x \in A_m} x$$

lies in the line through the origin and direction $(1, 1, -1)$*.* $\square$

The statement of Theorem 2 may look somewhat artificial. In Section 2 we will see what it means in plain words. There might be other easier ways of expressing those conditions that may allow to systematically build and geometrically represent symmetric gradients, but this is the formulation we have found more convenient. By using this theorem we can provide examples of gradients like the one in Figure 2. Notice the inner subcube. Matrices participating in the gradient correspond to black dots in the figure. The different lines will be explained in Section 2. Any of these theorems enables us to think about 2×2 gradients, at least

Figure 2. An example for $m = 4$.

for symmetric gradients, in terms of such sets of points in three dimensional space, facilitating the search for counterexamples to the fact that rank-one convexity implies quasiconvexity or else looking for a proof that every probability measure (counting appropriately multiplicity)

$$\nu = \frac{1}{2m^2} \sum_{x \in A_m} \delta_x$$

is a laminate for every m and every admissible A_m. We believe these results help in understanding the structure of such gradients becoming more transparent and clear.

Section 3 contains the proofs of Theorems 1 and 2, while Section 2 is concerned with the visualization of admissible sets for small values of m. Several interesting examples are also proposed.

2. Some examples

Before proving Theorems 1 and 2, we think it is worthwhile to investigate the structure of admissible sets $\mathcal{A}_m$ for small values of m. In particular, it is quite interesting to explore the structure of such sets when some of the numbers $r_j^{(i)}$ collapse and appear repeated.

One further important observation concerns the first moment of such sets of points. Without loss of generality, since the structure of gradients is invariant under translations, we can assume that the first moment of such sets is the origin. This simplifies somewhat our discussion as we need not be concerned about the condition $r_1^{(3)} = 0$. In fact, the following corollary is a direct consequence of Theorem 1.

Corollary 1 *A set $\mathcal{A}_m$ of at most $2m^2$ points of $\mathbb{R}^3$ corresponds to a translation of a gradient in Γ_m if and only if*
1. there exist at most $3m$ numbers

$$r_j^{(i)}, \quad i = 1, 2, 3, j = 1, 2, \ldots m,$$

with

$$r_j^{(i)} \leq r_{j+1}^{(i)},$$

for all i and j, such that

$$\mathcal{A}_m \subset \left\{ \left(r_{j_1}^{(1)}, r_{j_2}^{(2)}, r_{j_3}^{(3)} \right) : 1 \leq j_1, j_2, j_3 \leq m \right\};$$

2. if we put

$$x = \left(x^{(1)}, x^{(2)}, x^{(3)} \right)$$

for vectors in $\mathbb{R}^3$, then for each possible pair $i_1 \neq i_2$ and every i, j, we must have

$$\left| \mathcal{A}_m \cap \left\{ x \in \mathbb{R}^3 : x^{(i_1)} = r_i^{(i_1)}, x^{(i_2)} = r_j^{(i_2)} \right\} \right| = 2,$$

counting appropriately multiplicity when some of the $r_j^{(i)}$'s are repeated;
3. for $i_1 \neq i_2$, we must have

$$\left(\Psi^{(i_1)} \cdot \Psi^{(i_2)} \right)^j \equiv \textit{identity}$$

if and only if j is a multiple of m.

For $m = 2$, it is elementary to check that

$$\mathcal{A}_2 = \left\{ \left(r_{j_1}^{(1)}, r_{j_2}^{(2)}, r_{j_3}^{(3)} \right) : 1 \leq j_1, j_2, j_3 \leq 2 \right\},$$

the eight points of a prism or cube in $\mathbb{R}^3$ centered at the origin (Figure 3). All points have the same weight $1/8$.

Figure 3. The case $m = 2$.

The case $m = 3$ is much more interesting. Except for permutations of indeces, the only non-trivial possibility is the one in Figure 4.

Figure 4. The case $m = 3$.

In all of our figures, thick lines (either continuous or dashed) indicate rank-one directions joining two black dots. These, in turn, represent matrices participating in the symmetric gradient. Other lines have been drawn to facilitate the understanding of the figure.

As remarked before, it is also important to realize what can be obtained by taking admissible sets when the numbers $r_j^{(i)}$ are different, but some of them start to approach each other until they become the same. One of these situations was somehow the main idea examined in [11]. For instance, if we take $r_2^{(2)} = r_3^{(2)}$, then we obtain the gradient in Figure 5.

Figure 5. A limit case for $m = 3$. Points with a 2 indicate weight $2/18$.

In trying to look for a counterexample of a gradient that is not a laminate, it is important to look for sets with as few points as possible. If for $m = 3$ we take

$$r_1^{(1)} = r_2^{(1)}, \quad r_1^{(2)} = r_2^{(2)}, \quad r_2^{(3)} = r_3^{(3)},$$

we have the set with suitable weights drawn on Figure 6.

Figure 6. A limit case for $m = 3$. Weights are indicated.

The question is whether such gradient is a laminate or not. Notice that this time the first moment is the point $(1/3, 1/3, 2/3)$ if we place the origin at the most inner point of the cube. As one can see, many different situations are possible. Another interesting example correspond to the choice (Figure 7)

$$r_1^{(1)} = r_2^{(1)}, \quad r_1^{(2)} = r_2^{(2)}, \quad r_1^{(3)} = r_2^{(3)}.$$

Figure 7. A limit case for $m = 3$. Weights are indicated.

The first moment is $(1/3, 1/3, 1/3)$ if the inner most point of the cube is taken as the origin.

For $m = 4$ the situation is much more complicated. One example is shown in Figure 2 in the Introduction. This case ($m = 4$) is the first one where condition 3 in Corollary 1 can be explained and appreciated. If we drop such condition, the set in Figure 8 would be elligible for a symmetric gradient. However this is not true. This will be shown in Section 3. Note the difference with Figure 2.

Figure 8. A non-admissible set.

A section of this cube through an intermediate plane would be the one in Figure 9.

Figure 9. An intermediate section of Figure 8.

This violates condition 3 in Corollary 1 as we would have

$$\left(\Psi^{(i_1)} \cdot \Psi^{(i_2)}\right)^2 \equiv \text{identity},$$

for appropriate choices of i_1 and i_2. On the other hand the same section for Figure 2 is shown in Figure 10. Observe the subtle difference between the two.

Figure 10. An intermediate section of Figure 2.

If in that example we put

$$r_1^{(i)} = r_2^{(i)}, \quad r_3^{(i)} = r_4^{(i)},$$

for all i, we obtain the example examined in [11] (Figure 11).

Figure 11. A limit case for $m = 4$.

Would it be possible to prove that all sets verifying the hypotheses of Corollary 1 correspond to laminates? It is hard to say.

3. Proofs

We divide the discussion in this section in several steps.

1. Description of a 2×2 discrete, T-periodic gradient. As pointed out in the Introduction, we will consider the unit square

$$T = (0, 1) \times (0, 1)$$

divided in $2m^2$ subtriangles

$$T_{i,j}^{(1)} \cup T_{i,j}^{(2)} = (i - 1/m, j - 1/m) \times (i/m, j/m),$$

obtained by dividing this small square in two triangles along the diagonal with normal $(1, 1)$.

Our gradients will be piecewise-affine, T-periodic, taking on constant values on each of the triangles

$$T_{i,j}^{(k)}, \quad 1 \le i, j \le m, k = 1, 2.$$

Let these matrices be denoted $F_{i,j}^{(k)}$.

We shall prove the following lemma, which is interesting in itself.

Lemma 1 *A set of matrices*

$$F_{i,j}^{(k)}, \quad 1 \le i,j \le m, k = 1,2,$$

are the values taken on by a piecewice-affine, T-periodic 2×2 gradient as described above, if and only if there exist vectors

$$d_{i,j}, \quad a_i, \quad b_j, \qquad 1 \le i,j \le m,$$

such that

$$\sum_{k=1}^{m} d_{k,j} = \sum_{l=1}^{m} d_{i,l} = 0,$$
$$\sum_{k=1}^{m} a_k = \sum_{l=1}^{m} b_l = 0,$$

for all i,j and

$$F_{i,j}^{(1)} = \begin{pmatrix} a_i + \sum_{l=1}^{j-1} d_{i,l} \\ b_j + \sum_{k=1}^{i-1} d_{k,j} \end{pmatrix},$$
$$F_{i,j}^{(2)} = \begin{pmatrix} a_i + \sum_{l=1}^{j} d_{i,l} \\ b_j + \sum_{k=1}^{i} d_{k,j} \end{pmatrix}.$$

PROOF. It is clear that

$$F_{i,j}^{(2)} - F_{i,j}^{(1)} = \begin{pmatrix} d_{i,j} \\ d_{i,j} \end{pmatrix}, \tag{2}$$

where d_{ij} are vectors such that, due to the periodicity restriction,

$$\sum_{k=1}^{m} d_{kj} = 0, \quad \sum_{l=1}^{m} d_{il} = 0,$$

for all i,j. In particular, if we imagine that the these vectors are given, then we can regard (2), as providing the passage from $F_{i,j}^{(1)}$ to $F_{i,j}^{(2)}$ by putting

$$F_{i,j}^{(2)} = F_{ij}^{(1)} + \begin{pmatrix} d_{i,j} \\ d_{i,j} \end{pmatrix}.$$

Similarly,

$$F_{i,j}^{(1)} - F_{i,j-1}^{(2)} = \begin{pmatrix} 0 \\ b_{i,j} \end{pmatrix}, \qquad F_{i,j}^{(1)} - F_{i-1,j}^{(2)} = \begin{pmatrix} a_{i,j} \\ 0 \end{pmatrix},$$

for certain vectors $a_{i,j}$, $b_{i,j}$.

These rules can be interpreted in a "row-wise" fashion by saying:

1. any row of $F_{i,j}^{(2)}$ is obtained from the same row of $F_{i,j}^{(1)}$ by adding $d_{i,j}$;
2. $F_{i,j-1}^{(2)}$ and $F_{i,j}^{(1)}$ share the same first row;
3. $F_{i-1,j}^{(2)}$ and $F_{i,j}^{(1)}$ share the same second row.

If we further put, for $i,j \ge 2$,

$$F_{i,1}^{(1)} - F_{i-1,1}^{(2)} = \begin{pmatrix} a_i \\ 0 \end{pmatrix}, \qquad F_{1,j}^{(1)} - F_{1,j-1}^{(2)} = \begin{pmatrix} 0 \\ b_j \end{pmatrix},$$

and

$$F_{1,1}^{(1)} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix},$$

we again have, by the periodicity, that

$$\sum_{k=1}^{m} a_k = \sum_{l=1}^{m} b_l = 0.$$

By recursion, bearing in mind the three rules above, it is not difficult to show the statement of the lemma. ■

Notice that this lemma implies that the probability measure

$$\frac{1}{2m^2} \sum_{i,j} \left(\delta_{F^{(1)}_{i,j}} + \delta_{F^{(2)}_{i,j}} \right)$$

where the participating matrices are constructed from any set of vectors

$$\{d_{i,j}, a_i, b_j : 1 \leq i, j \leq m\}$$

as in the statement, is a homogeneous gradient Young measure ([3]).

2. Imposing the symmetry constraint. We would like to impose the symmetry restriction on the form of the gradient in Lemma 1 by requiring that all matrices involved be symmetric. This amounts to asking for the symmetry of $F^{(2)}_{i,j}$ and the difference

$$F^{(2)}_{i,j} - F^{(1)}_{i,j},$$

for every i, j.

Since

$$F^{(2)}_{i,j} - F^{(1)}_{i,j} = \begin{pmatrix} d_{i,j} \\ d_{i,j} \end{pmatrix},$$

the symmetry of this matrix can only happen if

$$d_{i,j} = c_{i,j}(1,1)$$

where now $c_{i,j}$ are scalars with

$$\sum_{k=1}^{m} c_{k,j} = \sum_{k=1}^{m} c_{i,k} = 0,$$

for all i, j. If we take this conclusion into the symmetry of $F^{(2)}_{i,j}$, we must enforce

$$a^{(2)}_i + \sum_{k \leq j} c_{i,k} = b^{(1)}_j + \sum_{k \leq i} c_{k,j},$$

for all i, j. Upper indeces for vectors denote components. This condition implies that the expressions

$$b^{(1)}_j + \sum_{k \leq i} c_{k,j} - \sum_{k \leq j} c_{i,k}$$

should be independent of j for i fixed. By equating two such sums

$$b^{(1)}_j - b^{(1)}_l = \sum_{k \leq i} c_{k,l} - \sum_{k \leq l} c_{i,k} - \sum_{k \leq i} c_{k,j} + \sum_{k \leq j} c_{i,k},$$

for all i, j, l. In particular, this last number ought to be independent of i. Hence by taking $j = l + 1$, we must have

$$\sum_{k \leq i} (c_{k,l} - c_{k,l+1}) + c_{i,l+1} = \sum_{k \leq i+1} (c_{k,l} - c_{k,l+1}) + c_{i+1,l+1}.$$

Simplifying, we get

$$c_{i,l+1} = c_{i+1,l},$$

for all i, l. It is then elementary to check that the numbers $c_{i,j}$ are determined by m numbers C_k such that

$$\sum_k C_k = 0,$$

and

$$c_{i,j} = C_{i+j-1}, \quad C_{m+k} = C_k.$$

Since the symmetry do not pose any constraint on the first component of a_i and the second component of b_j, these can be arbitrarily selected as long as their sum vanishes. If we take all this information back on the matrices of our gradient we have the following lemma. Computations are elementary. The converse is also an easy, direct compuation.

Lemma 2 *A set of matrices*

$$F_{i,j}^{(k)}, \quad 1 \le i, j \le m, k = 1, 2,$$

are the values taken on by a piecewice-affine, T-periodic, symmetric 2×2 gradient, if and only if there exist $3m$ numbers

$$A_i, \quad B_i, \quad C_i, \qquad 1 \le i \le m,$$

such that

$$\sum_i A_i = \sum_i B_i = \sum_i C_i = 0$$

and

$$F_{i,j}^{(1)} = \begin{pmatrix} A_i + \sum_{k \le j-1} C_{i+k-1} & -\sum_{k \ge i+j-1} C_k \\ -\sum_{k \ge i+j-1} C_k & B_j + \sum_{k \le i-1} C_{k+j-1} \end{pmatrix},$$
$$F_{i,j}^{(2)} = \begin{pmatrix} A_i + \sum_{k \le j} C_{i+k-1} & -\sum_{k \ge i+j} C_k \\ -\sum_{k \ge i+j} C_k & B_j + \sum_{k \le i} C_{k+j-1} \end{pmatrix}. \quad \square$$

3. An alternative description in $\mathbb{R}^3$. Under our identification (1), the symmetric matrices of Lemma 2 become the vectors

$$F_{i,j}^{(1)} \mapsto \left(A_i + \sum_{k \ge i} C_k, B_j + \sum_{k \ge j} C_k, -\sum_{k \ge i+j-1} C_k \right),$$
$$F_{i,j}^{(2)} \mapsto \left(A_i + \sum_{k \ge i} C_k, B_j + \sum_{k \ge j} C_k, -\sum_{k \ge i+j} C_k \right).$$

This involves easy computations.

If we now put

$$r_j^{(1)} = A_j + \sum_{k \ge j} C_k,$$
$$r_j^{(2)} = B_j + \sum_{k \ge j} C_k,$$
$$r_j^{(3)} = -\sum_{k \ge j} C_k,$$

we can solve for A_j, B_j, C_j in terms of $r_j^{(i)}$ as follows

$$C_j = r_{j+1}^{(3)} - r_j^{(3)}, \quad 1 \le j \le m-1,$$
$$r_1^{(3)} = 0, \quad C_m = -r_m^{(3)},$$

and

$$A_j = r_j^{(1)} + r_j^{(3)},$$
$$B_j = r_j^{(2)} + r_j^{(3)},$$

for all j. Notice that the constraints

$$\sum_j A_j = \sum_j B_j = \sum_j C_j = 0$$

translates into

$$r^{(3)} = 0, \quad \sum_j r_j^{(1)} = \sum_j r_j^{(2)} = -\sum_j r_j^{(3)}.$$

4. PROOF OF THEOREM 1. Let $\mathcal{A}_m$ be the set of points corresponding to the matrices $F_{i,j}^{(k)}$ of an admissible symmetric gradient. By all of our previous discussion

$$\mathcal{A}_m = \left\{ \left(r_i^{(1)}, r_j^{(2)}, r_{i+j-1}^{(3)} \right), \left(r_i^{(1)}, r_j^{(2)}, r_{i+j}^{(3)} \right) : 1 \le i, j \le m \right\}, \tag{3}$$

where the $r_j^{(i)}$ are given by the formulas in Step 4. Keep in mind that

$$r_{m+j}^{(3)} = r_j^{(3)}. \quad \blacksquare$$

5. PROOF OF THEOREM 2. The necessity is clear from Theorem 1. Notice that condition 3 in the statement reflects the fact that in following each vertical or horizontal strip in the discretization of Figure 1, we cannot have two disjoint sets of matrices: all must be connected through the axes among themselves. See Figure 12 and relate it to Figures 9 and 10.

Figure 12. All matrices must be connected.

For the sufficiency we could try to express a set under the conditions of Theorem 1 in the form given in (3), for suitable permutations of indeces. However, we believe it is more transparent and intuitive to directly build the gradient associted with such set of points. Indeed, assuming that all points $r_j^{(i)}$ are distinct among themselves, hypothesis 2 in the statement of Theorem 1 implies that any point in $\mathcal{A}_m$ is "surrounded" by another three points along the three coordinate axes. In terms of symmetric gradients, this means that the matrices corresponding to such four points can be placed in one of the diagrams as in Figure 13.

Globally, the result is a T-periodic, symmetric gradient with the appropriate first moment. Keep in mind the comments about translations at the beginning of Section 2.

A situation where some of the numbers $r_j^{(i)}$ are repeated is a limit of the above. $\blacksquare$

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Figure 13. Four related matrices in an admissible set.

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