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A singular equation with positive and free boundary solutions

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Abstract. Let $0 < \beta < 1$. The equation $-\Delta u = \chi_{\{u>0\}}\big(-u^{-\beta} + \lambda f(x,u)\big)$ in Ω with Dirichlet boundary condition on $\partial\Omega$ has a maximal solution $u_\lambda \geq 0$ for every $\lambda > 0$. For λ less than a constant λ^* the solution vanishes inside the domain and for $\lambda > \lambda^*$ the solution is positive and stable. We obtain optimal regularity of u_λ even in the presence of a free boundary. If $\lambda \geq \lambda^*$ the solutions of the singular parabolic equation $u_t - \Delta u + u^{-\beta} = \lambda f(u)$ are positive and globally defined while for $0 < \lambda < \lambda^*$ there is no positive global solution.

Soluciones positivas y con frontera libre de una ecuación singular

Resumen. Para $0 < \beta < 1$ consideramos la ecuación $-\Delta u = \chi_{\{u>0\}}\big(-u^{-\beta} + \lambda f(x,u)\big)$ en Ω con condición de borde tipo Dirichlet. Esta ecuación posee una solución maximal $u_\lambda \geq 0$ para todo $\lambda > 0$. Si λ es menor que una cierta constante λ^*, u_λ se anula en el interior del dominio creando una frontera libre, y para $\lambda > \lambda^*$ esta solución es positiva en Ω y estable. Establecemos la regularidad de u_λ incluso en presencia de una frontera libre. Para $\lambda \geq \lambda^*$ la solución del problema parabólico singular $u_t - \Delta u + u^{-\beta} = \lambda f(u)$ es global y positiva mientras que si $0 < \lambda < \lambda^*$ no existe solución global positiva.

1. Introduction

We study the elliptic problem

$$\begin{cases} -\Delta u = g_\lambda(x,u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

on a smooth, bounded domain $\Omega \subset \mathbb{R}^n$ with a singular nonlinearity g_λ given by

$$g_\lambda(x,u) = \chi_{\{u>0\}}\big(-u^{-\beta} + \lambda f(x,u)\big), \tag{2}$$

where $\chi_{\{u>0\}}$ is the characteristic function of the set $\{u > 0\}$ and by convention $g_\lambda(x,0) = 0$. Henceforth $0 < \beta < 1$, $\alpha = 2/(1+\beta)$, $\lambda > 0$ is a parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in x, $f \geq 0$, $f \not\equiv 0$, and is nondecreasing, concave and sublinear in the second variable u uniformly in x, that is, $\lim_{u\to\infty} f(x,u)/u = 0$ uniformly for $x \in \Omega$. We also assume that $f_u(x,\cdot)$ is continuous on $(0,\infty)$ for a.e. $x \in \Omega$.

Equation (1) arises as limit of some equations modelling catalytic and enzymatic reactions, see [1] and [7] for an account.

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Definition 1 Let $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. We say that $u \in H_0^1(\Omega)$, $u \geq 0$, is a solution of (1) if $g_\lambda(x,u)\delta \in L^1(\Omega)$, and

$$\int_\Omega \nabla u \nabla \varphi = \int_{\{u>0\}} \big(- u^{-\beta} + \lambda f(x,u)\big)\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

By a positive classical solution we mean a function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ which is positive in Ω and satisfies (1) in the usual sense.

This note is only intended as a summary of our results, while complete proofs will appear elsewhere [5]. We have also included some remarks and examples that are not in [5].

2. Existence of a maximal solution and its regularity

Theorem 1 *For all $\lambda > 0$ there is a unique maximal solution u_λ to (1). Moreover there exists $\lambda^* \in (0,\infty)$ such that for $\lambda > \lambda^*$ the maximal solution u_λ is positive in Ω and belongs to $C(\overline{\Omega}) \cap C^{1,\mu}_{\mathrm{loc}}(\Omega)$ for all $0 < \mu < 1$. We also deduce that $a\delta \leq u_\lambda \leq b\delta$ in Ω where a, b are positive constants depending only on Ω, $\lambda > 0$ and f. If $f \in C^1(\overline{\Omega} \times [0,\infty))$ then actually u_λ is a classical solution.*

For $0 < \lambda \leq \lambda^*$ the maximal solution u_λ has optimal regularity $C(\overline{\Omega}) \cap C^{1,\frac{1-\beta}{1+\beta}}_{\mathrm{loc}}(\Omega)$. If $0 < \lambda < \lambda^*$ then the set $\{u_\lambda = 0\}$ has positive measure. On the other hand, u_{λ^*} is positive a.e. and is unique in the class of solutions which are positive a.e.. $\square$

Some problems similar to (1) were already considered in the literature [3, 4, 6, 8, 12]. Optimal interior $C^{1,\frac{1-\beta}{1+\beta}}$ estimates were established in [11] for local minimizers of the energy functional $\int \frac{1}{2}|\nabla u|^2 + (u^+)^{1-\beta}$ in the convex set $\{u \in H^1(\Omega) \,:\, u = 1 \text{ on } \partial\Omega\}$.

We obtain the maximal solution u_λ as the (decreasing) limit as $\varepsilon \to 0$ of the maximal solutions $u_{\lambda,\varepsilon}$ to

$$\begin{cases} -\Delta u + \dfrac{u}{(u+\varepsilon)^{1+\beta}} = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

This approach is inspired by [7]. First we show that $u_{\lambda,\varepsilon}$ converges pointwisely to the maximal subsolution u of the following problem

$$\begin{cases} -\Delta u + \chi_{\{u>0\}} u^{-\beta} = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The techniques of [11] can be adapted to obtain precise estimates of the derivatives of the maximal subsolution u. This enables us to verify that the function u satisfies (1) and we deduce that $u_\lambda = u$. A byproduct of these estimates is the uniform convergence $u_{\lambda,\varepsilon} \to u_\lambda$ in $\overline{\Omega}$ as $\varepsilon \to 0$ (and not only a.e.).

3. Stability

The question of stability of the maximal solution u_λ for $\lambda \geq \lambda^*$ leads us to define, for a function $u \in L^1_{\mathrm{loc}}(\Omega)$, $u > 0$ a.e. in Ω, the expression

$$\Lambda(u) = \inf_{\|\varphi\|_{L^2} \leq 1} \int_\Omega |\nabla\varphi|^2 - \big(\beta u^{-\beta-1} + \lambda f_u(x,u)\big)\varphi^2 \tag{4}$$

(for a general $u > 0$ a.e. $\Lambda(u)$ makes sense, but can be $-\infty$). This is the first eigenvalue of the linearization of problem (1). The stability property allows us to obtain the positivity for $u^* := u_{\lambda^*}$ under some restrictions on β. In [10] the authors deal with stability questions for a singular equation.

Theorem 2 *For $\lambda > \lambda^*$ the maximal solution u_λ of (1) is stable, that is, $\Lambda(u_\lambda) > 0$. For $\lambda = \lambda^*$ the solution u^* is weakly stable, in the sense that $\Lambda(u^*) \geq 0$. Conversely, if u is a solution of (1) for some $\lambda \geq \lambda^*$ such that u is positive a.e. and $\Lambda(u) \geq 0$, then u coincides with the maximal solution (i.e. $u = u_\lambda$).* □

Theorem 3 *If $(3\beta + 1 + 2\sqrt{\beta^2 + \beta})/(\beta + 1) > n/2$, then there exists $c > 0$ depending only on Ω, n and β such that $u^* \geq c\delta^\alpha$. In particular u^* is positive in Ω (and not only a.e.).* □

4. Remarks and examples

In this section we discuss the optimality of our results, and we give examples illustrating various situations.

(A) There are examples where u_λ is not identically zero for some $0 < \lambda < \lambda^*$.

Let $f = \chi_{(-A,A)}$ for some suitable $A > 0$ to be chosen later. First fix $\eta > 1$ such that $\eta - \frac{\eta^{1-\beta}}{1-\beta} > 0$ and consider the ODE

$$-u'' = -u^{-\beta} + 1 \tag{5}$$

with initial conditions $u(0) = \eta$, $u'(0) = 0$. Standard results of ODE theory imply that u is defined on a maximal open interval, say $(-x_0, x_0)$ (at the end of this example we present a more explicit expression of u in the case $\beta = 1/2$). The solution u is symmetric with respect to 0 and is decreasing in the nonempty interval $x \in (0, x_0)$. Moreover, $\lim_{x \nearrow x_0} u(x) = 0$. Therefore there exists some $A > 0$ (unique) such that

$$\eta - \frac{\eta^{1-\beta}}{1-\beta} - u(A) = 0. \tag{6}$$

Note that the expression $\frac{1}{2}(u')^2 - \frac{u^{1-\beta}}{1-\beta} + u$ is a constant in the interval $(0, A)$, and therefore condition (6) is equivalent to $\frac{1}{2}u'(A)^2 = \frac{u(A)^{1-\beta}}{1-\beta}$. Define $c = \big(\alpha(\alpha-1)\big)^{-1/(1+\beta)}$ and $B = (\frac{1}{c}u(A))^{1/\alpha} + A$. Extend $u(x)$ by the formula

$$u(x) = \begin{cases} \text{solution of (5)}, & x \in [-A, A] \\ c(B - |x|)^\alpha, & x \in (-B, -A) \cup (A, B) \\ 0, & x \notin (-B, B). \end{cases}$$

If $R > B$, then u is a nontrivial solution to (1) corresponding to $\lambda = 1$. We will see that the maximal solution $\bar{u}$ has compact support in $\Omega = (-R, R)$ if R is large enough. To accomplish this, set

$$\overline{B} = \sup\{\, t \in (A, R) \mid \bar{u} > 0 \text{ on } (0, t) \,\} > 0$$

and let us show that

$$\overline{B} \leq \left[\frac{1}{c}\Big(\frac{1-\beta}{2}A^2\Big)^{1/(1-\beta)}\right]^{1/\alpha} + A. \tag{7}$$

Indeed, integrate (5) over $(-A, 0)$ to get $\bar{u}'(0) - \bar{u}'(-A) = \int_0^A \bar{u}'' \geq -A$. Since $\bar{u}'(0) = 0$ we get the estimate

$$\bar{u}'(-A) \leq A. \tag{8}$$

Observe that on $(-\overline{B}, -A)$, $\bar{u}$ satisfies $\bar{u}'' = \bar{u}^{-\beta}$. Multiplying this equation by $\bar{u}'$ and integrating we find $\frac{1}{2}(\bar{u}')^2 - \frac{\bar{u}^{1-\beta}}{1-\beta} = D$ on $(-\overline{B}, -A)$, where D is a constant. Since $\bar{u}(-\bar{B}) = 0$ we must have $D \geq 0$ and this implies that

$$\frac{\bar{u}^{1-\beta}}{1-\beta} \leq \frac{1}{2}(\bar{u}')^2 \quad \text{on } (-\overline{B}, -A). \tag{9}$$

It is not difficult then to check that

$$\bar{u}(x) \geq c(x + \overline{B})^\alpha, \quad \forall x \in (-\overline{B}, -A). \tag{10}$$

In particular combining (10) at $x = -A$, (9) and (8) we get

$$c(\overline{B} - A)^\alpha \le \bar{u}(-A) \le \Big(\frac{1-\beta}{2}\bar{u}'(-A)^2\Big)^{1/(1-\beta)} \le \Big(\frac{1-\beta}{2}A^2\Big)^{1/(1-\beta)}$$

from which (7) follows.

When $\beta = 1/2$ we have a more explicit expression of the solution of problem (5). Multiplying the equation by u' and integrating one finds $u' = (4u^{1/2} - 2u + c)^{1/2}$, where $c > 0$ is a constant depending only on β and η. Set $h(s) = (4s^{1/2} - 2s + c)^{-1/2}$ and integrate it from 0 to ξ. One obtains

$$H(\xi) = \sqrt{c} - \sqrt{2}\arcsin\Big(\frac{\sqrt{2}}{\sqrt{2+c}}\Big) - \sqrt{c + 4\sqrt{\xi} - 2\xi} - \sqrt{2}\arcsin\Big(\frac{\sqrt{2}(1-\sqrt{\xi})}{\sqrt{2+c}}\Big).$$

Our equation transforms into $(H(u(x)))' = -1$ for $x > 0$. Integrating and applying the inverse function H^{-1} we obtain $u(x) = H^{-1}(H(\eta) - x)$, $0 < x < H(\eta)$. We remark that when $\beta = 1/2$, it is proved in [3] that there is a unique correspondence between $\eta \ge 4$ and x_0, where $u(x_0) = 0$ and u solves (5). Thus $u(x) = H^{-1}(H(\eta) - x)$ is the maximal solution in $(-x_0, x_0)$ with $f \equiv 1$ and is stable.

(B) Is the branch $\lambda \mapsto u_\lambda$ continuous?

The answer is negative in general, and examples can be readily constructed. For instance, let Ω be an interval in $\mathbb{R}$ and assume that f depends only on u. Then for any $\lambda > 0$ the maximal solution is either identically zero or positive in Ω (see [5]). We stress that the stability characterization of Theorem 2 implies that $\lambda \mapsto u_\lambda$ is continuous on $[\lambda^*, \infty)$.

(C) Can one characterize the maximal solution when $\lambda < \lambda^*$ in a way similar to Theorem 2?

One possible approach would be to say that a solution $u \in C(\Omega)$ to (1) is weakly stable if

$$\int_\omega \frac{\partial g_\lambda}{\partial u}(x,u)\varphi^2 \le \int_\omega |\nabla\varphi|^2 \quad \forall \varphi \in C_0^\infty(\omega), \tag{11}$$

where $\omega = \{x \in \Omega \mid u(x) > 0\}$. Assume now that $u \in C(\Omega)$ is a weakly stable solution of (1) in the sense of relation (11). Is it true that it has to be the maximal solution? It turns out that the answer is negative in general, that is, there are examples of solutions satisfying (11) which are not the maximal one, see details in [5].

(D) The condition on β in Theorem 3 is almost optimal.

Assume that $(3\beta + 1 + 2\sqrt{\beta^2 + \beta})/(\beta + 1) < n/2$ and let B be the unit ball of $\mathbb{R}^n$. We will see that there exists $f = f(x) \in C^\infty(B) \cap L^\infty(B)$ with $f \ge 0$ such that the solution $u^* = u_{\lambda^*}$ satisfies $u^* > 0$ in $B \setminus \{0\}$ and $u^*(x) = const\,|x|^\alpha$ for x near the origin. Indeed, let $v(x) = c|x|^\alpha$ where the constant $c > 0$ is chosen so that $\alpha(\alpha + n - 2) = c^{-1-\beta}$. Then it is easy to verify that v satisfies $\Delta v = v^{-\beta}$ in $\mathbb{R}^n$. Let $0 < R < 1$ to be fixed later and let $h(r)$ be a smooth function defined for $r \in [0,1]$ such that $0 \le h \le c$, $h', h'' \ge 0$ on $[0,1]$, $h \equiv 0$ on $[0,R]$ and $h(1) = c$. Then $\Delta h = h'' + \frac{n-1}{r}h' \ge 0$ in $\mathbb{R}^n$. Set $u(x) = v(x) - h(|x|)$. We find $-\Delta u = -v^{-\beta} + \Delta h = -u^{-\beta} + f(x)$ where $f(x) = u^{-\beta} - v^{-\beta} + \Delta h \ge 0$. We claim that u is weakly stable if R and h are chosen appropriately. Let $\varphi \in C_0^\infty(B)$ and let us consider

$$\beta\int_B u^{-1-\beta}\varphi^2 = \beta\alpha(\alpha + n - 2)\int_B r^{-2}\varphi^2 + \beta\int_B \Big((cr^\alpha - h)^{-1-\beta} - (cr^\alpha)^{-1-\beta}\Big)\varphi^2 = I_1 + I_2. \tag{12}$$

A computation shows that $\beta\alpha(\alpha + n - 2) < \frac{(n-2)^2}{4}$. Thus by Hardy's inequality with the weight r^{-2},

$$I_1 \le (1-\varepsilon)\int_B |\nabla\varphi|^2 \tag{13}$$

for some $\varepsilon > 0$ depending only on β and n. To estimate I_2 observe that we can choose h in such a way that $cr^\alpha - h \ge \frac{1}{C}\delta$, where $\delta(x) = \mathrm{dist}(x, \partial B) = 1 - |x|$, and the constant C is independent of R. In this way

$(cr^\alpha - h)^{-1-\beta} \le C\delta^{-1-\beta}$ and therefore, using Hardy's inequality with weight δ^{-2} we get

$$\begin{aligned} I_2 &\le C \int_{B\setminus B_R} \delta^{-1-\beta}\varphi^2 \le C\Big(\int_B \delta^{-2}\varphi^2\Big)^{(1+\beta)/2}\Big(\int_{B\setminus B_R}\varphi^2\Big)^{(1-\beta)/2} \\ &\le \frac{\varepsilon}{2}\int_B |\nabla\varphi|^2 + C(\varepsilon)\int_{B\setminus B_R}\varphi^2. \end{aligned}$$

But

$$\int_{B\setminus B_R}\varphi^2 \le C(1-R)^2\int_B |\nabla\varphi|^2 \quad \forall \varphi \in C_0^\infty(B), \tag{14}$$

where C is independent of φ and R. Hence, by choosing $R < 1$ with $1 - R$ small enough and combining (12), (13) and (14) we conclude that

$$\beta\int_B u^{-1-\beta}\varphi^2 \le \int_B |\nabla\varphi|^2 \quad \forall\varphi\in C_0^\infty(B).$$

By Theorem 2 u is the maximal solution u^* on B with data f. Moreover $\lambda^* = 1$ in this situation.

Let us mention that the conclusion $u^* \ge c\delta^\alpha$ in Theorem 3 is optimal, in the sense that a one can not hope to have a similar inequality with a smaller exponent, see [5].

5. The parabolic problem

We are also interested in studying the singular parabolic problem

$$\begin{cases} u_t - \Delta u = g_\lambda(u) & \text{in } \Omega\times(0,\infty), \\ u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{15}$$

where for simplicity we consider the function f depending only on u. The papers [10, 9] treat a singular parabolic equation with the opposite sign in front of the singular term $u^{-\beta}$.

The quantity λ^* given in Theorem 1 is a critical parameter for the elliptic problem (1), but it is also a borderline for the existence of global positive solutions of (15) with a suitable fixed initial data u_0. This kind of interplay between stationary and evolution problem was undertaken in [2] for $g_\lambda(x,u) = \lambda f(u)$ with f positive, increasing and convex.

Let us fix θ such that $1 < \theta < \alpha$. We begin establishing the existence of a solution locally in time and its uniqueness.

Theorem 4 *The parabolic problem (15) has a local solution u defined in an interval $(0,T)$, provided that the initial data u_0 is bounded and $u_0 \ge c\delta^\theta$ for some $c > 0$. Moreover, u belongs to $L^\infty(\Omega\times(0,T))\cap C^1(\Omega\times(0,T))$ and satisfies $u \ge c'\delta^\theta$ in (0,T) for some $c' > 0$ (T and c' depend on c and θ).* $\square$

Theorem 5 *If $u_0 \in L^\infty(\Omega)$ and $u_0 \ge c\delta^\theta$ for some $c > 0$. Then the local solution u is unique in the set*

$$\mathcal{M}_T = \big\{u\in L^\infty(\Omega\times(0,T)) \,:\, \forall S\in(0,T) \text{ there exists } c>0 \text{ such that } u(t)\ge c\delta^\theta \text{ for } t\in(0,S)\big\}. \quad \square$$

Let $T(t)$ be the heat semigroup in Ω with zero Dirichlet boundary condition. A function $u \in \mathcal{M}_T$ is regarded as a solution to (15) if

$$u(t) = T(t)u_0 + \int_0^t T(t-s)g_\lambda(u(s))\,ds \qquad \forall t\in(0,T). \tag{16}$$

If $u \in \mathcal{M}_T$ then $u^{-\beta} \in L^\infty((0,T'),L^p(\Omega))$ for some $p > 1$ and all $0 < T' < T$. Hence (16) makes sense in $L^p(\Omega)$. The above result is a consequence of a comparison principle and a smoothing effect for the heat semigroup $T(t)$ with weights involving powers of δ.

We close this section stating our global existence results.

Theorem 6 *If the elliptic problem (1) has a solution w which is positive a.e., then for any initial data $u_0 \in L^\infty(\Omega)$ satisfying $u_0 \geq w$ and $u_0 \geq c\delta^\theta$ for some $c > 0$, the solution of the parabolic problem (15) is global and positive, in the sense that* $\sup \left\{ T > 0 \,|\, \exists c > 0 \; u(t) \geq c\delta^\theta \; \forall t \in (0,T) \right\} = \infty$. □

Theorem 7 *Assume that $0 < \beta < 1$ and that the parabolic problem (15) has a positive global classical solution. Then the elliptic problem (1) has a solution which is positive a.e. Otherwise, for $\beta \geq 1$ there is no positive global classical solution of (15).* □

Remark 1 Suppose that $\lambda < \lambda^*$. Then, since the maximal solution of (1) vanishes on a set of positive measure, by the above theorem problem (15) cannot have a positive global classical solution. ■

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