RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **97** (1), 2003, pp. 69–72
Geometría y Topología / Geometry and Topology
Comunicación Preliminar / Preliminary Communication

Topological types of symmetries of elliptic-hyperelliptic Riemann surfaces and an application to moduli spaces

José A. Bujalance, Antonio F. Costa and Ana M. Porto

Abstract. Let X be a Riemann surface of genus g. The surface X is called elliptic-hyperelliptic if it admits a conformal involution h such that the orbit space $X/\langle h\rangle$ has genus one. The involution h is then called an elliptic-hyperelliptic involution. If $g > 5$ then the involution h is unique, see [1]. We call symmetry to any anticonformal involution of X. Let $Aut^{\pm}(X)$ be the group of conformal and anticonformal automorphisms of X and let σ, τ be two symmetries of X with fixed points and such that $\{\sigma, h\sigma\}$ and $\{\tau, h\tau\}$ are not conjugate in $Aut^{\pm}(X)$. We describe all the possible topological conjugacy classes of $\{\sigma, \sigma h, \tau, \tau h\}$. As consequence of our study we obtain that, in the moduli space of complex algebraic curves of genus g (g even > 5), the subspace whose elements are the elliptic-hyperelliptic real algebraic curves is not connected. This fact contrasts with the result in [12]: the subspace whose elements are the hyperelliptic real algebraic curves is connected.

Tipos topológicos de simetrías de superficies de Riemann elípticas-hiperelípticas y una aplicación a los espacios de moduli

Resumen. Sea X una superficie de Riemann de género g. Diremos que la superficie X es elíptica-hiperelíptica si admite una involución conforme h de modo que $X/\langle h\rangle$ tenga género uno. La involución h se llama entonces involución elíptica–hiperelíptica. Si $g > 5$ entonces la involución h es única, ver [1]. Llamamos simetría a toda involución anticonforme de X. Sea $Aut^{\pm}(X)$ el grupo de automorfismos conformes y anticonformes de X y σ, τ dos simetrías de X con puntos fijos y tales que $\{\sigma, h\sigma\}$ y $\{\tau, h\tau\}$ no son conjugados en $Aut^{\pm}(X)$. Describimos las clases de conjugación topológicas de $\{\sigma, \sigma h, \tau, \tau h\}$. Como aplicación obtenemos que el subespacio del espacio de móduli de las curvas algebraicas complejas de género g (g par y mayor que 5) formado por las curvas algebraicas reales elípticas-hiperelípticas no es conexo. Este hecho contrasta con el resultado en [12]: el subespacio del espacio de móduli formado por las curvas algebraicas reales hiperelípticas es conexo.

A Riemann surface X is called *elliptic-hyperelliptic* if it admits a conformal involution h such that the orbit space $X/\langle h\rangle$ is an elliptic Riemann surface, i.e. a genus one Riemann surface. The involution h is then called an *elliptic-hyperelliptic involution*. The study of the elliptic-hyperelliptic Riemann surfaces started with Schottky ([10], [11]) and Roth ([9]).

A *symmetry* σ of X is an anticonformal involution. The topological type of σ is given by properties of the fixed point set $Fix(\sigma)$. The set $Fix(\sigma)$ consists of k disjoint Jordan curves, $0 \leq k \leq g+1$ ([8]). We shall say that the *species* of σ is $+k$ or $-k$ according $X - Fix(\sigma)$ is connected or not. There are several studies on symmetry types for families of Riemann surfaces: for genus two surfaces in reference [7]; for

Presentado por José María Montesinos.
Recibido: 30 de Octubre de 2002. *Aceptado*: 15 de Enero de 2003.
Palabras clave / Keywords: Riemann surface, anticonformal automorphism, algebraic curve, moduli space
Mathematics Subject Classifications: Primary 30F99, 14H37; Secondary 32G15, 14H15

$PSL(2,q)-$Hurwitz Riemann surfaces in [2]; for Accola-Maclachlan and Kulkarni surfaces in [3] and for hyperelliptic surfaces in [4].

Let X an elliptic-hyperelliptic Riemann surface of genus $g > 5$ and h the elliptic-hyperelliptic involution. If σ is a symmetry of X, in the article [5] the possible species for $\{\sigma, \sigma h\}$ are obtained. Now we complete the study of symmetries with fixed points of elliptic-hyperelliptic Riemann surfaces.

Let $Aut^{\pm}(X)$ be the group of conformal and anticonformal automorphisms of X. The first result that we announce is:

Theorem 1 *An elliptic-hyperelliptic Riemann surface X of genus > 5 has at most eigth conjugacy classes of symmetries (in $Aut^{\pm}(X)$) with fixed points.* $\square$

The proof is a consequence of the fact that in an elliptic surface $X/\langle h\rangle$ there are at most four conjugacy classes of symmetries with fixed points.

Let g be an integer, $g > 5$. In order to have a clear statement for the next Theorem, we define the following functions:

1. $SEH_g^{(1)}(t_1, t_2, x_1, x_2, y)$ for t_1, t_2 positive integers and $x_1, x_2, y \in \{0,1\}$, $0 \le t_1 + t_2 \le 2g-2$ and t_1, t_2 even.

(i) $t_1 > 0, t_2 > 0:$

If $t_1 + t_2 < 2g-2$, $SEH_g^{(1)}(t_1, t_2, x_1, x_2, y) = (-\frac{(t_1+t_2)}{2}, -\frac{(t_1+t_2)}{2})$.

If $t_1 + t_2 = 2g-2$, $SEH_g^{(1)}(t_1, t_2, x_1, x_2, y) = \begin{cases} (-(g-1), -(g-1)) \text{ if } x_1 = 1 \\ (+(g-1), +(g-1)) \text{ if } x_1 = 0 \end{cases}$

(ii) $SEH_g^{(1)}(t_1, t_2, x_1, x_2, y) = SEH_g^{(1)}(t_2, t_1, x_2, x_1, y)$, $t_1 = 0, t_2 > 0:$

If $t_2 < 2g-2$, $SEH_g^{(1)}(0, t_2, x_1, x_2, y) = \begin{cases} (-(\frac{t_2}{2}+1), -\frac{t_2}{2}) \text{ if } x_1 = 1 \\ (-(\frac{t_2}{2}+2), -\frac{t_2}{2}) \text{ if } x_1 = 0 \end{cases}$

If $t_2 = 2g-2$, $SEH_g^{(1)}(0, 2g-2, x_1, x_2, y) = \begin{cases} (-g, -(g-1)) \text{ if } x_1 = 1 \\ (+(g+1), +(g-1)) \text{ if } x_1 = 0 \end{cases}$

(iii) $t_1 = t_2 = 0:$

If g is even $SEH_g^{(1)}(0, 0, x_1, x_2, y) = \begin{cases} (-1, -2) \text{ if } y = 1 \\ (0, +3) \text{ if } y = 0 \end{cases}$,

If g is odd $SEH_g^{(1)}(0, 0, x_1, x_2, y) = \begin{cases} (-1, -1) \text{ if } x_1 = 1,\ y = 1 \\ (-2, -2) \text{ if } x_1 = 0,\ y = 1 \\ (0, +2) \text{ if } x_1 = 1,\ y = 0 \\ (0, +4) \text{ if } x_1 = 0,\ y = 0 \end{cases}$.

2. $SEH_g^{(2)}(t)$ for t an even integer such that $0 \le t \le 2g-2$,

(i) $t > 0:$

If $t_1 < 2g-2$, $SEH_g^{(2)}(t) = (-\frac{t}{2}, -\frac{t}{2})$.

If $t = 2g-2$, $SEH_g^{(2)}(2g-2) = (-(g-1), +(g-1))$.

(ii) $t = 0:$

If g is even, $SEH_g^{(2)}(0) = (0, -1)$.

If g is odd, $SEH_g^{(2)}(0) = (0, -2)$.

Theorem 2 *Let X be an elliptic-hyperelliptic Riemann surface of genus $g > 5$, and let h be the elliptic-hyperelliptic involution. Assume that X has two symmetries σ, τ, with fixed points such that $\{\sigma, \sigma h\}$ and $\{\tau, \tau h\}$ are not conjugate in $Aut^{\pm}(X)$. Then the possible species for the symmetries $((\sigma, \sigma h), (\tau, \tau h))$ are in one of the following four cases.*

(I) Let T_i, $i = 1, 2$ be positive even integers, $x \in \{0,1\}$, $y^{(i)} \in \{0,1\}$, $y^{(i)} = 0$ if $T_i = 0$ and R be a positive integer such that: $2R + T_1 + T_2$ divides $g-1$. The following species are possible: $(SEH_g^{(1)}(t_1^{(1)}, t_2^{(1)}, x, x, y^{(1)}), SEH_g^{(1)}(t_1^{(2)}, t_2^{(2)}, x, x, y^{(2)}))$.

(II) Let T_i, $i = 1, ..., 4$ be positive integers, $i = 1, ..., 4$, $x_j^{(i)}, y^{(i)} \in \{0,1\}$, and $\varepsilon \in \{1,2\}$ such that:

(a) $g+1-2\varepsilon-\Sigma T_i \in 2\mathbb{Z}^+$,
(b) $T_{i+1}+T_{i+3}+x_1^{(k)}+x_2^{(k)}$ *is even,*
we define: $t_i^{(j)} = 2T_{2i+j-2}+2(\varepsilon-1)$, $i=1,2$, $j=1,2$. *The following species are possible:* $(SEH_g^{(1)}(t_1^{(1)},t_2^{(1)},x_1^{(1)},x_2^{(1)},y^{(1)}), SEH_g^{(1)}(t_1^{(2)},t_2^{(2)},x_1^{(2)},x_2^{(2)},y^{(2)}))$.
(III) Let T_i, $i=1,2$ *be positive integers and* $\varepsilon,\delta \in \{1,2\}$, *such that:*
(a) $g-1-\Sigma T_i-(\varepsilon-1) \in 2\mathbb{Z}^+$,
(b) $\delta \geq \varepsilon$.
The following species are possible:
$(SEH_g^{(2)}(2T_1+2(\delta-1)), SEH_g^{(2)}(2T_2+2(\delta-1)))$.
(IV) Let T_i, $i=1,2,3$ *be positive integers,* $x_1,x_2,y \in \{0,1\}$ *and* $\varepsilon \in \{1,2\}$ *such that:*
(a) $g+1-2\Sigma T_i-2\varepsilon \in 4\mathbb{Z}^+$,
(b) $x_1+x_2+T_1+T_3$ *is even,*
we define: $t_1=2T_1+2(\varepsilon-1)$; $t_2=2T_3+2(\varepsilon-1)$. *The following species are possible:*

$$(SEH_g^{(1)}(t_1,t_2,x_1,x_2,y), SEH_g^{(2)}(2T_2+2(\varepsilon-1))). \quad \square$$

SKETCH OF THE PROOF OF THEOREM 2. The orbifold $X/\langle h,\sigma,\tau\rangle$ admits two uniformizations, one as quotient of the complex disc: $X/\langle h,\sigma,\tau\rangle = \mathbb{D}/\Phi$, where Φ is a non-euclidean crystallographic group ([6]); and other one as quotient of the complex plane: $X/\langle h,\sigma,\tau\rangle = X/\langle h\rangle/\langle\sigma,\tau\rangle = \mathbb{C}/\Psi$ where Ψ is an euclidean plane crystallographic group. The possible signatures of Ψ and monodromies $\Psi \to \langle\sigma,\tau\rangle$ can be listed analyzing the actions of symmetries of tori. The signature of Φ can be deduced from the signature of Ψ and the monodromy $\Phi \to \langle h,\sigma,\tau\rangle$ can be deduced from $\Psi \to \langle\sigma,\tau\rangle$. Using a similar technique that in [5] the species of σ and τ can be obtained from $\Phi \to \langle h,\sigma,\tau\rangle$.

As an application is possible to obtain an interesting property of the moduli space of complex algebraic curves. Let $\mathcal{M}_{EH}^R$ be the set of points corresponding to elliptic-hyperelliptic real algebraic curves in the moduli space of complex algebraic curves of genus g, $g>5$, and let $\mathcal{M}_H^R$ be the set of points corresponding to hyperelliptic real algebraic curves. If g is even then the set $\mathcal{M}_{EH}^R$ is not connected. This result contrasts with the fact that $\mathcal{M}_H^R$ is connected (see [12]). The idea of the proof is the existence of symmetries with species appearing in the Theorems 3.3 and 3.4 (for $p=1$) of [5] but not appearing in the list of the Theorem 2.

References

[1] Accola R. D. M. (1984). On cyclic trigonal Riemann surfaces, I, *Trans. Amer. Math. Soc.*, **283**, 423–449.

[2] Broughton S. A., Bujalance E., Costa A. F., Gamboa J. M. and Gromadzki G. (1996). Symmetries of Riemann surfaces on which PSL(2,q) acts as Hurwitz automorphism group, *J. Pure Appl. Alg.*, **106**, 113–126.

[3] Broughton S. A., Bujalance E., Costa A. F., Gamboa J. M. and Gromadzki G. (1999). Symmetries of Accola-Maclachlan and Kulkarni surfaces, *Proc. Amer. Math. Soc.*, (3) **127**, 637–646.

[4] Bujalance E., Cirre J., Gamboa J. M. and Gromadzki G. (2001). Symmetry types of hyperelliptic Riemann surfaces, *Mém. Soc. Math. Fr. (N. S.)*, **86**, vi+122 pp.

[5] Bujalance E. and Costa A. F. (1997) On symmetries of p-hyperelliptic Riemann surfaces, *Math. Ann.*, **308**, 31–45.

[6] Bujalance E., Etayo J. J., Gamboa J. M. and Gromadzki G. (1990), *Automorphism Groups of Compact Bordered Klein Surfaces*, Springer-Verlag, Berlin-Heidelberg.

[7] Bujalance E. and Singerman D. (1985). The symmetry type of a Riemann surface, *Proc. London Math. Soc.*, **51**, 501–519.

[8] Harnack A. (1876). Über die Vieltheiligkeit der ebenen algebraischen Kurven, *Math. Ann.*, **10**, 189–199.

[9] Roth P. (1912). Über elliptisch-hyperelliptische Funktionen, *Monatsh. Math. Physik*, **22**, 106–160.

[10] Schottky F. (1890). Über die charakteristischen Gleichungen symmetrischer ebener Flächen. . . , *J. reine und angew. Math.*, **106**, 199–268.

[11] Schottky F. (1891). Theorie der elliptisch-hyperelliptischen Funktionen von vier Argumenten. I, *J. reine und angew. Math.*, **108**, 147–178 and II, ibid. **108**, 193–255.

[12] Seppälä M. (1990). Real algebraic curves in the moduli space of complex curves, *Comp. Math.*, **74**, 259–283.

J. A. Bujalance, A. F. Costa and A. M. Porto
Departamento de Matemáticas Fundamentales
Facultad de Ciencias
UNED
Senda del rey, 9
28040 Madrid, Spain
jbujalan@mat.uned.es, acosta@mat.uned.es, asilva@mat.uned.es