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Decomposable subspaces of Banach spaces

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Abstract. We introduce and study the notion of hereditarily A-indecomposable Banach space for A a space ideal. For a hereditarily A-indecomposable space X we show that the operators from X into a Banach space Y can be written as the union of two sets $\mathsf{A}\Phi_+(X,Y)$ and $\mathsf{A}(X,Y)$. For some ideals A defined in terms of incomparability, the first set is open, the second set correspond to a closed operator ideal and the union is disjoint.

Subespacios descomponibles de espacios de Banach

Resumen. Introducimos y estudiamos la noción de hereditibilidad A-indescomponible espacio de Banach para un espacio ideal A. Demostramos que para un espacio A-indescomponible X los operadores de X en un espacio de Banach Y pueden ser escritos como la unión de dos conjuntos $\mathsf{A}\Phi_+(X,Y)$ y $\mathsf{A}(X;Y)$. Para algunos ideales A definidos en términos de incomparabilidad, el primer conjunto es abierto, el segundo conjunto corresponde a un operador cerrado ideal y la unión es disjunta.

Let A be a space ideal in the sense of Pietsch [3]. For each Banach space X we consider

$$S_{\mathsf{A}}(X) := \{M \subset X : M \text{ is a subspace of } X \text{ and } M \notin \mathsf{A}\}.$$

Definition 1 *A Banach space X is said to be* A-indecomposable *if there are no subspaces M and N in S_{A} such that $X = M \oplus N$.*

The space X is said to be hereditarily A-indecomposable (HAI) *if every subspace M of X is* A-*indecomposable.*

Let us see some examples. Note that a nontrivial example should include a A-indecomposable space which is not in A.

Let A $=$ F, the finite dimensional spaces. The existence of infinite dimensional, hereditarily F-indecomposable spaces has been a long-standing open problem in Banach space theory. Finally, Gowers and Maurey gave an example in [2], that we denote X_{GM}.

Let A $=$ R be the reflexive spaces and let A $=$ WSC be the weakly sequentially complete spaces. James' space J is hereditarily R-indecomposable and hereditarily WSC-indecomposable space, but it is neither reflexive, nor weakly sequentially complete. The reason is that $\dim(J^{**}/J) = 1$.

From the previous examples we can derive new examples as follows.

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Let $Q : X^{**} \longrightarrow X^{**}/X$ denote the quotient map. Given a closed subspace M of X, we can identify M^{**}/M with $Q(M^{\perp\perp})$. Thus,

$$\mathsf{A}^{co} := \{X : X^{**}/X \in \mathsf{A}\}$$

is a space ideal.

Let A be one of the space ideals F, R or WSC. Let X be a Banach space such that X^{**}/X is isomorphic to X_{GM}, J or J, respectively. The space X is a hereditarily A^{co}-indecomposable space which is not in A^{co}.

Remark 1 The HI spaces contain no unconditional basic sequence [2]. Similarly, every unconditional basic sequence in a HRI space X generates a reflexive subspace. ■

Recall that the injection modulus of an operator $T \in L(X, Y)$ is defined by

$$j(T) := \inf\{\|Tx\| : x \in X, \|x\| = 1\}.$$

We will consider here the following two derived quantities

$$sj_{\mathsf{A}}(T) := \sup\{j(TJ_M) : M \in S_{\mathsf{A}}(X)\} \text{ and}$$
$$in_{\mathsf{A}}(T) := \inf\{\|TJ_M\| : M \in S_{\mathsf{A}}(X)\}.$$

Definition 2 *Suppose that $S_{\mathsf{A}}(X) \neq \emptyset$ and let Y be a Banach space. We define*

1. $\mathsf{A}SS(X, Y) := \{T \in L(X, Y) : sj_{\mathsf{A}}(T) = 0\}$.
2. $\mathsf{A}\Phi_+(X, Y) := \{T \in L(X, Y) : in_{\mathsf{A}}(T) > 0\}$.

For $S_{\mathsf{A}}(X)$ empty we define $\mathsf{A}SS(X, Y) = \mathsf{A}\Phi_+(X, Y) = L(X, Y)$.

In the case $\mathsf{A} = \mathsf{F}$, the finite dimensional spaces, the quantities in_{F} and sj_{F} were introduced in [5]. In this case $\mathsf{F}\Phi_+ = \Phi_+$, the upper semi-Fredholm operators, and $\mathsf{F}SS = SS$, the strictly singular operators.

Theorem 1 *For a Banach space X the following assertions are equivalent:*

1. *X is* HAI.
2. *For every space Y and every $T \in L(X, Y)$, $sj_{\mathsf{A}}(T) \leq in_{\mathsf{A}}(T)$.*
3. *For every space Y, $L(X, Y) = \mathsf{A}\Phi_+(X, Y) \cup \mathsf{A}SS(X, Y)$.* □

The proof is based on the following fact that X is HAI and if $M, N \in S_{\mathsf{A}}(X)$, then $\operatorname{dist}(S_M, S_N) = 0$, where S_M is the unit sphere in M.

We say that two Banach spaces X and Y are *totally incomparable* [4] if no infinite dimensional subspace of X is isomorphic to a subspace of Y. Given a class $\mathcal{C}$ of Banach spaces, the class of incomparability $\mathcal{C}_i$ was defined in [1] as follows:

$$\mathcal{C}_i := \{X : X \text{ is totally incomparable with every } Y \in \mathcal{C}\}.$$

The class $\mathcal{C}_i$ is a space ideal. Moreover it is not difficult to see that $X \in \mathcal{C}_{ii}$ if and only if X has no infinite dimensional subspace in $\mathcal{C}_i$, and that $\mathcal{C}_{iii} = \mathcal{C}_i$.

Theorem 2 *Let X and Y be Banach spaces. Suppose that $\mathsf{A} = \mathsf{A}_{ii}$ and $S_{\mathsf{A}}(X) \neq \emptyset$. Then*

$$\mathsf{A}\Phi_+(X, Y) \cap \mathsf{A}SS(X, Y) = \emptyset$$

If, additionally, X is a HAI *space, then the union $L(X, Y) = \mathsf{A}\Phi_+(X, Y) \cup \mathsf{A}SS(X, Y)$ is disjoint.* □

The proof of the previous result is based on the fact that, under the hypothesis of the statement, if $M \in S_{\mathsf{A}}(X)$, then $sj_{\mathsf{A}}(TJ_M) = sj_{\mathsf{F}}(TJ_M)$.

Remark 2 In the case $\mathsf{A} = \mathsf{A}_{ii}$, we get two additional facts:

1. The components $\mathsf{A}\Phi_+(X,Y)$ are open and the class $\mathsf{A}\Phi_+$ is closed under products: $T \in \mathsf{A}\Phi_+(X,Y)$ and $S \in \mathsf{A}\Phi_+(Y,Z)$ imply $ST \in \mathsf{A}\Phi_+(X,Z)$.
2. The class $\mathsf{A}SS$ is a closed operator ideal. ■

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