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Bounded Linear Maps between (LF)–spaces

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Abstract. Characterizations of pairs (E, F) of complete (LF)–spaces such that every continuous linear map from E to F maps a 0–neighbourhood of E into a bounded subset of F are given. The case of sequence (LF)–spaces is also considered. These results are similar to the ones due to D. Vogt in the case E and F are Fréchet spaces. The research continues work of J. Bonet, A. Galbis, S. Önal, T. Terzioğlu and D. Vogt.

Aplicaciones lineales acotadas entre espacios (LF)

Resumen. Se dan caracterizaciones de pares (E, F) de espacios (LF) tales que toda aplicación de E en F que aplica un intervalo de 0 de E en un subconjunto acotado de F. Se considera también el caso de una sucesión de espacios (LF). Los resultados son similares a los obtenidos por D. Vogt para el caso en que E y F son espacios de Fréchet. Esta investigación continua el trabajo de J. Bonet, A. Galbis, S. Önal, T. Terzioğlu and D. Vogt.

The problem of the characterization of those pairs of locally convex spaces E and F such that every continuous linear map from E to F maps a 0–neighbourhood into a bounded subset of F (denoted by $L(E, F) = LB(E, F)$) has been extensively considered in the literature with different purposes (e.g., see [1, 4, 5, 6, 10, 9, 12, 13]). Pairs of Fréchet spaces E and F for which the identity holds have been completely characterized by D. Vogt in [12]. For pairs of barrelled (DF)–spaces a similar result has been provided by A. Galbis in [6] (for further characterizations see also [1, 4]).

It is worth noting that the theory of pairs of Fréchet spaces between which every continuous linear map is bounded turned to be a powerful tool in the study of the topological structure of Fréchet spaces: for example, it is strongly related to the important topological invariants (DN) and (Ω) introduced by D. Vogt (see [12]). On the other hand, in [2, 3] J. Bonet and P. Domański used such a theory to clarify the relation between the various notions of vector–valued real analytic functions.

Motivated by these facts, we continue here the research on this topic, in particular giving a complete characterization of the pairs of (LF)–spaces E and F such that the identity $L(E, F) = LB(E, F)$ holds. Our characterization is similar to the one given in [12] for the case of Fréchet spaces.

The article is divided in three sections. In section 1 we collect some general results on $L(E, F) = LB(E, F)$ with E and F locally convex spaces. In section 2 we characterize the pairs of (LF)–spaces E and F for which $L(E, F) = LB(E, F)$. As consequences, we derive similar results for other cases, for example with E (LF)–space and F (LB) or DF–space, with E (LB) or Fréchet space and F (LF)–space, etc. Finally, in section 3 we apply our results to concrete sequence (LF)–spaces.

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Our notation is standard. If E is a locally convex space, the family of all bounded absolutely convex sets in E is denoted by $\mathcal{B}(E)$. If $B \in \mathcal{B}(E)$, E_B denotes $\mathrm{span}B$ equipped with the gauge functional of B as a norm. The inclusion map $E_B \hookrightarrow E$ is clearly linear and continuous. The family of all absolutely convex 0–neighbourhoods in E is denoted by $\mathcal{U}(E)$.

If E and F are locally convex spaces, a linear map $T\colon E \to F$ is called *bounded* if there is $U \in \mathcal{U}(E)$ so that $T(U) \in \mathcal{B}(F)$. Of course, every bounded map is continuous. The space of all linear and continuous maps (the space of all linear and bounded maps, resp.) from E into F is denoted by $L(E,F)$ (by $LB(E,F)$, resp.). Clearly $LB(E,F) \subset L(E,F)$. $L_b(E,F)$ denotes the space $L(E,F)$ endowed with the topology of uniform convergence on the bounded sets of E.

We refer the reader to [7, 8, 11] for other undefined notations and for the general theory of locally convex spaces.

1. General Results

In the sequel $(E_n)_n$ always denotes a sequence of locally convex spaces. Put $E = \oplus_n E_n$ (the locally convex direct sum of $(E_n)_n$), the map $i_m\colon E_m \to E$, $x \to (x\delta_{mn})_n$, is an isomorphism onto its range for every $m \in \mathbb{N}$. On the other hand, the map $p_m\colon E \to E_m$, $(x_n)_n \to x_m$, is an homomorphism onto and $p_m \circ i_m = id_{E_m}$ for every $m \in \mathbb{N}$.

Proposition 1 *Let $(E_n)_n$ be a sequence of locally convex spaces and let F be a Fréchet space. Then the following conditions are equivalent:*

(i) $L(\oplus_n E_n, F) = LB(\oplus_n E_n, F)$;

(ii) $L(E_n, F) = LB(E_n, F), \forall n \in \mathbb{N}$.

PROOF. (i)⇒(ii). Let $m \in \mathbb{N}$ and $T \in L(E_m, F)$. Then $T \circ p_m \in L(\oplus_n E_n, F) = LB(\oplus_n E_n, F)$; hence there is $U \in \mathcal{U}(\oplus_n E_n)$ such that $T(p_m(U)) \in \mathcal{B}(F)$. Since p_m is an homomorphism onto, $p_m(U)$ is also a 0–neighbourhood in E_m and hence $T \in LB(E_m, F)$.

(ii)⇒(i). Let $T \in L(\oplus_n E_n, F)$. Then $T \circ i_n \in L(E_n, F) = LB(E_n, F)$ for all $n \in \mathbb{N}$; hence, for each $n \in \mathbb{N}$, there is $U_n \in \mathcal{U}(E_n)$ such that $T(i_n(U_n)) \in \mathcal{B}(F)$. Since F is a Fréchet space, we can find a sequence of scalars $(\lambda_n)_n$ such that $\cup_n \lambda_n T(i_n(U_n)) = T(\cup_n \lambda_n i_n(U_n))$ is also a bounded set in F, where $\Gamma(\cup_n \lambda_n i_n(U_n)) \in \mathcal{U}(\oplus_n E_n)$. Then the result follows. ■

Proposition 2 *Let $(E_n)_n$ be a sequence of locally convex spaces and let F be a Fréchet space. Then the following conditions are equivalent:*

(i) $L(F, \oplus_n E_n) = LB(F, \oplus_n E_n)$;

(ii) $L(F, E_n) = LB(F, E_n), \forall n \in \mathbb{N}$.

PROOF. (i)⇒(ii). Let $m \in \mathbb{N}$ and $T \in L(F, E_m)$. Then $i_m \circ T \in L(F, \oplus_n E_n) = LB(F, \oplus_n E_n)$; hence, there is $U \in \mathcal{U}(F)$ such that $i_m(T(U))$ is a bounded set of $\oplus_n E_n$. Since i_m is an isomorphism into, $T(U)$ is also a bounded set in E_m and the result follows.

(ii)⇒(i). Let $T \in L(F, \oplus_n E_n)$. Then:

$$\exists k \in \mathbb{N} \;\; T(U_k) \subset \oplus_{n \le k} E_n, \tag{1}$$

where $(U_k)_k$ is a decreasing basis of 0-neighbourhoods in F.

Suppose that (1) is not true. Then, for each $k \in \mathbb{N}$, there is $x_k \in U_k$ such that $T(x_k) \notin \oplus_{n \le k} E_n$. Clearly, $(x_k)_k$ converges to 0 in F and hence $(T(x_k))_k$ converges to 0 in $\oplus_n E_n$ too. Thus $(T(x_k))_k \subset \oplus_{n \le m} E_n$ for some $m \in \mathbb{N}$, obtaining a contradiction.

By (1), $T(F) \subset \oplus_{n\le k}E_n$ and hence $T \in L(F, \oplus_{n\le k}E_n)$. Now, for each $n \le k$ $p_n \circ T \in L(F, E_n) = LB(F, E_n)$ and hence there is $k_n \in \mathbb{N}$ such that $B_n = p_n(T(U_{k_n}))$ is a bounded set of E_n. Put $U = \cap_{n\le k}U_{k_n}$, then $U \in \mathcal{U}(F)$ and $T(U) \subset \oplus_{n\le k}B_n$. This completes the proof. ■

Remark 1 It is clear from the above proofs that implication (i)⇒(ii) in Propositions 1 and 2 always holds for any locally convex space F. But the converse generally does not hold as the following example shows. Let $F = (\ell^p)^{(\mathbb{N})}$, $1 \le p \le \infty$, and let $E = (\omega)^{(\mathbb{N})}$. Clearly, the inclusion map $F \hookrightarrow E$ is linear and continuous, but not bounded.

On the other hand, $L(F, \omega) = LB(F, \omega)$ and $L(\ell^p, (\omega)^{(\mathbb{N})}) = LB(\ell^p, (\omega)^{(\mathbb{N})})$. ■

Next, put $E = \prod_n E_n$, for each $m \in \mathbb{N}$ the map $j_m\colon E_m \to E$, $x \to (x\delta_{mn})_n$, is an isomorphism onto its range and the map $r_m\colon E \to E_m$, $(x_n)_n \to x_m$, is an homomorphism onto such that $r_m \circ j_m = id_{E_m}$.

Proposition 3 *Let $(E_n)_n$ be a sequence of locally convex spaces and let F be a DF–space. Then the following conditions are equivalent:*

(i) $L(\prod_n E_n, F) = LB(\prod_n E_n, F)$*;*

(ii) $L(E_n, F) = LB(E_n, F), \forall n \in \mathbb{N}$.

PROOF. (i)⇒(ii). Let $m \in \mathbb{N}$ and $T \in L(E_m, F)$. Then $T \circ r_m \in L(\prod_n E_n, F) = LB(\prod_n E_n, F)$; hence there is $U \in \mathcal{U}(\prod_n E_n)$ such that $T(r_m(U)) \in \mathcal{B}(F)$. Since r_m is an homomorphism onto, $r_m(U)$ is also a 0–neighbourhood of E_m and the result follows.

(ii)⇒(i). Let $T \in L(\prod_n E_n, F)$. Then:

$$\exists k \in \mathbb{N} \quad T(\{0\}^k \times \prod_{n>k} E_n) \subset B_k, \tag{2}$$

where $(B_k)_k$ is a fundamental increasing sequence of bounded sets of F.

Suppose that (2) is not true. Then, for each $k \in \mathbb{N}$ there is $x_k \in \{0\}^k \times \prod_{n>k} E_n$ such that $T(x_k) \notin B_k$. Clearly, $(x_k)_k$ converges to 0 in $\prod_n E_n$ and hence $(T(x_k))_k$ converges to 0 in F too. Since $(B_m)_m$ is a fundamental system of bounded subsets of F, $(T(x_k))_k \subset B_m$ for some $m \in \mathbb{N}$, obtaining a contradiction.

Condition (2) implies that $T(\{0\}^k \times \prod_{n>k} E_n) = \{0\}$, being it a bounded subspace of F. On the other hand, for each $n \le k$, $T \circ j_n \in L(E_n, F) = LB(E_n, F)$ and hence there is $U_n \in \mathcal{U}(E_n)$ such that $T(j_n(U_n)) \in \mathcal{B}(F)$.

Put $U = \prod_{n\le k} U_n \times \prod_{n>k} E_n$, then U is a 0–neighbourhood of $\prod_n E_n$ and $T(U)$ is a bounded set of F because

$$U = \prod_{n\le k} U_n \times \prod_{n>k} E_n \subset \sum_{n\le k} j_n(U_n) + \{0\}^k \times \prod_{n>k} E_n$$

and hence

$$T(U) \subset \sum_{n\le k} T(j_n(U_n)),$$

where $\sum_{n\le k} T(j_n(U_n))$ is a bounded subset of F. Then the result follows. ■

Proposition 4 *Let $(E_n)_n$ be a sequence of locally convex spaces and let F be a DF–space. Then the following conditions are equivalent:*

(i) $L(F, \prod_n E_n) = LB(F, \prod_n E_n)$*;*

(ii) $L(F, E_n) = LB(F, E_n), \forall n \in \mathbb{N}$.

PROOF. (i)$\Rightarrow$(ii). Let $m \in \mathbb{N}$ and $T \in L(F, E_m)$. Then $j_m \circ T \in L(F, \prod_n E_n) = LB(F, \prod_n E_n)$; thus there is $U \in \mathcal{U}(F)$ such that $j_m(T(U)) \in \mathcal{B}(\prod_n E_n)$. Since j_m is an isomorphism into, $T(U)$ is also a bounded set of E_m and the result follows.

(ii)$\Rightarrow$(i). Let $T \in L(F, \prod_n E_n)$. Then, for each $n \in \mathbb{N}$, $r_n \circ T \in L(F, E_n) = LB(F, E_n)$ and hence there is $U_n \in \mathcal{U}(F)$ such that $r_n(T(U_n)) \in \mathcal{B}(E_n)$. Since F is a DF–space, there is a sequence of scalars $(\lambda_n)_n$ such that $U = \cap_n \lambda_n U_n$ is also a 0–neighbourhood of F. On the other hand, for each $n \in \mathbb{N}$ $r_n(T(U))$ is bounded in E_n, implying that $T(U)$ is a bounded subset of $\prod_n E_n$. Then the result follows. ∎

Remark 2 It is clear from the above proofs that implication (i)$\Rightarrow$(ii) in Propositions 3 and 4 always holds for any locally convex space F. In general the converse is not true as the following example shows.

Let $F = (\ell^p)^{\mathbb{N}}$ and $E = (\ell^q)^{\mathbb{N}}$, $1 \leq p < q \leq \infty$. Clearly, the inclusion map $F \hookrightarrow E$ is linear and continuous, but not bounded. On the other hand, $L(F, \ell^q) = LB(F, \ell^q)$ and $L(\ell^p, E) = LB(\ell^p, E)$. ∎

Let $(E_n, i_{n+1,n})_n$ be a sequence of locally convex spaces and linear and continuous inclusion maps $i_{n+1,n} \colon E_n \hookrightarrow E_{n+1}$. Let $E := \mathrm{ind}_n E_n$, i.e. $E = \cup_n E_n$ and it is endowed with the finest locally convex topology for which all the inclusion maps $i_n \colon E_n \hookrightarrow E$ are continuous. Then

Proposition 5 *Let $E = \mathrm{ind}_n E_n$ be an inductive limit of locally convex spaces and let F be a Fréchet space. Then:*

(i) $L(E_n, F) = LB(E_n, F), \forall n \in \mathbb{N} \Rightarrow L(E, F) = LB(E, F)$*;*

(ii) E is regular and $L(F, E_n) = LB(F, E_n), \forall n \in \mathbb{N} \Rightarrow L(F, E) = LB(F, E)$.

PROOF. (i). Let $T \in L(E, F)$. Then, for each $n \in \mathbb{N}$, $T \circ i_n \in L(E_n, F) = LB(E_n, F)$ and hence there is $U_n \in \mathcal{U}(E_n)$ such that $T(i_n(U_n)) \in \mathcal{B}(F)$. Since F is a Fréchet space, there is a sequence of scalars $(\lambda_n)_n$ for which $T(\cup_n \lambda_n i_n(U_n)) = \cup_n \lambda_n T(i_n(U_n))$ is also a bounded set of F. On the other hand, $\Gamma(\cup_n \lambda_n i_n(U_n)) \in \mathcal{U}(E)$ and then $T \in LB(E, F)$.

(ii). Let $T \in L(F, E)$. Then:

$$\exists m \in \mathbb{N} \;\; T(F) \subset E_m \text{ and } T \colon F \to E_m \text{ is continuous.} \tag{3}$$

To prove (3) it suffices to repeat the same proof of Gronthendieck– Floret's factorization theorem [11, 8.5.38] . For the sake of completeness, we give here the proof of (3).

Let $\mathcal{F}$ be the filter generated by $(T(U_k))_k$, where $(U_k)_k$ is a decreasing basis of 0–neighbourhoods in F. For each $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the filter generated by a basis of 0–neighbourhoods in E_n.

Let $(x_k)_k$ be a $\mathcal{F}$–convergent sequence in E. Then there is $(y_k)_k \subset F$ so that $T(y_k) = x_k$ and $(y_k)_k$ is also a convergent sequence in F. Since F is a Fréchet space, there is an unbounded sequence of positive scalars $(\lambda_k)_k$ such that $(\lambda_k y_k)_k$ is bounded in F. It follows that $(\lambda_k x_k)_k = (T(\lambda_k y_k))_k$ is a bounded set of E. Since E is a regular inductive limit, there is $n \in \mathbb{N}$ such that $(\lambda_k x_k)_k \subset E_n$ and bounded here. Thus $(x_k)_k$ converges to 0 in E_n and $(x_k)_k$ is a $\mathcal{F}_n$–convergent sequence. By [11, 8.5.35], there is $m \in \mathbb{N}$ such that $\mathcal{F}_m$ is coarser than $\mathcal{F}$ and then (3) follows.

By (3), $T \in L(F, E_m) = LB(F, E_m)$ and hence $T \in LB(F, E)$. ∎

Remark 3 The converse of Proposition 5–(i) does not hold in general as the following example shows.

For each $n \in \mathbb{N}$ we define $E_n := \prod_{k<n} \omega \times \prod_{k \geq n} s$, where s is the Fréchet space of all rapidly decreasing sequences. Put $E = \mathrm{ind}_n E_n$. By [11, 8.7.2] E is a dense topological subspace of $\omega^{\mathbb{N}}$. If $T \in L(E, s)$, there is then a unique linear continuous extension $S \in L(\omega^{\mathbb{N}}, s)$. Therefore S, and hence T, is bounded because $\omega^{\mathbb{N}}$ is a quojection and s has a continuous norm. On the other hand, it is plain that $L(E_n, s) \neq LB(E_n, s)$ for every $n \in \mathbb{N}$. ∎

Let $(E_n, r_{n,n+1})_n$ be a sequence of locally convex spaces and $r_{n,n+1}\colon E_{n+1} \to E_n$ linear and continuous maps. Let $E := \operatorname{proj}_n E_n$. Then all the maps $r_m\colon E \to E_m$, $(x_n)_n \to x_m$, are continuous. We have

Proposition 6 *Let $E = \operatorname{proj}_n E_n$ be a projective limit of locally convex spaces and let F be a DF–space. Then:*

$$L(F, E_n) = LB(F, E_n), \forall n \in \mathbb{N} \Rightarrow L(F, E) = LB(F, E).$$

PROOF. Let $T \in L(F, E)$. Then, for each $n \in \mathbb{N}$ $r_n \circ T \in L(F, E_n) = LB(F, E_n)$ and hence there is $U_n \in \mathcal{U}(F)$ such that $B_n = r_n(T(U_n)) \in \mathcal{B}(E_n)$. Since F is a DF–space, there is a sequence of scalars $(\lambda_n)_n$ such that $U = \cap_n \lambda_n U_n$ is a 0–neighbourhood in F, where $T(U) \subset \prod_n \lambda_n B_n$; thus $T(U)$ is bounded in E. ∎

2. The case of (LF)–spaces

In this section we consider the equality $L(E, F) = LB(E, F)$ in case E is a regular (LF)–space and F is a complete (LF)–space or a DF–space. We find a characterization similar to the one given by Vogt [12] when E and F are Fréchet spaces. Applications of our results for particular cases will be given.

Let $(E_m, i_{m+1,m})_m$ be a sequence of Fréchet spaces and $i_{m+1,m}\colon E_m \hookrightarrow E_{m+1}$ linear and continuous inclusion maps. Let $E = \operatorname{ind}_m E_m$ so that all inclusion maps $i_m\colon E_m \hookrightarrow E$ are continuous and $i_{m+1} \circ i_{m+1,m} = i_m$. We always assume that, for each $m \in \mathbb{N}$, $E_m = \operatorname{proj}_k(E_{mk}, p_k^m)$ is the reduced projective limit of the sequence $(E_{mk}, p_k^m)_k$ of Banach spaces and linear and continuous maps $s^m_{k,k+1}\colon E_{mk+1} \to E_{mk}$ with dense range. Let $s_k^m\colon E_m \to E_{mk}$ be the canonical projection such that $s^m_{k,k+1} \circ s^m_{k+1} = s^m_k$. Put $U_k^m = \{x \in E_m : p_k^m(s_k^m(x)) \leq 1\}$. Then $(U_k^m)_k$ is a basis of 0–neighbourhoods in E_m. Without loss of generality, we can also suppose that $U_k^m \subset U_k^{m+1}$ for every m and $k \in \mathbb{N}$. Consequently, for each m and $k \in \mathbb{N}$ there is a linear and continuous map $i^k_{m+1,m}\colon E_{mk} \to E_{m+1k}$ such that the following diagram commutes

$$\begin{array}{ccccccc} & E_m & \overset{s_k^m}{\to} & E_{mk} & \overset{s^m_{k-1,k}}{\to} & E_{mk-1} & \\ i_{m+1,m} & \downarrow & & \downarrow i^k_{m+1,m} & & \downarrow & i^{k-1}_{m+1,m} \\ & E_{m+1} & \overset{s_k^{m+1}}{\to} & E_{m+1k} & \overset{s^{m+1}_{k-1,k}}{\to} & E_{m+1k-1} & . \end{array} \tag{4}$$

We denote by p'^m_k the gauge functional of $\overset{\circ}{U}{}^m_k$ (the polar of U_k^m with respect to the duality $< E_m, E'_m >$).

Let $(F_r, j_{r+1,r})_r$ be another sequence of Fréchet spaces and linear and continuous inclusion maps $j_{r+1,r}\colon F_r \hookrightarrow F_{r+1}$. Let $F = \operatorname{ind}_r F_r$ so that all inclusion maps $j_r\colon F_r \hookrightarrow F$ are continuous and $j_{r+1} \circ j_{r+1,r} = j_r$. We always assume that, for each $r \in \mathbb{N}$, $F_r = \operatorname{proj}_h(F_{rh}, q_h^r)$ is the reduced projective limit of the sequence $(F_{rh}, q_h^r)_h$ of Banach spaces and linear and continuous maps $t^r_{h,h+1}\colon F_{rh+1} \to F_{rh}$ with dense range. Let $t_h^r\colon F_r \to F_{rh}$ be the canonical projection so that $t^r_{h,h+1} \circ t^r_{h+1} = t^r_h$. Put $V_h^r = \{x \in F_r : q_h^r(t_h^r(x)) \leq 1\}$. Then $(V_h^r)_h$ is a basis of 0-neighbourhoods in F_r. We observe that:

Remark 4 (i) For each $r \in \mathbb{N}$ we define $J_{r+1,r}\colon L_b(E, F_r) \to L_b(E, F_{r+1})$ by $J_{r+1,r}(T) := j_{r+1,r} \circ T$. Clearly, $J_{r+1,r}$ is an injective, linear and continuous map. We can then consider the inductive limit $\operatorname{ind}_r L_b(E, F_r)$ of the sequence $(L_b(E, F_r))_r$ of locally convex spaces. If for any $r \in \mathbb{N}$ we define $J_r\colon L_b(E, F_r) \to L_b(E, F)$ by $J_r(T) := j_r \circ T$, J_r is also an injective, linear and continuous map and $J_{r+1} \circ J_{r+1,r} = J_r$. Thus there is an injective, linear and continuous map $J\colon \operatorname{ind}_r L_b(E, F_r) \to L_b(E, F)$.

On the other hand, if for any $r \in \mathbb{N}$ we define $I^r_{m,m+1}\colon L_b(E_{m+1}, F_r) \to L_b(E_m, F_r)$ by $I^r_{m+1,m}(T) := T \circ i_{m+1,m}$, $I^r_{m,m+1}$ is linear and continuous. We can then consider the projective limit $\operatorname{proj}_m L_b(E_m, F_r)$ of the projective sequence $(L_b(E_m, F_r), I^r_{m,m+1})$. Thus the map $I^r\colon L_b(E, F_r) \to \operatorname{proj}_m L_b(E_m, F_r)$ defined by $I^r(T) := (T \circ i_m)_m$ is an isomorphism onto. Moreover, if $J'_{r+1,r}\colon \operatorname{proj}_m L_b(E_m, F_r) \to$

$\mathrm{proj}_m L_b(E_m, F_{r+1})$ is the map given by $J'_{r+1,r}((T_m)_m) := (j_{r+1,r} \circ T_m)_m$, $J'_{r+1,r}$ is injective, linear and continuous and the following diagram is commutative

$$\begin{array}{cccccc} & L_b(E,F_r) & \stackrel{I^r}{\to} & \mathrm{proj}_m L_b(E_m,F_r) & \\ J_{r+1,r} & \downarrow & & \downarrow & J'_{r+1,r} \\ & L_b(E,F_{r+1}) & \stackrel{I^{r+1}}{\to} & \mathrm{proj}_m L_b(E_m,F_{r+1}) & . \end{array}$$

Consequently, we can consider the inductive limit $\mathrm{ind}_r \, \mathrm{proj}_m L_b(E_m, F_r)$ and the map

$$I\colon \operatorname*{ind}_r \operatorname*{proj}_m L_b(E_m,F_r) \to \operatorname*{ind}_r L_b(E,F_r)$$

given by $((I^r)^{-1})_r$ (i.e., $I((T_m)_m) := (I^r)^{-1}((T_m)_m)$ if $(T_m)_m \in \mathrm{proj}_m L_b(E_m, F_r)$), which is an isomorphism onto.

Now, given any r and m in $\mathbb{N}$, for each $k \in \mathbb{N}$ we define $S^{m,r}_{k+1,k}\colon L_b(E_{mk}, F_r) \to L_b(E_{mk+1}, F_r)$ by $S^{m,r}_{k+1,k}(T) := T \circ s^m_{k,k+1}$. Since $s^m_{k,k+1}$ has dense range, $S^{m,r}_{k+1,k}$ is injective, linear and continuous. We can then consider the inductive limit $\mathrm{ind}_k L_b(E_{mk}, F_r)$ of the inductive sequence $(L_b(E_{mk}, F_r), S^{m,r}_{k+1,k})$. If for $k \in \mathbb{N}$ we define $S^{m,r}_k\colon L_b(E_{mk}, F_r) \to L_b(E_m, F)$ by $S^{m,r}_k(T) := T \circ s^m_k$, $S^{m,r}_k$ is also an injective, linear and continuous map. Moreover, $S^{m,r}_{k+1} \circ S^{m,r}_{k+1,k} = S^{m,r}_k$. Then there is an injective, linear and continuous map $S^{m,r}\colon \mathrm{ind}_k L_b(E_{mk}, F_r) \to L_b(E_m, F_r)$.

Also, if $I^{k,r}_{m,m+1}\colon L_b(E_{m+1k}, F_r) \to L_b(E_{mk}, F_r)$ is the map given by $I^{k,r}_{m,m+1}(T) := T \circ i^k_{m+1,m}$, $I^{k,r}_{m,m+1}$ is linear and continuous and turns to be commutative the following diagram because of (4)

$$\begin{array}{ccccc} L_b(E_{mk-1},F_r) & \stackrel{S^{m,r}_{k+1,k}}{\to} & L_b(E_{mk},F_r) & \stackrel{S^{m,r}_k}{\to} & L_b(E_m,F_r) \\ I^{k-1,r}_{m,m+1}\uparrow & & \uparrow I^{k,r}_{m,m+1} & & \uparrow I^r_{m,m+1} \\ L_b(E_{m+1k-1},F_r) & \stackrel{S^{m+1,r}_{k+1,k}}{\to} & L_b(E_{m+1k},F_r) & \stackrel{S^{m+1,r}_k}{\to} & L_b(E_{m+1},F_r). \end{array}$$

Therefore the map $I'^r_{m,m+1}\colon \mathrm{ind}_k L_b(E_{m+1k}, F_r) \to \mathrm{ind}_k L_b(E_{mk}, F_r)$, given by $I'^r_{m,m+1}(T) := T \circ i^k_{m+1,m}$ for $T \in L_b(E_{m+1k}, F_r)$, is well–defined, linear and continuous and such that the following diagram also commutes

$$\begin{array}{cccccc} & \mathrm{ind}_k L_b(E_{mk},F_r) & \stackrel{S^{m,r}}{\to} & L_b(E_m,F_r) & \\ I'^r_{m,m+1} & \uparrow & & \uparrow & I^r_{m,m+1} \\ & \mathrm{ind}_k L_b(E_{m+1k},F_r) & \stackrel{S^{m+1,r}}{\to} & L_b(E_{m+1},F_r) & . \end{array}$$

This means that the map $S^r\colon \mathrm{proj}_m \mathrm{ind}_k L_b(E_{mk}, F_r) \to \mathrm{proj}_m L_b(E_m, F_r)$ defined by $S^r((T_m)_m) := (S^{m,r}(T_m))_m$ is linear and continuous. Since $S^{m,r}$ is injective for every $m \in \mathbb{N}$, S^r is also injective.

Moreover, if $J''_{r+1,r}\colon \mathrm{proj}_m \mathrm{ind}_k L_b(E_{mk}, F_r) \to \mathrm{proj}_m \mathrm{ind}_k L_b(E_{mk}, F_{r+1})$ is the map such that $J''_{r+1,r}((T_m)_m) = (j_{r,r+1} \circ T_m)_m$, then $J''_{r+1,r}$ is injective, linear and continuous and turns to be commutative the following diagram

$$\begin{array}{cccccc} & \mathrm{proj}_m \mathrm{ind}_k L_b(E_{mk},F_{r+1}) & \stackrel{S^{r+1}}{\to} & \mathrm{proj}_m L_b(E_m,F_{r+1}) & \\ J''_{r+1,r} & \uparrow & & \uparrow & J'_{r+1,r} \\ & \mathrm{proj}_m \mathrm{ind}_k L_b(E_{mk},F_r) & \stackrel{S^r}{\to} & \mathrm{proj}_m L_b(E_m,F_r) & . \end{array}$$

The map $S\colon \mathrm{ind}_r \mathrm{proj}_m \mathrm{ind}_k L_b(E_{mk}, F_r) \to \mathrm{ind}_r \mathrm{proj}_m L_b(E_m, F_r)$, given by $S((T_m)_m) := S^r((T_m)_m)$ if $(T_m)_m \in \mathrm{proj}_m \mathrm{ind}_k L_b(E_{mk}, F_r)$, is then well-defined, injective, linear and continuous.

Finally, given k, m and $r \in \mathbb{N}$, if for any $h \in \mathbb{N}$ we define $T^{m,r}_{h,h+1}\colon L_b(E_{mk}, F_{rh+1}) \to L_b(E_{mk}, F_{rh})$ by $T^{m,r}_{h,h+1}(T) := t^r_{h,h+1} \circ T$, $T^{m,r}_{h,h+1}$ is also linear and continuous. Let $\mathrm{proj}_h L_b(E_{mk}, F_{rh})$ be the projective limit of the projective sequence $(L_b(E_{mk}, F_{rh}), T^{m,r}_{h,h+1})_h$. Then the map $T^{m,r}_h\colon L_b(E_{mk}, F_r) \to L_b(E_{mk}, F_{rh})$ defined by $T^{m,r}_h(T) = (t^r_h \circ T)_h$ is an isomorphism onto.

Now, proceeding exactly as we did before, one shows that we can consider the space

$$\operatorname*{ind}_r \operatorname*{proj}_m \operatorname*{ind}_k \operatorname*{proj}_h L_b(E_{mk}, F_{rh})$$

and define in a canonical way an injective, linear and continuous map T from this space to

$$\operatorname*{ind}_r \operatorname*{proj}_m \operatorname*{ind}_k L_b(E_{mk}, F_r).$$

Thus we have proved that the map

$$R := J \circ I \circ S \circ T\colon \operatorname*{ind}_r \operatorname*{proj}_m \operatorname*{ind}_k \operatorname*{proj}_h L_b(E_{mk}, F_{rh}) \to L_b(E, F)$$

is well–defined, injective, linear and continuous.

(ii) Suppose that F is a regular (LF)–space. Then, as it is easy to see, the image of the map R above constructed is the space $LB(E, F)$ of all linear bounded maps from E into F.

(iii) Suppose that E is a regular (LF)–space and F is a complete (LF)–space. Let $((k(n, j))_j)_n$ be a countable set of increasing sequences of positive integers and let $(r(n))_n$ be another increasing sequence of positive integers. Then the space

$$H := \{T \in L(E, F) : ||T||_{j,n} := \sup_{x \in U^n_{k(n,j)}} q_j^{r(n)}(T(x)) < +\infty\ \forall n,\ j \in \mathbb{N}\}$$

is a Fréchet space with respect to the topology generated by the sequence $(||\ ||_{j,n})_{j,n}$ of seminorms.

To see this, it suffices to show that the inclusion map $H \hookrightarrow L_b(E, F)$ is continuous.

Let $B \in \mathcal{B}(E)$ and $V \in \mathcal{U}(F)$. Since E is a regular (LF)–space, there is $n_0 \in \mathbb{N}$ such that $B \subset E_{n_0}$ and bounded here. Now, $V \cap F_{r(n_0)}$ is a 0–neighbourhood in $F_{r(n_0)}$ and hence $V \cap F_{r(n_0)} \supset \mu V_{j_0}^{r(n_0)}$ for some $j_0 \in \mathbb{N}$ and $\mu > 0$. Since B is a bounded set of E_{n_0}, we find a $\lambda > 0$ such that $B \subset \lambda U^{n_0}_{k(n_0,j_0)}$. Then

$$\begin{aligned} W &= \{T \in H : T(U^{n_0}_{k(n_0,j_0)}) \subset V_{j_0}^{r(n_0)}\} \\ &\subset \mu^{-1}\lambda\{T \in L(E, F) : T(B) \subset V\} = \mu^{-1}\lambda M, \end{aligned}$$

where W and M are 0–neighbourhoods in H and in $L_b(E, F)$, respectively. In fact, if $T \in W$, then $T(U^{n_0}_{k(n_0,j_0)}) \subset V_{j_0}^{r(n_0)}$ and hence $T(B) \subset \lambda T(U^{n_0}_{k(n_0,j_0)}) \subset \lambda V_{j_0}^{r(n_0)} \subset \lambda\mu^{-1} V \cap F_{r(n_0)} \subset \lambda\mu^{-1} V$: therefore $T \in \mu^{-1}\lambda M$.

Since $L_b(E, F)$ is complete and the inclusion map $H \hookrightarrow L_b(E, F)$ is continuous, we obtain that H is a Fréchet space. ■

Now, we are ready to state and prove:

Theorem 1 *Let $E = \operatorname{ind}_m E_m$ be a regular (LF)–space and let $F = \operatorname{ind}_r F_r$ be a complete (LF)–space. Then the following conditions are equivalent:*

(i) $L(E, F) = LB(E, F)$*;*

(ii) for each sequence $((k(n, j))_j)_n$ of increasing sequences of positive integers and for each increasing sequence $(r(n))_n$ of positive integers

$$\begin{gathered} \exists r \in \mathbb{N}\ \forall m \in \mathbb{N}\ \exists k \in \mathbb{N}\ \forall h \in \mathbb{N}\ \exists j_0,\ n_0 \in \mathbb{N}\ \exists C > 0 : \\ \sup_{x \in U^m_k} q^r_h(T(x)) \le C \max_{\substack{1 \le n \le n_0 \\ 1 \le j \le j_0}} \sup_{x \in U^n_{k(n,j)}} q_j^{r(n)}(T(x)), \end{gathered} \tag{5}$$

for every $T \in L(E, F)$ with $\sup_{x \in U^n_{k(n,j)}} q_j^{r(n)}(T(x)) < +\infty$ for each $j, n \in \mathbb{N}$.

PROOF. (i)⇒(ii). Let $(k(n,j))_j)_n$ be a sequence of increasing sequences of positive integers and let $(r(n))_n$ be another increasing sequence of positive integers. We consider the space

$$H = \{T \in L(E,F) : ||T||_{j,n} = \sup_{x \in U^n_{k(n,j)}} q_j^{r(n)}(T(x)) < +\infty,\ \forall j,\ n \in \mathbb{N}\}.$$

By Remark 4–(iii) , H is a Fréchet space with respect to the topology generated by the sequence $(||\ ||_{j,n})_{j,n}$ of seminorms and the inclusion map $H \hookrightarrow L_b(E,F)$ is continuous.

On the other hand, by hypothesis $L(E,F) = LB(E,F)$; thus, by Remark 4–(i) and (ii), the linear map

$$R\colon \operatorname*{ind}_r \operatorname*{proj}_m \operatorname*{ind}_k \operatorname*{proj}_h L_b(E_{mk}, F_{rh}) \to L_b(E,F)$$

is a continuous algebraical isomorphism onto.

Endowed the space $L(E,F)$ with the topology induced via the map R, it follows that the inclusion map

$$H \hookrightarrow L(E,F) = \operatorname*{ind}_r \operatorname*{proj}_m \operatorname*{ind}_k \operatorname*{proj}_h L_b(E_{mk}, F_{rh})$$

has closed graph. Since H is a Fréchet space and

$$L(E,F) = \operatorname*{ind}_r \operatorname*{proj}_m \operatorname*{ind}_k \operatorname*{proj}_h L_b(E_{mk}, F_{rh})$$

is a strictly webbed, we can apply the Wilde's closed graph theorem [7, 5.4.1] to conclude that this inclusion map is continuous. Thus, by the Localization Theorem [7, 5.6.3], there is $r \in \mathbb{N}$ such that $H \subset \operatorname{proj}_m \operatorname{ind}_k \operatorname{proj}_h L_b(E_{mk}, F_{rh})$ and the map $H \hookrightarrow \operatorname{proj}_m \operatorname{ind}_k \operatorname{proj}_h L_b(E_{mk}, F_{rh})$ is also continuous. Therefore for each $m \in \mathbb{N}$ the map

$$H \to \operatorname*{ind}_k \operatorname*{proj}_h L_b(E_{mk}, F_{rh}),\ \ T \to T \circ i_m,$$

is continuous. Finally, by Grothendieck's factorization theorem [11, 8.5.38], there is $k \in \mathbb{N}$ such that

$$H \to \operatorname*{proj}_h L_b(E_{mk}, F_{rh}) = L_b(E_{mk}, F_r),\ \ T \to \tilde{T},$$

is also continuous, where $\tilde{T} \circ s_k^m = T \circ i_m$.

We have shown that

$$\begin{gathered}\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists j_0,\ n_0 \in \mathbb{N}\, \exists C > 0 : \\ \sup_{x \in U_k^m} q_h^r(T(x)) \le C \max_{\substack{1 \le n \le n_0 \\ 1 \le j \le j_0}} \sup_{x \in U^n_{k(n,j)}} q_j^{r(n)}(T(x)),\end{gathered}$$

for every $T \in H$. This completes the proof.

(ii)⇒(i). Let $T \in L(E,F)$. Then $T \circ i_n \in L(E_n, F)$ for all $n \in \mathbb{N}$. By Grothendieck's factorization theorem [11, 8.5.38], we can find an increasing sequence $(r(n))_n$ of positive integers such that $T \circ i_n \in L(E_n, F_{r(n)})$ for all $n \in \mathbb{N}$. Since $T \circ i_n \in L(E_n, F_{r(n)})$, we can find another increasing sequence $(k(n,j))_j$ of positive integers such that, for each $j \in \mathbb{N}$,

$$\sup_{x \in U^n_{k(n,j)}} q_j^{r(n)}(T(x)) = d_{nj} < +\infty.$$

On the other hand, by assumption there is $r \in \mathbb{N}$ so that (5) holds. Consequently, we get:

$$\begin{gathered}\forall m \in \mathbb{N}\, \exists k_m \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists j_0,\ n_0 \in \mathbb{N}\, \exists C > 0 : \\ \sup_{x \in U^m_{k_m}} q_h^r(T(x)) \le C \max_{\substack{1 \le n \le n_0 \\ 1 \le j \le j_0}} d_{nj} < +\infty.\end{gathered}$$

This implies that $T(U^m_{k_m})$ is a bounded subset of F_r for all $m \in \mathbb{N}$. Since F_r is a metrizable space, there is a sequence of scalars $(\lambda_m)_m$ such that $T(\cup_m \lambda_m U^m_{k_m}) = \cup_m \lambda_m T(U^m_{k_m})$ is bounded in F_r, where $\Gamma(\cup_m \lambda_m U^m_{k_m})$ is a 0–neighbourhood in E. Then the proof is complete. ■

With obvious changes the previous characterization holds for other cases, e.g., taking E (LF)–space or (LB)–space and F Fréchet space or complete DF–space. To obtain this, it is enough to do only some remarks.

Consider first the case where E is a regular (LF)–space and F is a Fréchet space.

Let F be a Fréchet space. Assume that $F = \mathrm{proj}_h(F_h, q_h)$ is the reduced projective limit of the projective sequence $(F_h, q_h)_h$ of Banach spaces and linear and continuous maps $t_{h,h+1}\colon F_{h+1} \to F_h$ with dense range. Let $t_h\colon F \to F_h$ be the canonical projection such that $t_{h,h+1} \circ t_{h+1} = t_h$. Put $V_h = \{x \in F : q_h(t_h(x)) \leq 1\}$. Then $(V_h)_h$ is a basis of 0–neigbourhoods in F.

For each $r \in \mathbb{N}$ put $F_r = F$ and $j_{r+1,r} = j_r = id_F$. Clearly, $F = \mathrm{ind}_r F_r$. Therefore:

Remark 5 (i) Let $E = \mathrm{ind}_m E_m$ be an (LF)–space and let F be a Fréchet space. By Remark 4–(i) and (ii), the map R turns to be a linear and continuous map from the space $\mathrm{proj}_m \mathrm{ind}_k \mathrm{proj}_h L_b(E_{mk}, F_h)$ into the space $L_b(E, F)$ and its image is exactly the space $LB(E, F)$ of all linear bounded maps from E into F.

(ii) Let E be a regular (LF)–space and let F be a Fréchet space. Let $(k(n))_n$ be an increasing sequence of positive integers. Proceeding exactly as we did in Remark 4–(iii), one shows that the space

$$H := \{T \in L(E, F) : ||T||_n = \sup_{x \in U^n_{k(n)}} q_n(T(x)) < +\infty,\ \forall n \in \mathbb{N}\}$$

is a Fréchet space with respect to the topology generated by the sequence $(||\ ||_n)_n$ of seminorms and the inclusion map $H \hookrightarrow L_b(E, F)$ is continuous. ■

Proceeding as we did to prove Theorem 1, we easily get:

Proposition 7 *Let $E = \mathrm{ind}_m E_m$ be a regular (LF)–space and let F be a Fréchet space. Then the following conditions are equivalent:*

(i) $L(E, F) = LB(E, F)$;

(ii) for each increasing sequence $(k(n))_n$ of positive integers and for each $m \in \mathbb{N}$

$$\exists k \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \exists C > 0 : \tag{6}$$
$$\sup_{x \in U^m_k} q_h(T(x)) \leq C \max_{1 \leq n \leq n_0} \sup_{x \in U^n_{k(n)}} q_n(T(x)),$$

for every $T \in L(E, F)$ with $\sup_{x \in U^n_{k(n)}} q_n(T(x)) < +\infty$ for each $n \in \mathbb{N}$. □

Next, consider the case where E is a Fréchet space and F is a regular (LF)–space.

Let E be a Fréchet space. Assume that $E = \mathrm{proj}_k(E_k, p_k)$ is the reduced projective limit of the projective sequence $(E_k, p_k)_k$ of Banach spaces and linear and continuous maps $s_{k,k+1}\colon E_{k+1} \to E_k$ with dense range. Let $s_k\colon E \to E_k$ be the canonical projection such that $s_{k,k+1} \circ s_{k+1} = s_k$. Put $U_k = \{x \in E : p_k(s_k(x)) \leq 1\}$. Then $(U_k)_k$ is a basis of 0–neighbourhoods in E.

For each $m \in \mathbb{N}$ let $E_m = E$ and $i_{m+1,m} = i_m = id_E$. Clearly, $E = \mathrm{ind}_m E_m$. We have:

Remark 6 (i) Let E be a Fréchet space and let $F = \mathrm{ind}_r F_r$ be a regular (LF)–space. By Remark 4–(i) and (ii), the map R turns to be a linear and continuous map from the space $\mathrm{ind}_r \mathrm{ind}_k \mathrm{proj}_h L_b(E_k, F_{rh})$ into the space $L_b(E, F)$ and its image is the space $LB(E, F)$ of all linear and bounded maps from E into F.

(ii) Let E be a Fréchet space and let $F = \operatorname{ind}_r F_r$ be a complete (LF)–space. Let $(k(n))_n$ be an increasing sequence of positive integers and $r \in \mathbb{N}$. Proceeding as we did in Remark 4–(iii), one shows that the space

$$H := \{T \in L(E,F) : ||T||_n = \sup_{x \in U_{k(n)}} q_n^r(T(x)) < +\infty,\ \forall n \in \mathbb{N}\}$$

is a Fréchet space with respect to the topology generated by the sequence $(||\ ||_n)_n$ of seminorms and the inclusion map $H \hookrightarrow L_b(E,F)$ is continuous. ■

Argumenting as we did to prove Theorem 1, we then obtain:

Proposition 8 *Let E be a Fréchet space and let $F = \operatorname{ind}_r F_r$ be a complete (LF)–space. Then the following conditions are equivalent:*

(i) $L(E,F) = LB(E,F)$*;*

(ii) for each increasing sequence $(k(n))_n$ of positive integers and for each $r \in \mathbb{N}$

$$\begin{gathered}\exists k,\ r_0 \in \mathbb{N}\ \forall h \in \mathbb{N}\ \exists n_0 \in \mathbb{N}\ \exists C > 0 : \\ \sup_{x \in U_k} q_h^{r_0}(T(x)) \le C \max_{1 \le n \le n_0} \sup_{x \in U_{k(n)}} q_n^r(T(x)),\end{gathered} \tag{7}$$

for every $T \in L(E,F)$ with $\sup_{x \in U_{k(n)}} q_n^r(T(x)) < +\infty$ for each $n \in \mathbb{N}$. □

Suppose that F is a complete DF–space. Let $(B_r)_r$ be a fundamental system of absolutely convex closed bounded sets of F with $B_r \subset B_{r+1}$. Denote by q_r the gauge of B_r and by F_r the linear span of B_r. Then (F_r, q_r) is a normed space and the inclusion map $j_r \colon (F_r, q_r) \hookrightarrow F$ is continuous. Actually, also the inclusion maps $j_{r+1,r} \colon F_r \hookrightarrow F_{r+1}$ are continuous. Let $F_i := \operatorname{ind}_r(F_r, q_r)$. Thus the inclusion map $F_i \hookrightarrow F$ is continuous (algebraically $F_i = F$). Since F is a complete DF–space, (F_r, q_r) is a Banach space and F_i is a complete (LB)–space. Moreover, if E is a locally convex space, the inclusion map $L_b(E, F_i) \hookrightarrow L_b(E,F)$ is linear and continuous and $LB(E,F_i) = LB(E,F)$. We have:

Remark 7 (i) Let $E = \operatorname{ind}_m E_m$ be an (LF)–space and let $F = \operatorname{ind}_r F_r$ be a regular (LB)–space (F is the inductive limit of the inductive sequence $(F_r, j_{r+1,r})$ of Banach spaces and linear and continuous inclusion maps $j_{r+1,r} \colon F_r \hookrightarrow F_{r+1}$ so that all inclusion maps $j_r \colon F_r \hookrightarrow F$ are continuous) or let F be a complete DF–space. By Remarks 4–(i) and (ii), the map R turns to be a linear and continuous map from the space $\operatorname{ind}_r \operatorname{proj}_m \operatorname{ind}_k L_b(E_{mk}, F_r)$ into the space $L_b(E,F)$ and its image is the space $LB(E,F)$.

(ii) Let $E = \operatorname{ind}_m E_m$ be a regular (LF)–space and let $F = \operatorname{ind}_r F_r$ be a complete (LB)–space or let F be a complete DF–space. Let $(k(n))_n$ and $(r(n))_n$ be two increasing sequences of positive integers. Proceeding again as we did in Remark 4–(iii), one shows that the space

$$H = \{T \in L(E,F) : ||T||_n := \sup_{x \in U^n_{k(n)}} q_{r(n)}(T(x)) < +\infty,\ \forall n \in \mathbb{N}\}$$

is a Fréchet space with respect to the topology generated by the sequence $(||\ ||_n)_n$ of seminorms and the inclusion map $H \hookrightarrow L_b(E,F)$ is continuous. ■

Repeating the proof of Theorem 1 with obvious changes, we get:

Proposition 9 *Let $E = \operatorname{ind}_m E_m$ be a regular (LF)–space. Let $F = \operatorname{ind}_r F_r$ be a complete (LB)–space or let F be a complete DF–space. Then the following conditions are equivalent:*

(i) $L(E,F) = LB(E,F)$*;*

(ii) for each pair of increasing sequences $((k(n))_n,\ (r(n))_n)$ *of positive integers*

$$\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists k,\ n_0 \in \mathbb{N}\, \exists C > 0 : \qquad (8)$$
$$\sup_{x \in U_k^m} q_r(T(x)) \leq C \max_{1 \leq n \leq n_0} \sup_{x \in U_{k(n)}^n} q_{r(n)}(T(x)),$$

for every $T \in L(E, F)$ *with* $\sup_{x \in U_{k(n)}^n} q_{r(n)}(T(x)) < +\infty$ *for each* $n \in \mathbb{N}$. $\square$

It also holds a result similar to the previous ones in case E is a regular (LB)–space and F a complete (LF) or (LB)–space.

Let $E = \operatorname{ind}_m(E_m, p_m)$ be the inductive limit of the sequence $(E_m, i_{m+1,m})$ of Banach spaces ($U_m = \{x \in E_m : p_m(x) \leq 1\}$ denotes the closed unit ball of E_m) and linear and continuous inclusion maps $i_{m+1,m} \colon E_m \hookrightarrow E_{m+1}$, where all inclusion maps $i_m \colon E_m \hookrightarrow E$ are continuous. Then:

Remark 8 (i) Let $E = \operatorname{ind}_m E_m$ be an (LB)–space and let $F = \operatorname{ind}_r F_r$ be a regular (LF)–space (let $F = \operatorname{ind}_r F_r$ be a regular (LB)–space or let F be a complete DF–space, resp.). By Remarks 4–(i) and (ii), the map R turns to be a linear and continuous map from the space $\operatorname{ind}_r \operatorname{proj}_m \operatorname{proj}_h L_b(E_m, F_{rh})$ (from the space $\operatorname{ind}_r \operatorname{proj}_m L_b(E_m, F_r)$, resp.) into the space $L_b(E, F)$ and its image is the space $LB(E, F)$.

(ii) Let $E = \operatorname{ind}_m E_m$ be a regular (LB)–space and let $F = \operatorname{ind}_r F_r$ be a complete (LF)–space (let F be a complete (LB) or DF space, resp.). Let $(r(n))_n$ be an increasing sequence of positive integers. Then

$$H = \{T \in L(E, F) : ||T||_{j,n} := \sup_{x \in U_n} q_j^{r(n)}(T(x)),\ \forall j,\ n \in \mathbb{N}\}$$

(the space

$$H = \{T \in L(E, F) : ||T||_n := \sup_{x \in U_n} q_{r(n)}(T(x)),\ \forall n \in \mathbb{N}\},$$

resp.) is a Fréchet space with respect to the topology generated by the sequence $(||\ ||_{j,n})_{j,n}$ ($(||\ ||_n)_n$, resp.) of seminorms and the inclusion map $H \hookrightarrow L_b(E, F)$ is continuous. $\blacksquare$

By repeating again the same proof of Theorem 1 with simple changes, we obtain:

Proposition 10 *Let* $E = \operatorname{ind}_m E_m$ *be a regular (LB)–space and let* $F = \operatorname{ind}_r F_r$ *be a complete (LF)–space. Then the following conditions are equivalent:*

(i) $L(E, F) = LB(E, F)$*;*

(ii) for each increasing sequence $(r(n))_n$ *of positive integers*

$$\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists j_0,\ n_0 \in \mathbb{N}\, \exists C > 0 : \qquad (9)$$
$$\sup_{x \in U_m} q_h^r(T(x)) \leq C \max_{\substack{1 \leq n \leq n_0 \\ 1 \leq j \leq j_0}} \sup_{x \in U_n} q_j^{r(n)}(T(x)),$$

for every $T \in L(E, F)$ *with* $\sup_{x \in U_n} q_j^{r(n)}(T(x)) < +\infty$ *for each* j *and* $n \in \mathbb{N}$. $\square$

Proposition 11 *Let* $E = \operatorname{ind}_m E_m$ *be a regular (LB)–space and let* $F = \operatorname{ind}_r F_r$ *be a complete (LB)–space or let* F *be a complete DF–space. Then the following conditions are equivalent:*

(i) $L(E, F) = LB(E, F)$*;*

(ii) for each increasing sequence $(r(n))_n$ *of positive integers*

$$\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \exists C > 0 : \qquad (10)$$
$$\sup_{x \in U_m} q_r(T(x)) \leq C \max_{1 \leq n \leq n_0} \sup_{x \in U_n} q_{r(n)}(T(x)),$$

for every $T \in L(E, F)$ *with* $\sup_{x \in U_n} q_{r(n)}(T(x)) < +\infty$ *for each* $n \in \mathbb{N}$. $\square$

3. The case of sequence (LF)–spaces

In this section we consider concrete sequence (LF)–spaces which were introduced by Vogt [13, §5] and we apply our results of § 2.

Let $(a^k_{i,m})_{i,m,k}$ be a matrix with non negative real valued entries and with the following properties:

$$\begin{array}{ll} \forall i,\, k,\, m \in \mathbb{N} & a^k_{i,m} \le a^{k+1}_{i,m}, \\ \forall i,\, k,\, m \in \mathbb{N} & a^k_{i,m+1} \le a^k_{i,m}, \\ \forall i,\, m \in \mathbb{N}\, \exists k \in \mathbb{N} & a^k_{i,m} > 0. \end{array}$$

For each $m \in \mathbb{N}$ put $A_m = (a^k_{i,m})_{i,k}$ and

$$\lambda_1(A_m) = \{x = (x_1, x_2, \ldots) : p^m_k(x) = \sum_i |x_i| a^k_{i,m} < +\infty, \forall k \in \mathbb{N}\}.$$

Then $\lambda_1(A_m)$ is a Fréchet space with fundamental system of seminorms $(p^m_k)_k$ and the inclusion map $i_{m+1,m} \colon \lambda_1(A_m) \hookrightarrow \lambda_1(A_{m+1})$ is continuous for every $m \in \mathbb{N}$. Following [13, § 5], we set $E^1 = \cup_m \lambda_1(A_m)$ endowed with the finest topology for which all the inclusion maps $i_m \colon \lambda_1(A_m) \hookrightarrow E^1$ are continuous, i.e. $E^1 = \mathrm{ind}_m\, \lambda_1(A_m)$ is an (LF)–space. In the sequel, e_i always denotes the i–th unit vector of E^1. Recall that in [13, 5.14] it has been shown that E^1 is a complete (LF)–space if, and only if, it is a regular (LF)–space if, and only if, the matrix $(a^k_{i,m})_{i,m,k}$ is of type (WQ), i.e. $(a^k_{i,m})_{i,m,k}$ satisfies the following condition:

$$\forall \mu \in \mathbb{N}\, \exists n,\, k \in \mathbb{N}\, \forall m,\, K \in \mathbb{N}\, \exists N \in \mathbb{N}, S > 0\, \forall i \in \mathbb{N} :\; a^m_{i,n} \le S(a^k_{i,\mu} + a^N_{i,K}).$$

We have:

Theorem 2 *Let F be a complete (LF)–space. Suppose that E^1 is a regular (LF)–space. Then the following conditions are equivalent:*

(i) $L(E^1, F) = LB(E^1, F)$;

(ii) for each sequence $((k(n,j))_j)_n$ of increasing sequences of positive integers and for each increasing sequence $(r(n))_n$ of positive integers

$$\begin{gathered} \exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists j_0,\, n_0 \in \mathbb{N}\, \exists C > 0 : \\ (a^k_{i,m})^{-1} q^r_h(y) \le C \max_{\substack{1 \le n \le n_0 \\ 1 \le j \le j_0}} (a^{k(n,j)}_{i,n})^{-1} q^{r(n)}_j(y), \end{gathered} \tag{11}$$

for every $y \in F_{r(1)}$ and $i \in \mathbb{N}$.

PROOF. (i)⇒(ii). Let $((k(n,j))_j)_n$ be a countable set of increasing sequences of positive integers. Let $(r(n))_n$ be another increasing sequence of positive integers. Since $L(E,F) = LB(E,F)$, we can apply Theorem 1 to conclude that condition (5) holds. Accordingly, we show that condition (11) holds too.

Let $y \in F_{r(1)}$ and let $i \in \mathbb{N}$. We define $T \colon E^1 \to F$ by $T(x) := x_i y$. Clearly, T is a linear and continuous map. Moreover, for each j and $n \in \mathbb{N}$, we have:

$$\begin{aligned} \sup_{x \in U^n_{k(n,j)}} q^{r(n)}_j(T(x)) &= \sup_{x \in U^n_{k(n,j)}} (a^{k(n,j)}_{i,n})^{-1} a^{k(n,j)}_{i,n} |x_i| q^{r(n)}_j(y) \\ &\le (a^{k(n,j)}_{i,n})^{-1} q^{r(n)}_j(y) < +\infty. \end{aligned}$$

Therefore, by (5) we obtain that

$$\begin{aligned}(a_{i,m}^k)^{-1}q_h^r(y) &= q_h^r(T((a_{i,m}^k)^{-1}e_i)) \le \sup_{x\in U_k^m} q_h^r(T(x)) \\ &\le C \max_{\substack{1\le n\le n_0\\1\le j\le j_0}} \sup_{x\in U^n_{k(n,j)}} q_j^{r(n)}(T(x)) \\ &\le C \max_{\substack{1\le n\le n_0\\1\le j\le j_0}} (a_{i,m}^{k(n,j)})^{-1} q_j^{r(n)}(y).\end{aligned}$$

Then (ii) holds.

(ii)⇒(i). Let $T \in L(E^1, F)$. Then $T \circ i_n \in L(\lambda_1(A_n), F)$ for every $n \in \mathbb{N}$; hence, by Grothendieck's factorization theorem [11, 8.5.38] we can find an increasing sequence $(r(n))_n$ of positive integers such that $T \circ i_n \in L(\lambda_1(A_n), F_{r(n)})$ for every $n \in \mathbb{N}$. It follows that, for each $n \in \mathbb{N}$, we can find another sequence $(k(n,j))_j$ of positive integers such that

$$\forall j \in \mathbb{N} \quad \sup_{x\in U^n_{k(n,j)}} q_j^{r(n)}(T(x)) = d_{nj} < +\infty. \tag{12}$$

By (ii), condition (11), accordingly there is $r \in \mathbb{N}$ such that

$$\begin{gathered}\forall m \in \mathbb{N}\ \exists k_m \in \mathbb{N}\ \forall h \in \mathbb{N}\ \exists j_0,\ n_0 \in \mathbb{N}\ \exists C > 0 : \\ (a_{i,m}^{k_m})^{-1} q_h^r(y) \le C \max_{\substack{1\le n\le n_0\\1\le j\le j_0}} (a_{i,n}^{k(n,j)})^{-1} q_j^{r(n)}(y),\end{gathered} \tag{13}$$

for every $y \in F_{r(1)}$ and $i \in \mathbb{N}$.

This implies that $T(U^m_{k_m}) \subset F_r$ and bounded here. Indeed, if $x \in U^m_{k_m}$ and $x = \sum_i x_i e_i$, then $T(x) = \sum_i x_i T(e_i)$; hence, by (12) and (13) we get:

$$\begin{aligned} q_h^r(T(x)) &\le \sum_i |x_i| q_h^r(T(e_i)) = \sum_i |x_i| a_{i,m}^{k_m} (a_{i,m}^{k_m})^{-1} q_h^r(T(e_i)) \\ &\le \sum_i |x_i| a_{i,m}^{k_m} C \max_{\substack{1\le n\le n_0)\\1\le j\le j_0}} (a_{i,n}^{k(n,j)})^{-1} q_j^{r(n)}(T(e_i)) \\ &\le \sum_i |x_i| a_{i,m}^{k_m} C \max_{\substack{1\le n\le n_0\\1\le j\le j_0}} d_{nj} = L_{mh} p^m_{k_m}(x) \le L_{mh}\end{aligned}$$

(because $(e_i)_i \subset \lambda_1(A_1)$, $(T(e_i))_i \subset F_{r(1)}$). This means that $T(U^m_{k_m}) \subset F_r$ and bounded here. Since F_r is a metrizable space, there is a sequence of scalars $(\lambda_m)_m$ such that $T(\cup_m \lambda_m U^m_{k_m}) = \cup_m \lambda_m T(U^m_{k_m})$ is also bounded in F_r, where $\Gamma(\cup_m \lambda_m U^m_{k_m})$ is a 0–neighbourhood of E^1. Now, the proof is complete. ■

Now, let $(b^h_{j,r})_{j,h,r}$ be another matrix with nonnegative real valued entries and with the following properties:

$$\begin{array}{ll} \forall j,\ h,\ r \in \mathbb{N} & b^h_{j,r} \le b^{h+1}_{j,r}, \\ \forall j,\ h,\ r \in \mathbb{N} & b^h_{j,r+1} \le b^h_{j,r}, \\ \forall j,\ r \in \mathbb{N}\ \exists h \in \mathbb{N} & b^h_{j,r} > 0. \end{array}$$

For each $r \in \mathbb{N}$ put $B_r := (b^h_{j,r})_{j,h}$ and

$$\lambda_\infty(B_r) = \{x = (x_1, x_2, \ldots) : q_h^r(x) := \sup_j |x_j| b^h_{j,r} < +\infty, \forall h \in \mathbb{N}\}.$$

Then the space $\lambda_\infty(B_r)$ is a Fréchet space with fundamental system of seminorms $(q_h^r)_h$ and the inclusion map $j_{r+1,r} \colon \lambda_\infty(B_r) \hookrightarrow \lambda_\infty(B_{r+1})$ is continuous for every $r \in \mathbb{N}$. Following [13, § 5], we set $E^\infty =$

$\cup_r \lambda_\infty(B_r)$ endowed with the finest topology for which all the inclusion maps $j_r\colon \lambda_\infty(B_r) \hookrightarrow E^\infty$ are continuous, i.e. $E^\infty = \mathrm{ind}_r\, \lambda_\infty(B_r)$. As before, we denote by e_j the j–th unit vector of E^∞. Moreover, in [13, 5.14] it has been shown that E^∞ is a complete (LF)–space if, and only if, it is a regular (LF)–space if, and only if, the matrix $(b^h_{j,r})_{j,h,r}$ is of type (WQ), i.e. $(b^h_{j,r})_{j,h,r}$ satisfies the following condition:

$$\forall \mu \in \mathbb{N}\; \exists n,\, k \in \mathbb{N}\; \forall m,\, K \in \mathbb{N}\; \exists N \in \mathbb{N}, S > 0\; \forall i \in \mathbb{N}:\; b^m_{i,n} \le S(b^k_{i,\mu} + b^N_{i,K}).$$

We have:

Theorem 3 *Let $E = \mathrm{ind}_m\, E_m$ be a regular (LF)–space. Suppose that $E^\infty = \mathrm{ind}_r\, \lambda_\infty(B_r)$ is a complete (LF)–space. Then the following conditions are equivalent:*

(i) $L(E, E^\infty) = LB(E, E^\infty)$;

(ii) for each sequence $((k(n,j))_j)_n$ of increasing sequences of positive integers and for each increasing sequence $(r(n))_n$ of positive integers

$$\begin{gathered}\exists r \in \mathbb{N}\; \forall m \in \mathbb{N}\; \exists k \in \mathbb{N}\; \forall h \in \mathbb{N}\; \exists j_0,\, n_0 \in \mathbb{N}\; \exists C > 0 : \\ b^h_{l,r} p'^m_k(u) \le C \max_{\substack{1\le n\le n_0\\ 1\le j\le j_0}} b^j_{l,r(n)} p'^n_{k(n,j)}(u),\end{gathered} \tag{14}$$

for every $l \in \mathbb{N}$ and $u \in E'$ with $p'^n_{k(n,j)}(u) < +\infty$ for each j and $n \in \mathbb{N}$.

PROOF. (i)⇒(ii). Let $(k(n,j))_j)_n$ be a countable set of increasing sequences of positive integers. Let $(r(n))_n$ be another increasing sequence of positive integers. Since $L(E, E^\infty) = LB(E, E^\infty)$, we can apply Theorem 1 to conclude that condition (5) holds. Accordingly, we show that condition (14) holds too.

Let $u \in E'$ with $p'^m_{k(n,j)}(u) < +\infty$ for every j and $n \in \mathbb{N}$. Let $l \in \mathbb{N}$. Consider the map $T\colon E \to E^\infty$ defined by $T(x) := u(x)e_l$. Clearly, T is linear and continuous. In particular, for each j and $n \in \mathbb{N}$:

$$\begin{aligned}\sup_{x\in U^n_{k(n,j)}} q^{r(n)}_j(T(x)) &= \sup_{x\in U^n_{k(n,j)}} |u(x)| q^{r(n)}_j(e_l) \\ &= \sup_{x\in U^n_{k(n,j)}} |u(x)| b^j_{l,r(n)} \\ &= p'^n_{k(n,j)}(u) b^j_{l,r(n)} < +\infty\end{aligned}$$

Therefore, by condition (5) we obtain that:

$$\begin{aligned}b^h_{l,r} p'^m_k(u) &= \sup_{x\in U^m_k} q^r_h(T(x)) \le C \max_{\substack{1\le n\le n_0\\ 1\le j\le j_0}} \sup_{x\in U^n_{k(n,j)}} q^{r(n)}_j(T(x)) \\ &= C \max_{\substack{1\le n\le n_0\\ 1\le j\le j_0}} b^j_{l,r(n)} p'^n_{k(n,j)}(u).\end{aligned}$$

This complete the proof.

(ii)⇒(i). Let $T \in L(E, E^\infty)$. Then $T \circ i_n \in L(E_n, E^\infty)$ for every $n \in \mathbb{N}$. Since E_n is a Fréchet space and E^∞ is a (LF)–space, we can apply Grothendieck's factorization theorem [11, 5.8.38] to find an increasing sequence $(r(n))_n$ of positive integers such that $T \circ i_n \in L(E_n, \lambda_\infty(B_{r(n)}))$ for every $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$ there is another increasing sequence $(k(n,j))_j$ of positive integers such that

$$\forall j \in \mathbb{N} \quad \sup_{x\in U^n_{k(n,j)}} q^{r(n)}_j(T(x)) = d_{nj} < +\infty. \tag{15}$$

By condition (14), accordingly there is $r \in \mathbb{N}$ such that

$$\begin{gathered}\forall m \in \mathbb{N}\, \exists k_m \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists j_0,\, n_0 \in \mathbb{N}\, \exists C > 0 : \\ b^h_{l,r} p'^m_{k_m}(u) \le C \max_{\substack{1 \le n \le n_0 \\ 1 \le j \le j_0}} b^j_{l,r(n)} p'^n_{k(n,j)}(u),\end{gathered} \tag{16}$$

for every $l \in \mathbb{N}$ and $u \in E'$ with $p'^n_{k(n,j)}(u) < +\infty$ for each j and $n \in \mathbb{N}$.

This implies that $T(U^m_{k_m}) \subset \lambda_\infty(B_r)$ and bounded here. To see this we proceed as follows.

Let $g_l \colon E^\infty \to \mathbb{K}$, $x \to x_l$. Put $u_l = g_l \circ T$. Then $T(x) = (u_l(x))_l$. Moreover, for each j and $n \in \mathbb{N}$, by (15) we get:

$$\begin{aligned}\sup_l b^j_{l,r(n)} p'^n_{k(n,j)}(u_l) &= \sup_l b^j_{l,r(n)} \sup_{x \in U^n_{k(n,j)}} |u_l(x)| = \sup_{x \in U^n_{k(n,j)}} \sup_l b^j_{l,r(n)} |u_l(x)| \\ &= \sup_{x \in U^n_{k(n,j)}} q^{r(n)}_j (T(x)) = d_{nj} < +\infty\end{aligned}$$

which assures us that $p'^n_{k(n,j)}(u_l) < +\infty$ for all $l \in \mathbb{N}$. Therefore, we can apply (16) to obtain that:

$$\begin{aligned}\sup_{x \in U^m_{k_m}} q^r_h(T(x)) &= \sup_{x \in U^m_{k_m}} \sup_l |u_l(x)| b^h_{l,r} \le \sup_{x \in U^m_{k_m}} \sup_l p^m_{k_m}(x) p'^m_{k_m}(u_l) b^h_{lr} \\ &\le \sup_l p'^m_{k_m}(u_l) b^h_{lr} \le \sup_l C \max_{\substack{1 \le n \le n_0 \\ 1 \le j \le j_0}} b^j_{l,r(n)} p'^n_{k(n,j)}(u_l) \\ &= C \max_{\substack{1 \le n \le n_0 \\ 1 \le j \le j_0}} \sup_l b^j_{l,r(n)} p'^n_{k(n,j)}(u_l) \\ &= C \max_{\substack{1 \le n \le n_0 \\ 1 \le j \le j_0}} d_{nj} < +\infty.\end{aligned}$$

It follows that $T(U^m_{k_m}) \subset \lambda_\infty(B_r)$ and bounded here. Since $\lambda_\infty(B_r)$ is a metrizable space, there is a sequence of scalars $(\lambda_m)_m$ such that $T(\cup_m \lambda_m U^m_{k_m}) = \cup_m \lambda_m T(U^m_{k_m})$ is also a bounded set of $\lambda_\infty(B_r)$, where $\Gamma(\cup_m \lambda_m U^m_{k_m})$ is clearly a 0–neighbourhood in E. Now, the proof is complete. ■

Consequently, it is easily to prove:

Theorem 4 *Suppose that E^1 is a regular (LF)–space and that E^∞ is a nuclear (LF)–space. Then the following conditions are equivalent:*

(i) $L(E^1, E^\infty) = LB(E^1, E^\infty)$*;*

(ii) for each sequence $((k(n,j))_j)_n$ of increasing sequences of positive integers and for each increasing sequence $(r(n))_n$ of positive integers

$$\begin{gathered}\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists j_0,\ n_0 \in \mathbb{N}\, \exists C > 0 : \\ \forall i,\, l \in \mathbb{N}\ \ (a^k_{i,m})^{-1} b^h_{l,r} \le C \max_{\substack{1 \le n \le n_0 \\ 1 \le j \le j_0}} (a^{k(n,j)}_{i,n})^{-1} b^j_{l,r(n)}. \quad \square\end{gathered} \tag{17}$$

We now observe that, if the matrix $A_m = A = (a^k_i)_{i,k}$ for every $m \in \mathbb{N}$, then the space E^1 turns to be the Fréchet space $\lambda_1(A)$, where the sets defined by

$$U_k := \{x = (x_1, x_2, \ldots) : \sum_i |x_i| a^k_i \le 1\}$$

form a basis of 0–neighbourhoods in $\lambda_1(A)$. Then, argumenting as we did to prove Theorem 2 and using Proposition 8, we obtain:

Proposition 12 *Let F be a complete (LF)–space. Then the following condition are equivalent:*

(i) $L(\lambda_1(A), F) = LB(\lambda_1(A), F)$*;*

(ii) for each increasing sequence $(k(n))_n$ of positive integers and for each $r \in \mathbb{N}$

$$\begin{aligned}&\exists k,\ r_0 \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \exists C > 0 : \\ &(a_i^k)^{-1} q_h^{r_0}(y) \leq C \max_{1 \leq n \leq n_0} (a_i^{k(n)})^{-1} q_n^r(y),\end{aligned} \tag{18}$$

for every $y \in F_{r(1)}$ and $i \in \mathbb{N}$. □

On the other hand, if $(a_{i,m}^k)_i = (a_{i,m})_i = a_m$ for every k and $m \in \mathbb{N}$, the space $\lambda_1(A_m)$ turns to be the Banach space $l^1(a_m) = \{x = (x_1, x_2, \ldots) : ||x||_m = \sum_i |x_i| a_{i,m} < +\infty\}$ and hence $E^1 = k_1 = \operatorname{ind}_m l^1(a_m)$ is an (LB)–space. Then, argumenting again as we did to prove Theorem 2 and using Proposition 10, we get:

Proposition 13 *Let $k_1 = \operatorname{ind}_m l^1(a_m)$ be a regular (LB)–space and let $F = \operatorname{ind}_r F_r$ be a complete (LF)–space. Then the following conditions are equivalent:*

(i) $L(k_1, F) = LB(k_1, F)$*;*

(ii) for each increasing sequence $(r(n))_n$ of positive integers

$$\begin{aligned}&\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists j_0,\ n_0 \in \mathbb{N}\, \exists C > 0 : \\ &(a_{i,m})^{-1} q_h^r(y) \leq C \max_{\substack{1 \leq n \leq n_0 \\ 1 \leq j \leq j_0}} (a_{i,n})^{-1} q_j^{r(n)}(y),\end{aligned} \tag{19}$$

for every $y \in F_{r(1)}$ and $i \in \mathbb{N}$. □

Clearly, we can obtain results similar to the ones of Propositions 7, 9 and 11 in case $E = E^1$ and F is a Fréchet space or in case $E = E^1$ and F is a complete (LB) or DF–space, or in case $E = k_1$ and F is a complete (LB) or DF–spaces respectively. To prove this facts it suffices to argument as we did in the proof of Theorem 2 and to use Proposition 7, 9 and 11, respectively. Actually, one gets:

Proposition 14 *Let F be a Fréchet space. Suppose that E^1 is a regular (LF)–space. Then the following conditions are equivalent:*

(i) $L(E^1, F) = LB(E^1, F)$*;*

(ii) for each increasing sequence $(k(n))_n$ of positive integers and for each $m \in \mathbb{N}$

$$\begin{aligned}&\exists k \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \exists C > 0 : \\ &(a_{i,m}^k)^{-1} q_h(y) \leq C \max_{1 \leq n \leq n_0} (a_{i,n}^{k(n)})^{-1} q_n(y),\end{aligned} \tag{20}$$

for every $y \in F$ and $i \in \mathbb{N}$. □

Proposition 15 *Let F be a complete (LB) or DF–space. Suppose that E^1 is a regular (LF)–space. Then the following conditions are equivalent:*

(i) $L(E^1, F) = LB(E^1, F)$*;*

(ii) for each pair $((k(n))_n, (r(n))_n)$ *of increasing sequences of positive integers*

$$\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists k \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \exists C > 0 : \\ (a^k_{i,m})^{-1} q_r(y) \le C \max_{1 \le n \le n_0} (a^{k(n)}_{i,n})^{-1} q_{r(n)}(y), \tag{21}$$

for every $y \in F_{r(1)}$ *and* $i \in \mathbb{N}$. □

Proposition 16 *Let* $k_1 = \mathrm{ind}_m\, l^1(a_m)$ *be a regular (LB)–space and let* F *be a complete (LB) or DF–space. Then the following conditions are equivalent:*

(i) $L(k_1, F) = LB(k_1, F)$*;*

(ii) for each increasing sequence $(r(n))_n$ *of positive integers*

$$\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \exists C > 0 : \\ (a_{i,m})^{-1} q_r(y) \le C \max_{1 \le n \le n_0} (a_{i,n})^{-1} q_{r(n)}(y), \tag{22}$$

for every $y \in F_{r(1)}$ *and* $i \in \mathbb{N}$. □

Next, if the matrix $B_r = B = (b^h_j)_{j,h}$ for all $r \in \mathbb{N}$, then the space E^∞ turns to be a Fréchet space, i.e. $E^\infty = \lambda_\infty(B)$, where the sets so defined

$$V_h = \{x = (x_1, x_2, \dots) : \sup_j |x_j| b^h_j \le 1\}$$

form a basis of 0–neighbourhoods in $\lambda_\infty(B)$. Then, a proof similar to the one of Theorem 3 togheter with Proposition 7 gives:

Proposition 17 *Let* E *be a regular (LF)–space. Then the following conditions are equivalent:*

(i) $L(E, \lambda_\infty(B)) = LB(E, \lambda_\infty(B))$*;*

(ii) for each increasing sequence $(k(n))_n$ *of positive integers and for each* $m \in \mathbb{N}$

$$\exists k \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \exists C > 0 : \\ b^h_j p'^m_k(u) \le C \max_{1 \le n \le n_0} b^n_j p'^n_{k(n)}(u), \tag{23}$$

for every $u \in E'$ *with* $p'^n_{k(n)}(u) < +\infty$ *for all* $n \in \mathbb{N}$ *and* $j \in \mathbb{N}$. □

On the other hand, if $(b^h_{j,r})_j = (b_{j,r})_j = b_r$ for all $r \in \mathbb{N}$, the space $\lambda_\infty(B_r)$ turns to be the Banach space $l^\infty(b_r) = \{x = (x_1, x_2, \dots) : |x|_r = \sup_j |x_j| b_{j,r} < +\infty\}$ and hence the space $E^\infty = k_\infty = \mathrm{ind}_r\, l^\infty(b_r)$ is an (LB)–space. Then, by repeating the same argument used in the proof of Theorem 3 and by using Proposition 9, we get:

Proposition 18 *Let* E *be a regular (LF)–space. Suppose that* k_∞ *is a complete (LB)–space. Then the following conditions are equivalent:*

(i) $L(E, k_\infty) = LB(E, k_\infty)$*;*

(ii) for each pair $((k(n))_n, (r(n))_n)$ *of increasing sequences of positive integers*

$$\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists k,\ n_0 \in \mathbb{N}\, \exists C > 0 : \\ b_{j,r} p'^m_k(u) \le C \max_{1 \le n \le n_0} b_{j,r(n)} p'^n_{k(n)}(u), \tag{24}$$

for every $j \in \mathbb{N}$ *and* $u \in E'$ *with* $p'^n_{k(n)}(u) < +\infty$ *for all* $n \in \mathbb{N}$. □

Finally, we can also obtain results similar to the ones of Propositions 8, 10 and 11 in case E is a Fréchet space and $F = E^\infty$ or in case E is a regular (LB)–space and $F = E^\infty$, or in case E is a regular (LB)–space and $F = k_\infty$, respectively. The proof of this facts is based upon the same argument used in the proving Theorem 3 and upon Propositions 8, 10 and 11, respectively. Actually, we have:

Proposition 19 *Let E be a Fréchet space. Suppose that E^∞ is a complete (LF)–space. Then the following conditions are equivalent:*

(i) $L(E, E^\infty) = LB(E, E^\infty)$*;*

(ii) for each increasing sequence $(k(n))_n$ of positive integers and for each $r \in \mathbb{N}$

$$\begin{gathered}\exists k,\ r_0 \in \mathbb{N}\, \forall h \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \exists C > 0 : \\ b^h_{j,r_0} p'_k(u) \le C \max_{1\le n\le n_0} b^n_{j,r} p'_{k(n)}(u),\end{gathered} \tag{25}$$

for every $j \in \mathbb{N}$ and $u \in E'$ with $p'_{k(n)}(u) < +\infty$ for all $n \in \mathbb{N}$. □

Proposition 20 *Let E be a regular (LB)–space. Suppose that E^∞ is a complete (LF)–space. Then the following conditions are equivalent:*

(i) $L(E, E^\infty) = LB(E, E^\infty)$*;*

(ii) for each increasing sequence $(r(n))_n$ of positive integers

$$\begin{gathered}\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \forall h \in \mathbb{N} \exists j_0,\ n_0 \in \mathbb{N}\, \exists C > 0 : \\ b^h_{l,r} p'_m(u) \le c \max_{\substack{1\le n\le n_0\\ 1\le j\le j_0}} b^j_{l,r(n)} p'_n(u),\end{gathered} \tag{26}$$

for every $u \in E'$ and $l \in \mathbb{N}$. □

Proposition 21 *Let E be a regular (LB)–space. Suppose that k_∞ is a complete (LB)–space. Then the following conditions are equivalent:*

(i) $L(E, k_\infty) = LB(E, k_\infty)$*;*

(ii) for each increasing sequence $(r(n))_n$ of positive integers

$$\begin{gathered}\exists r \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \exists C > 0 : \\ b_{j,r} p'_m(u) \le C \max_{1\le n\le n_0} b_{j,r(n)} p'_n(u),\end{gathered} \tag{27}$$

for every $j \in \mathbb{N}$ and $u \in E'$. □

We also note that, by combining the various cases E^1, $\lambda_1(A)$ and k_1 together with the cases E^∞, k_∞ and $\lambda_\infty(B)$ (the latter spaces all have to be taken nuclear), we can obtain results similar to the one of Theorem 4. It is left to the reader the easy duty to find the appropriate characterizations in every possible and significant case.

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