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Non linear phenomena in Glaciology: ice-surging and streaming

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Abstract. In this survey we present some physically based models which have been recently proposed to deal with surging and streaming of ice sheets flowing along soft and deformable beds. These phenomena are related to a transition from slow (or zero) flow to fast flow and may result from the disruption of the normal subglacial drainage system. They advocate for an explanation in terms of continuum mechanics and generate some highly non linear coupled systems of mixed type equations. Some recent results on the free boundary nature of the models and their weak formulations are presented and some numerical strategies to solve the resulting mathematical obstacle problems are discussed.

Fenómenos no lineales en glaciología: flujo rápido y corrientes de hielo.

Resumen. En estas notas presentamos algunos modelos físicos que han sido propuestos recientemente para tratar el problema de los movimientos repentinos y casi periodicos del hielo, así como la aparición de corrientes de hielo rápidas en los grandes mantos glaciares que se deslizan sobre lechos blandos y deformables. Estos fenómenos están relacionados con la transición de un régimen de flujo lento a uno rápido y pueden aparecer debido a una modificación del sistema de drenaje del glaciar. Los fenómenos en cuestión se pueden justificar a través de la mecánica de medios continuos, desembocando en sistemas de ecuaciones altamente no lineales de tipo mixto. A continuación se presentan algunos resultados recientes, relacionados con la formulación débil de los modelos, considerados como problemas de frontera libre. También se incluye la discusión sobre algunas estrategias numéricas que se pueden emplear para resolver los *problemas de obstáculo* resultantes.

1. Introduction

One of the major current problems in glacier physics is to improve our understanding of sliding and processes in subglacial sediments. These notes contain a brief survey of some physically based models recently proposed in Glaciology to deal with such geophysical scenario and the resulting flow patterns. Modelling ice sheet dynamics has been a challenging problem since the beginning of the past century, but nowadays the scientific community is showing a renewed, growing interest toward this problem. This is not surprising because large ice sheets[1] influence, and are influenced by, climate and our understanding of the climate

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[1]By ice sheets we mean glaciers that are continuous sheets moving outwards in all directions and approximately of the size of Antarctica or Greenland.

system dynamics depends ultimately on the comprehension and predictability of the ice sheets dynamics. The intimate connections and feedbacks with the atmosphere, the lithosphere and the oceans are also essential for a global comprehension of the problem which could lead to a successful modelling. Consequently a multi disciplinary approach is required for solving ice sheets problems and in fact glaciology, usually introduced as a branch of fluid mechanics, involve oceanography, paleo-climatology, geology, geophysics, material sciences, hydrology and applied mathematics among other sciences.

As a consequence of this effort, various theories have appeared, during the last decades, in order to explain the flow of these large ice masses but a proper mathematical and numerical treatment is not yet available.

This introduces the main aim of these notes, which is to present the deduction and the mathematical analysis of some non linear obstacle problems related to the study of the flow of ice sheets along a soft, deformable bed.

This paper is organized as follows: after a brief introduction to the surging phenomenon we present the derivation of the Fowler and Johnson ice sheet model (Fowler and Johnson [20]). This is a simple but fundamental model which can be applied to different geophysical scenarios. As a first application, we consider the Hudson Strait model (Fowler and Schiavi [24]) and its numerical resolution. The free boundary nature of the problem and its mathematical analysis are then considered (Díaz and Schiavi [12]). As a second application, with a different geometry, the Siple Coast model (Fowler and Johnson [20]) is also analyzed and some recent numerical results (Muñoz, Schiavi and Kindelán[31], [32]) are presented.

The highly non linear character of the models equations, their couplings and the free boundary nature of the problems are considered (with a view to improve our understanding and numerical treatment of the possible singularities of the models). In this framework future research lines and open problems are finally discussed.

2. Surging models

In this section we introduce some basic terminology. For the geophysical aspects of the problem we shall follow Johnson [26]; more details can be found in Fowler [18], and in its wide bibliography part of which appears at the end of this survey.

A glacier, after showing little sign of activity for many years, suddenly starts to move rapidly and its surface is transformed to a chaotic mass of crevasses or ice pinnacles. Typically, the ice in the lower part of the glacier will move several kilometers in a few months, or at most, a few years. The rapid movement will then suddenly stop. This is very different to normal flow in which the terminus may advance a few meters for year. Such behavior is called *surging*.

It has been assumed for a long time that a glacier or ice sheet rests on an rigid, non deformable bed. But a glacier or ice sheet, wholly or in part, may rest on glacial till[2]. If the till does not deforme but it is permeable it will influence the sliding velocity by either allowing water to drain through it or by allowing additional water to reach the bed. If the basal shear stress and the pore pressure in the till are sufficiently large the till itself will deform. Hence, the problem arise of determining a sliding law for ice over a deformable till. The viscosity of the till is also likely to depend on water pressure p_w through the effective pressure N ($N = p_i - p_w$), where p_i is the ice overburden pressure) since a decrease in effective pressure allows increased mobility of the till constituents. Therefore the water pressure in the till must be determined and this involves a description of the subglacial drainage pattern. This is discussed in more detail in the following sections where a simplified model of a two-dimensional ice sheet is then described (see Fowler and Johnson [20], [19]). It includes basal ice sliding dependent on the basal water pressure, which itself is described by a simple theory of basal drainage. This model makes use of a lumped parameter

[2]For the purpose of these notes, till is defined as a sediment type of glacial origin, comprising relative coarse clasts in a finer-grained matrix. Till can creep slowly under the influence of stresses imposed by the overlying ice, and can be eroded and transported by flowing subglacial melt water (see [26]).

version of a boundary layer theory developed by Fowler [17], based on an original idea due to Nye [34] and subsequently developed by Lliboutry [29]. This is that large ice sheets have dynamics which can be characterized by *shear layers* near the base where the shear is largest due to the high stresses and high temperature there. In addition, relatively high velocities cause thermal gradients to be elevated in a thermal boundary layer near the base with the shear layer lying inside this. Because of the boundary layer nature of the flow, it is possible to produce a parameterized model which encapsulates the dynamics.

2.1. The Hudson Strait model

The Antarctic and Greenland ice sheets are the two mayor present day examples of ice sheets but during the last ice age (terminating about 10.000 years ago) ice sheets existed in North America (the Laurentide) and Northern Europe (the Fennoscandian), the ice extending into Southern England and Northern Europe. These ice sheets interact with climate, and their oscillations may be responsible for sudden shifts in climate in the recent geological past. Evidence of periodic deposition (Heinrich [25]) of ice-rafted debris (Heinrich events) in North Atlantic deep sea sediments suggests that the Laurentide ice sheet may have surged regularly with a time of order 10^4 years between surges (Andrews and Tedesco [2], Clark [10]). The primary candidate for such surges is the ice stream discharging down the Hudson Strait (Bond *et al.* [5]), although other ice streams may have been involved (Bond and Lotti [6]). MacAyeal [30], suggested a mechanism whereby the ice sheet might behave in this way. Briefly, if the ice is thin, then it is cold-based, sluggish and so thickens; this causes the base to melt, and if the resultant water production is sufficient to allow fast sliding to occur, then a surge may result, drawing the ice down to a thin state once more.

Fowler and Johnson [19], presented a physically based model of ice sheet motion controlled by basal sliding, which itself depends on basal water production. They showed that the basal water produced by frictional heat at the base could lead to a multivalued sliding law and to multiple ice flux/ice thickness relationships, and hence oscillatory behavior by what they called *hydraulic runaway*[3]. The idea that a multivalued sliding law could lead to surging behavior has been around for long time and it is perhaps due to Lliboutry [28]. It is well known (when studying nonlinear relaxation oscillators, e.g., the van der Pol oscillator) that such a multivaluedness will produce in a system a periodic solution of relaxation type, i.e., a *surge*. Current opinion is that the effective multiplicity is mediated through a switch in the subglacial drainage characteristics. The point is then to establish a realistic physical process whereby the sliding law can be multivalued. This is done in the Fowler and Johnson's ice sheet model which describes the interplay between ice dynamics, basal sliding (the relation between basal velocity, shear stress, water pressure and the characteristics of the bed) and the subglacial hydrology (the mode of flow of water at the beds of ice sheets). These concepts are elucidated further in the next sections where the Fowler and Schiavi [24], theory of the ice sheet surges is introduced.

2.2. The Fowler-Johnson ice sheet model

The model is described in more detail by Fowler and Johnson ([20], [19]), and is reviewed for completeness here where we follow Fowler and Schiavi's ([24]) application to the Hudson Strait flow area (where the flow is channelised). We consider a two-dimensional ice sheet on a flat base of thickness $h(x,t)$, where x is a horizontal coordinate, and we assume that $0 < x < l$, with $h_x = \partial h/\partial x = 0$ at $x = 0$ (the divide) and $h = h_l > 0$ at $x = l$ which represents the transition at $x = l$ from ice sheet to ice shelf. The basic equation is that of mass conservation,

$$h_t + (uh)_x = a, \tag{1}$$

where subscripts denote partial derivatives, a is the accumulation rate, and u is a depth-averaged horizontal velocity (essentially it is a sliding velocity at least while the base is at the melting point). The sliding law,

[3]The term hydraulic runaway was introduced by analogy with thermal runaway that has been previously suggested as an instability mechanism for ice sheets (Clarke *et al.* [11]). However, in the glaciological context, one can show (Fowler and Larson [21], [22]) that the free boundary nature of the problem renders the solution (in one approximate limit) both unique and linearly stable. Whether thermal runaway is viable is thus not clear.

relating shear stress τ to velocity u, is

$$\tau = cu^r N^s \tag{2}$$

(Fowler[14], Bindschadler [4]). Here r and s are exponents which may be between zero and one, and c is a frictional critical coefficient. The values $r = s = 1/2$ correspond to the view of Boulton and Hindmarsh [8], that coarse (sandy) tills have values of r, s of $O(1)$. The idea of Kamb [27] is that for clay-rich marine sediments, the sliding law is more nearly plastic, so that $s \approx 1$, $r \ll 1$. The basal shear stress is simply

$$\tau = -\rho g h h_x, \tag{3}$$

where ρ is ice density and g is the gravitational acceleration.

The extra variable N which appears in the sliding law (2) is the effective pressure (ice overburden pressure minus water pressure), introduced into basal sliding laws by Lliboutry [28]; it must be described by a basal drainage theory. It is fundamental to this theory of surges that a drainage law of the type predicted by Walder and Fowler [37], should be applicable. This was based on the physics of drainage over wet deformable sediments, such as the detrital carbonates of the Hudson Strait, and in its simplest form can be written as

$$N = c^*/Q^{1/3}, \tag{4}$$

where c^* (the drainage coefficient) is a constant relating to the deformable till properties, and Q is the water flux (per channel) in a distributed system of canals which drain the bed. A number of simplifications are built into (4), and we return to these later. In particular, note that (4) is an equilibrium theory (steady drainage), and also that (4) cannot apply as $Q \to 0$.

The final relation determines the water flux Q. For steady drainage, this is given by

$$\frac{\partial Q}{\partial x} = \frac{(G + \tau u - q)w_d}{\rho_w L}, \tag{5}$$

where G is geothermal heat flux, τu is the frictional heat generated by the basal ice motion, q is the cooling rate to the cold ice above, ρ_w is water density, L is latent heat, and w_d is the mean inter-canal spacing. Fowler and Johnson[20], parameterized the cooling rate to the ice by means of a thermal boundary layer description:

$$q = \Delta T \left(\frac{\rho c_p k}{\pi}\right)^{1/2} \frac{u}{\xi^{1/2}} + \frac{k \Delta T}{h}, \tag{6}$$

where

$$\xi = \int_0^x u dx, \tag{7}$$

$-\Delta T$ is the ice surface temperature, c_p is specific heat, and k is thermal conductivity. The first of these is a heat transfer term associated with rapid ice flow, the second is representative of conductive cooling.

The equations (1) to (7) are those posed by Fowler and Johnson (1996) when the basal ice is at the melting point. They must be supplemented by appropriate relationships when the basal ice is cold. We will consider this situation further in subsequent sections.

Scaling the model

The model consists of the equations (1)—(7) for the variables h, u, τ, N, Q, q and ξ, with boundary conditions

$$h_x = 0, \quad Q = 0 \text{ at } x = 0, \quad h = h_l \text{ at } x = l,$$

and an initial condition for h. The model is then non-dimensionalized as described by Fowler and Johnson [20], in terms of scales $[h], [u], [\tau]$, etc., where they defined:

$$[t] = \frac{[h]}{[a]} = \frac{[x]}{[u]}, \qquad [x] = l, \qquad [Q] = \frac{[\tau][u]w_d l}{\rho_w L}, \qquad [N] = \frac{c^*}{[Q]^{1/3}}$$

$$[\tau] = c[u]^r[N]^s, \qquad [\tau] = \rho g[h]^2/l, \qquad [q] = [\tau][u], \qquad \xi = [u]/l.$$

When the variables are written in terms of these scales, we find the dimensionless form of the model

$$\begin{aligned} h_t + (hu)_x &= a, \\ \tau &= u^r N^s, \\ \tau &= -hh_x, \\ Q_x &= \tau u + \gamma - \beta \frac{u}{\xi^{1/2}} - \frac{\lambda}{h}, \\ N &= Q^{-1/3}, \\ \xi &= \int_0^x u dx, \end{aligned} \tag{8}$$

where the dimensionless parameters γ, β, λ have typical values of order one. For example, taking $r = s = 1/2$, and choosing

$$[a] = 0.2 \text{ m y}^{-1}, \ \ l = 2000 \text{ km}, \ \ [N] = 0.4 \text{ bars},$$

where $[a]$ is the accumulation rate scale (a in (2.9) is the dimensionless accumulation rate). The value of $[N]$ corresponds to ice stream conditions in ice stream B (Bentley [3]), and if we choose c correspondingly, then we find

$$[h] \sim 1500 \text{ m}, \ \ [u] \sim 250 \text{ m y}^{-1}, \ \ [t] \sim 8000 \text{ y},$$

and

$$\lambda \approx 0.75, \ \ \gamma \approx 0.55, \ \ \beta \approx 1.2 \quad .$$

2.3. Numerical solution

A preliminary numerical resolution of the Hudson model can be found in Fowler and Johnson [19]. They just suggested the possibility of surging in their model by means of a crude 'lumped' version which essentially reduced the model (8) to a zero-dimensional one. Although they later (Fowler and Johnson [20]) offered some (right) comments on the possible form of spatially extended surges a proper numerical solution of the spatially extended model remained to be done. An attempt was done in Fowler and Schiavi [24], where they showed, numerically, the existence of *surging solutions*[4] and how the magnitude and duration of the surge is controlled by the drainage response time scale. The basic equations were written in the following form. Elimination of N and τ in (8) enables the sliding law to be written in the form

$$u = -Q^S h^R |h_x|^{R-1} h_x,$$

where $R = 1/r$, $S = s/(3r)$. The dimensionless equations are:

$$\begin{aligned} h_t + (hu)_x &= a, \\ u &= Q^s(-hh_x)^R, \\ Q_x &= \gamma + uhh_x - \frac{\beta u}{\left(\int_0^x u dx\right)^{1/2}} - \frac{\lambda}{h}. \end{aligned}$$

Defining the diffusion coefficient $D = h^{R+1}|h_x|^{R-1}Q^S$ the equation for h takes the form

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[D \frac{\partial h}{\partial x}\right] + a, \tag{9}$$

[4]By surging solutions we mean solutions of system (8) such that the evolution of the (maximum) thickness at the divide, $h(0, t)$ and of the outlet velocity at the margin, $u(1, t)$, have an oscillatory sine-like wave behavior. This is consistent with McAyeal´s idea of the North Atlantic major surging episodes and provides a mathematical formulation of the phenomenom.

and is of nonlinear diffusion type, with $D = D(h, h_x, Q)$, and Q is obtained by quadrature of $(8)_4$. We solved (9) using an implicit diffusion solver (also iterating the diffusion step by a simple predictor-corrector strategy) and an improved Euler discretization for Q and ξ (the results have been validated comparing them with those obtained using a more accurate fourth order Runge-Kutta solver).

Results

Numerical computations indicate a series of surges, the course of one of which is shown in Figure 1. We choose $\gamma = 0.3$, $\beta = 1.2$, $\lambda = 0.75$, and with $\Delta t = 10^{-4}$, $\Delta x = 10^{-3}$, $R = 1.1$, $S = 0.4$, we find that a front propagates backwards as shown in figure 1[5].

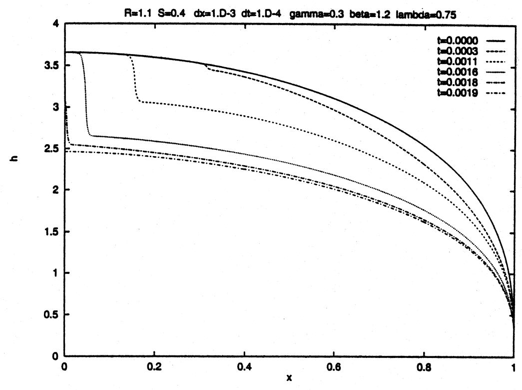

Figure 1. Propagation of a surge front backwards in the course of a numerical solution of the model (8), using parameter values $R = 1.1$, $S = 0.4$ (corresponding to $r = 0.91$ and $s = 1.09$), $\gamma = 0.3$, $\beta = 1.2$, and $\lambda = 0.75$. The apparent wave speed is about 200 and the shock width is about 0.01, whereas the space step is 10^{-3} and the time step is 10^{-4}. Thus the shock structure is resolved, but it moves by about 200×10^{-4}, i.e. about 0.02 in a single time step. The numerical solution is not solving the model.

Figures 2 and 3 show associated plots of u and Q. We see that very high spikes of u and a sharp jump of Q are associated with the propagation of this front. The jumps are in fact smooth for a sufficiently fine mesh, and they suggest the existence of a travelling wave. However, efforts to analyze the local structure of this travelling wave failed, and in fact a closer inspection reveals that the apparent wave speed V is greater than $\Delta x/\Delta t$, which indicates that the algorithm is not resolving the wave. Reduction to values $\Delta x = 10^{-4}$, $\Delta t = 10^{-6}$, only served to sharpen the profiles, and raise the maximum values of u and Q in the front. The implication is that the problem as formulated does not have smooth solutions, and we might conjecture that (weak) solutions in which h is discontinuous and u has delta function like behavior do exist.

2.4. Modifications to the model

Considering the solution type represented in figure 1, we see that it is realistic, but the model is unable to be solved satisfactorily, because of the absence of relaxation processes which allow for a switch in basal conditions. In fact the ice sheets can flow in either of two stable flow regimes (Fowler and Johnson [19]), which depend on the amount of basal water present. At low Q, slow flow prevails, while for higher Q, fast flow occurs. The occurrence of multiple flow states can lead to self-sustained oscillations, as explained by Fowler [15], but in order to prevent discontinuities in the flow, it is necessary to include relaxation processes

[5] All figures from 1 to 12, have been reprinted from the Journal of Glaciology, see Fowler and Schiavi [24], with permission of the International Glaciological Society.

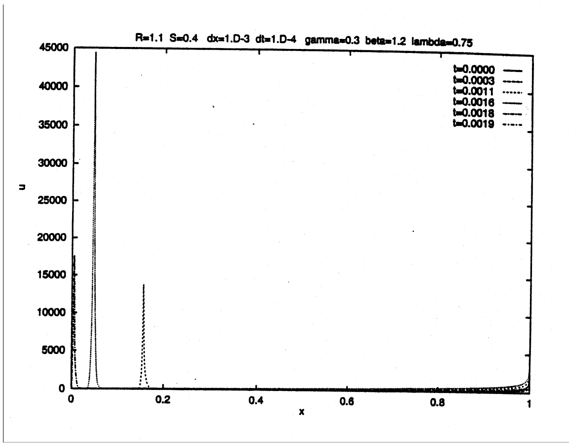

Figure 2. Propagation backwards of a velocity pulse in the surge. Note the scale of u, which exhibits delta function like behavior.

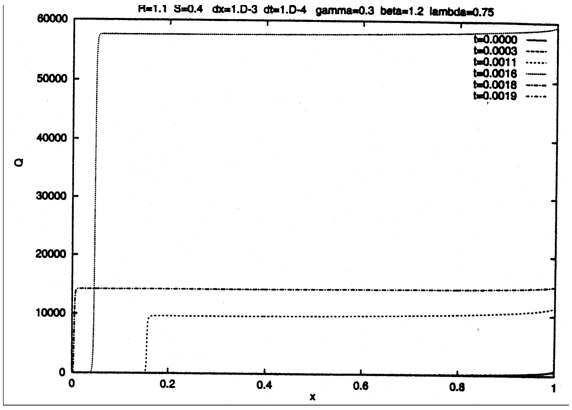

Figure 3. Growth of water flux and its backwards propagation in the surge of figure 1.

between the different regimes. Fowler [16], in a model very similar to this describing glacier surges, showed that inclusion of such terms allowed smooth solutions involving the rapid propagation of wave fronts in the surge, and we follow this suggestion here.

Drainage relaxation

The description of time dependent drainage is a complicated problem (Nye [35], Fowler and Ng [23]), and we choose to represent the response time of the drainage system by modifying $(8)_5$ as follows:

$$\delta N_t + N = [Q + \bar{Q}]^{-1/3}. \tag{10}$$

The term in δ is analogous to that introduced by Fowler [16], and is a simple representation of the relaxation of N towards equilibrium. δ is the ratio of the drainage response time (perhaps a month or less) to the ice sheet growth time scale, so that values of $\delta \sim 10^{-6}$ can be expected. The term $\bar{Q}$ is introduced so that when $Q \to 0$, the pore pressure cannot go below zero (or, N cannot go above the overburden ice pressure). In fact, it is convenient to choose $\bar{Q}$ (which is small) so that when $Q = 0$, the equilibrium value of N is such that the sliding law $(8)_2$ describes realistically the small amount of shearing in the ice, which is described below.

Freezing base

We now seek to extend the model more realistically to accommodate what happens when $Q = 0$. There are two principal basal situations to consider: till-based and sediment-based. By *till-based* we mean that a layer of basal deformable till lies between the ice and an essentially impermeable bedrock; while *sediment-based* means that the (permeable) sediments below the ice extend to a significant depth.

For a sediment-based ice sheet, let $s(x,t)$ denote the depth of the frost line below the ice-sediment interface. If the sediment porosity is ϕ, then the motion of the frost line is approximately governed by the Stefan condition

$$\rho_w \phi L \dot{s} = -[G - q],$$

and when this is scaled as in §2 (with $s \sim [h]$), we obtain the dimensionless equation

$$\lambda \phi St Pe \dot{s} = -(\gamma - q), \tag{11}$$

where the Stefan number St and Peclet number Pe are defined by

$$St = \frac{L}{c_p \Delta t}, \quad Pe = \frac{[a][h]}{\kappa_w},$$

and $\kappa_w = k/\rho_w c_p$. With previous estimates, and taking $\kappa_w = 38$ m^2 y^{-1}, $\Delta T = 50$ K, $c_p = 2 \times 10^3$ J kg^{-1} K^{-1}, we have $St \sim 3.4$, $Pe \sim 7.9$. With $\phi \approx 0.4$, we then have $\phi St Pe \sim 10.7$, which implies that (for times of $O(1)$) s is relatively small, and the conductive temperature in the frozen sediment implies the following (scaled) thermal relation for the basal ice temperature T_0:

$$T_0 = -qs/\lambda, \tag{12}$$

where q is the dimensionless heat flux, which is now taken to be given by

$$q = \frac{\lambda}{h+s} + \frac{\beta u}{\xi^{1/2}}. \tag{13}$$

We discuss the theory for a till-based ice sheet further below.

Shear flow in the ice

When the ice is cold at the bed, we may take the sliding velocity to be zero, but there is a small component of ice velocity due to shearing within the ice. Fowler[17], showed that the shear is concentrated in a shear layer near the base which gives an effective plug flow velocity in the ice of (with the present definition of the scales)

$$u \approx \frac{\bar{\varepsilon}\lambda}{\bar{\gamma}\gamma} \tau^n e^{\bar{\gamma} T_0}, \tag{14}$$

where $\bar{\gamma} = Q\Delta T/RT_m^2$, is the activation exponent defined by Fowler[17], and

$$\bar{\varepsilon} = \frac{A[\tau]^n [h]}{[u]}, \tag{15}$$

where A is the flow exponent. Taking $A = 0.2$ bar^{-3} y^{-1} (Paterson[36]), $n = 3$, $[\tau] = 0.1$ bar, $[h] = 1500$ m, $[u] = 250$ m y^{-1}, we have $\bar{\varepsilon} \sim 10^{-3}$, so that even if $T_0 \approx 0$, the shearing velocity is very small (note also that $\bar{\gamma} \approx 11$). We define

$$s = \nu \Sigma, \qquad \nu = \frac{1}{\lambda \phi St Pe} \sim 0.1,$$

$$\gamma^* = \nu\gamma \sim 1, \qquad \lambda^* = \frac{\bar{\varepsilon}\lambda}{\bar{\gamma}\gamma} \sim 10^{-3},$$

so that (12), (13) and (14) give, approximately,

$$u \approx \lambda^* \tau^n \exp(-\gamma^* \Sigma / h), \tag{16}$$

while (11) is (since $u \ll 1$)

$$\Sigma_t \approx -\left[\tau u + \gamma - \frac{\beta u}{\xi^{1/2}} - \frac{\lambda}{h + \nu\Sigma}\right]. \tag{17}$$

The equations (16) and (17) apply when $Q = 0$.

In solving the model, it is convenient to choose $\bar{Q}$ so that the equilibrium value of N as computed ($\bar{Q}^{-1/3}$ when $Q = 0$), causes the sliding law to give the shearing velocity (16). We find that this is the case if we choose

$$\bar{Q} \approx \left\{\lambda^* \tau^{n - \frac{1}{r}} \exp\left(-\frac{\gamma^* \Sigma}{h}\right)\right\}^{3r/s}, \tag{18}$$

which we promptly do.

Water flux adjustment

We may also modify the equation for Q by including a term $\delta' Q_t$ on the left hand side of $(8)_4$. In this case, a stable numerical method is to define the time step as $\Delta t = \delta' \Delta x$, so that integration from (i, j) to $(i + 1, j + 1)$ is along the characteristics, and is implemented with an improved Euler method.

Summary

The model we now solve is the following:

$$\begin{gathered} h_t + (hu)_x = a, \\ \tau = u^r N^s, \\ \tau = -h h_x, \\ \delta' Q_t + Q_x = f = \tau u + \gamma - \frac{\beta u}{\xi^{1/2}} - \frac{\lambda}{h + \nu\Sigma} \ \text{ if } \ Q > 0, \\ \Sigma_t = -f \ \text{ if } \ \Sigma > 0, \\ \delta N_t + N = (\bar{Q} + Q)^{-1/3}, \\ \xi = \int_0^x u\,dx, \end{gathered} \tag{19}$$

with $\bar{Q}$ given by (18). This reduces to (8) if $\delta, \delta', \lambda^*$ and ν are put to zero.

2.5. Numerical method

The model is solved for h as before. As mentioned, Q is solved along characteristics by choosing $\Delta t = \delta' \Delta x$; both N and Q are solved with the improved Euler method, and we also time step Σ forward using an improved Euler method from $(i, j - 1)$ to (i, j).

Results

In figures 4 and 5, we show the course of a surge using a value of $\delta = 10^{-2}$, with space step $\Delta x = 2 \times 10^{-4}$ and time step $\Delta t = 2 \times 10^{-6}$. Figures 4 and 5 show the evolution of the central thickness $h_{\max} = h(0, t)$ and the outlet velocity $u_{\text{out}} = u(1, t)$ as functions of time.

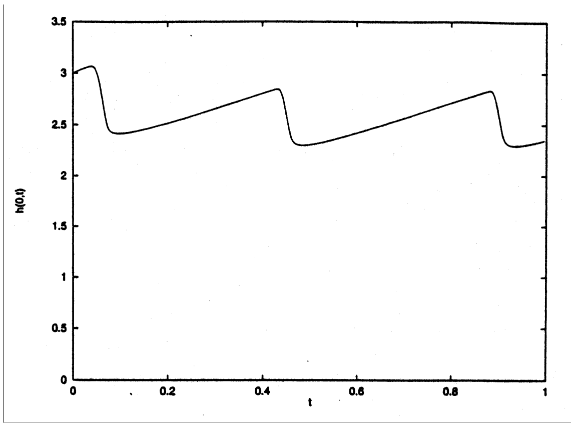

Figure 4. Evolution of the maximum thickness $h(0,t)$ at the ice divide, for values $\lambda = 0.75$, $\gamma = 0.55$, $\beta = 1.2$, $\delta = 10^{-2}$, $\delta' = 10^{-2}$.

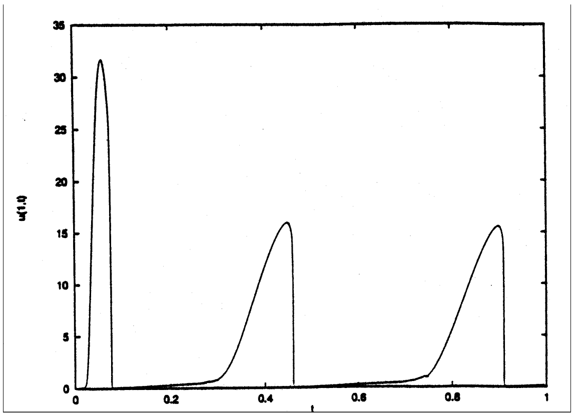

Figure 5. Evolution of the outlet velocity $u(1,t)$, as for figure 5.

We see a pattern of surging at regular intervals, with a period of the order of the convective time scale. The surge is initiated by an *activation wave* which propagates backwards from the margin at a rate controlled by the drainage pressure relaxation time, and which enables the transition from cold based to temperate-based dynamics. The propagation of this wave is shown in figures 6 to 8.

When the activation wave reaches the divide, a *deactivation wave* is formed which propagates forwards (at a faster speed which is controlled by the water flux relaxation time) causing the base to freeze again. The outlet ice flux at the margin is shut down very rapidly when this wave travels towards the front, and the slow build-up begins again. The propagation of this deactivation wave is shown in figures 9 to 12.

Surge activation and deactivation

During a surge, a wave front propagates backwards at a rate controlled by the drainage response time. The wave is illustrated in figures 6 to 8. We were not practically able to compute this for realistically small values of δ and δ', and therefore an useful asymptotic representation of this wave is deduced. Details can

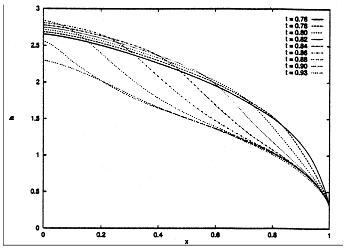

Figure 6. Backwards propagation of the activation wave in the depth profile.

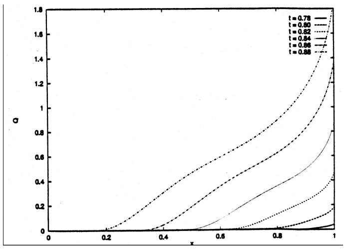

Figure 7. Backwards propagation of the activation wave in the water flux.

be found in Fowler and Schiavi [24].

When the activation wave reaches the divide, a deactivation wave propagates rapidly downstream, thus switching the surge off. We can see from figures 9 to 12 that this deactivation wave is associated with the propagation of the cold-temperate transition point downstream.

2.6. The multivalued Fowler-Johnson Ice Sheet model

The above numerical analysis shows that the treatment of cold-temperate transition points (the free boundary) is a delicate matter and that a more sophisticated numerical method should be employed due to the existence of weak, not classical, solutions of the model equations (we refer to Fowler and Schiavi [24], for the basic structure of the algorithm). This is subsequent to a correct weak formulation and analysis of the model and to the characterization of the associated free boundary in terms of an obstacle problem.

In this section we sketch the mathematical analysis (Díaz and Schiavi [12]) of the one dimensional hydrological flow model (8) with the term $Q + \bar{Q}$ or system (19) with $\delta = \delta' = \nu = 0$. Following a well established technique (see Duvaut and Lions [13]), Díaz and Schiavi [12], introduced a multivalued obstacle term (a maximal monotone graph) which allows to consider properly the interface between cold and temperate basal conditions. Both cold and temperate basal thermal regimes are then allowed at the ice/till interface so generalizing the previous theory on surges along soft temperate beds introduced by Fowler and Johnson [20]. Mathematically, a degenerate system of partial differential equations of mixed type is considered and, under certain circumstances, a moving boundary separating the cold and temperate basal regions can appear. By means of an iterative decoupling procedure and an implicit time discretization scheme we prove the existence of a weak bounded solution to the resulting semi-discretized (elliptic) system. The existence of a weak solution to the parabolic system is still an open problem (a partial result in this direction can be found in Díaz and Schiavi [12]). Moreover, by assuming some extra regularity on the solutions, a comparison property on the free boundary associated to the water flux in the basal hydrological system is also deduced. Sufficient conditions for the backward/forward propagation of the free boundary

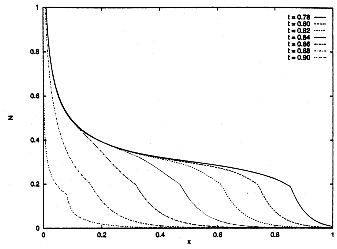

Figure 8. Backwards propagation of the activation wave in the effective pressure. Note that the values above the kink at about 0.2 apply when the ice is frozen at the base, and have no physical meaning. The value of N has been scaled down by a factor of 10^3.

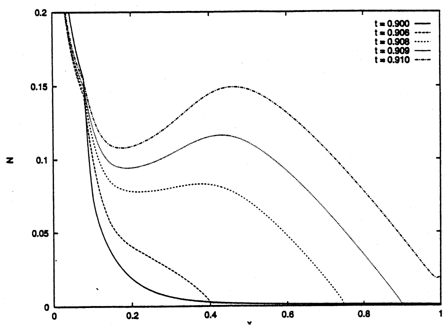

Figure 9. Forward propagation of the cold-temperate transition point in the deactivation wave. N is scaled as in figure 8.

are established.

2.7. The implicit discretized system

Let $T > 0$, $\Omega = (0,1)$, $R = 1/r$, $S = s/3r$ and define $p = R + 1$, $m = (2R+1)/R$. Notice that $p > 2$ and $m > 1$. In fact, our results will be valid for p and m arbitrary in this range of parameters. For $p \to \infty$ we have the pseudo-plastic behavior suggested by Kamb [27]. Elimination of N and τ in (8) allows the consideration of the following initial and boundary value problem: Given h_0, h_D, Q_D and an *accumulation rate* function $a(t,x)$, find three functions, h, Q and ξ satisfying

$$(S)\begin{cases} \partial_t h - \left[(\bar{Q}+Q)^S |(h^m)_x|^{p-2}(h^m)_x\right]_x = a \quad \text{in } (0,T)\times\Omega, \\ \partial_x Q + \beta(Q) \ni (\bar{Q}+Q)^S h^p |h_x|^p - \mu \xi_x \xi^{-1/2} + \gamma - \lambda h^{-1} \quad \text{in } (0,T)\times\Omega, \\ \partial_x \xi = (\bar{Q}+Q)^S h^{p-1}|h_x|^{p-1} \quad \text{in } (0,T)\times\Omega, \\ h(t,1) = h_D(t), \quad t \in (0,T), \\ h_x(t,0) = 0,\ Q(t,0) = Q_D(t),\ \xi(t,0) = 0 \quad t \in (0,T), \\ h(0,x) = h_0(x) \quad \text{on } \Omega. \end{cases}$$

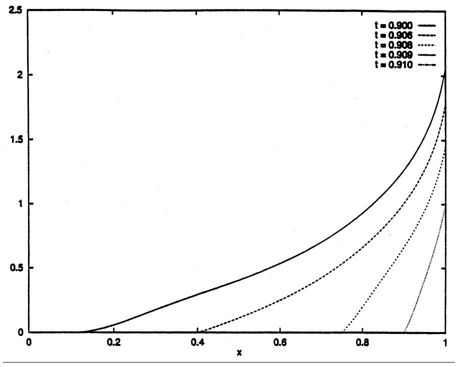

Figure 10. Forward propagation of water flux in the deactivation wave.

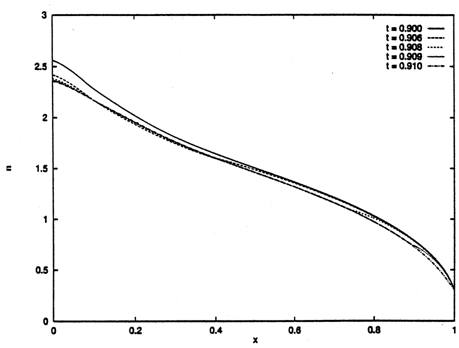

Figure 11. Adjustment of h in the deactivation wave. A slight adjustment of slope can be seen to propagate over the freezing front.

Here β denotes the maximal monotone graph defined by

$$\beta(r) = \emptyset \quad \text{if} \quad r < 0, \quad \beta(0) = (-\infty, 0], \quad \beta(r) = 0 \quad \text{if} \quad r > 0. \tag{20}$$

This graph is introduced to deal with the cold ($Q = 0$)/temperate ($Q > 0$) transition at the bed. It allows only non-negative (physically meaningful) water flux amounts. The coefficients γ, μ, λ are $O(1)$ dimensionless parameters. The small, real positive constant $\bar{Q}$, $0 < \bar{Q} \ll 1$ represents the ice shearing component in the flow when $Q = 0$. We shall indicate later the notion of solution for solving the system and, in particular, the multivalued equation for Q. Since $m > 1$ and $p > 2$ the equation for h becomes degenerate and so, as is well known, we cannot expect to have classical solutions. Notice also the singular term $\xi^{-1/2}$ arising in the equation for Q. In order to introduce the notion of weak solution we shall follow the usual procedure of the *abstract semigroup theory* which relies on the idea of considering the associated Euler implicit semidiscretized scheme. More precisely, given a positive integer number N and letting $k = T/N$ (the time step of the discretization) we denote by $I_{k,n} = (t_{n-1}, t_n) = ((n-1)k, nk)$, $(n = 1, ...N$, $t_n = nk)$ the associated sub-intervals of $(0, T)$. Let $V \doteq V_h \times V_\xi \times V_Q$ be the Banach space defined by $V_h \doteq \{\phi \in W^{1,p}(\Omega) : \phi(1) = 0\}$, $V_\xi \doteq \{\psi \in W^{1,p'}(\Omega) : \psi(0) = 0\}$, $V_Q \doteq \{\eta \in W^{1,1}(\Omega) : \eta(0) = 0\}$ (where, as usual, $p' \doteq p/(p-1)$). We shall assume (Fowler and Schiavi [24]) the following hypothesis on the data of the problem:

$$Q_D,\, h_D \in C[0, T], \quad h_0 \in W^{1,p}(0, 1), \quad h_D(0) = h_0(1) \tag{21}$$

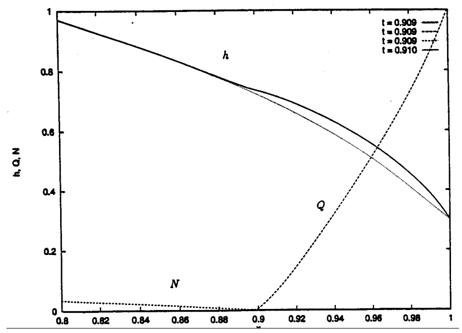

Figure 12. A snapshot of the wave, as exhibited by profiles of h, N, and Q. Also shown for comparison is the h profile following the passage of the wave. Note that N is scaled by 10^{-3}; behind the front at 0.9, $Q = 0$, while ahead of it, $N \approx 10^{-3}$.

$$\left.\begin{array}{l} M_D \geq h_D \geq m_D > 0,\ M_0 \geq h_0 \geq m_0 > 0, \text{ and } Q_D \geq 0, \\ \text{for some constants } M_D > m_D > 0 \text{ and } M_0 > m_0 > 0. \end{array}\right\} \tag{22}$$

$$a \in L^\infty((0,T)\times\Omega), \quad a > 0, \tag{23}$$

$$\left.\begin{array}{l} a(t,\cdot) \text{ is a non increasing function, for a.e. } t \in (0,T) \\ h_0'(x) < 0 \quad a.e.\, x \in (0,1) \end{array}\right\} \tag{24}$$

It is useful to introduce the following notation

$$A = (\bar{Q}+Q)^S,\ B = h^p|h_x|^p,\ C = \mu\,\xi_x\,\xi^{-1/2},\ D = \lambda h^{-1},\ E = h^{p-1}|h_x|^{p-1}. \tag{25}$$

Term A comes from the sliding law and reflects the importance of the lubricating water flux at the ice/till interface; B represents the frictional heating term, C and D are (respectively) the advective and conductive cooling terms while E is the amount of shear in the sliding law. Fixed $n < N$ and $k < T$, we define the piecewise constant in time approximations of the data in the usual manner (see, e.g., Alt and Luckhaus [1]):

$$h_{D_{k,n}}(t) \doteq \frac{1}{k}\int_{(n-1)k}^{nk} h_D(s)\,ds, \quad Q_{D_{k,n}}(t) \doteq \frac{1}{k}\int_{(n-1)k}^{nk} Q_D(s)\,ds \quad \forall\, t \in I_{k,n}.$$

Let also

$$a_{k,n}(t,x) \doteq \frac{1}{k}\int_{(n-1)k}^{nk} a(s,x)\,ds \quad a.e.\, t \in I_{k,n}, \text{ and a.e. } x \in \Omega.$$

We can now consider the stationary system

$$(\boldsymbol{S}_{k,n}) \begin{cases} \partial_t^{-k} h_{k,n} - \left[A_{k,n}|(h_{k,n}^m)_x|^{p-2}(h_{k,n}^m)_x\right]_x = a_{k,n} & \text{in } (0,T)\times\Omega, \\ \partial_x Q_{k,n} + \beta(Q_{k,n}) \ni (\bar{Q}+Q_{k,n})^S B_{k,n} - C_{k,n} + \gamma - D_{k,n} & \text{in } (0,T)\times\Omega, \\ \partial_x \xi_{k,n} = A_{k,n}E_{k,n} & \text{in } (0,T)\times\Omega, \\ h_{k,n}(t,1) = h_{D_{k,n}}(t) & t \in (0,T), \\ (h_{k,n})_x(t,0) = 0,\ Q_{k,n}(t,0) = Q_{D_{k,n}}(t),\ \xi_{k,n}(t,0) = 0 & t \in (0,T), \\ h_{k,0}(0,x) = h_0(x) & \text{on } \Omega, \end{cases}$$

where

$$\partial_t^{-k} h_{k,n}(t,.) \doteq \frac{h_{k,n}(.) - h_{k,n-1}(.)}{k} \quad \forall n = 1,..N, \ \text{ if } t \in I_{k,n}$$

and $A_{k,n}$, $B_{k,n}$, $C_{k,n}$, $D_{k,n}$ and $E_{k,n}$ are piecewise constant in time functions defined as in (25) replacing h, ξ and Q by $h_{k,n}$, $\xi_{k,n}$ and $Q_{k,n}$.

Definition 1 *Given a, h_D, Q_D and h_0 satisfying hypothesis (21), (22), (23), (24) and $a_{k,n}$, $h_{D_{k,n}}$, $Q_{D_{k,n}}$, $\xi_{D_{k,n}}$ the associated discretized functions, we say that $(h_{k,n}, \xi_{k,n}, Q_{k,n})$ is a* weak solution *of $(\boldsymbol{S}_{k,n})$ if*

$$(h^m_{k,n}(t,.), \xi_{k,n}(t,.), Q_{k,n}(t,.)) \in [h^m_D + V_h] \times V_\xi \times [Q_D + V_Q], \quad a.e.\ t \in (0,T),$$

there exists $b_{k,n}$ with $b_{k,n}(t,x) \in \beta(Q_{k,n}(t,x))$ a.e. $t \in (0,T)$ and $x \in (0,1)$, $b_{k,n}(t,\cdot) \in L^1(\Omega)$, $h^{-1}_{k,n}(t,\cdot) \in L^\infty(\Omega)$, $(\xi_{k,n}(t,\cdot))^{-\frac{1}{2}} \in L^1(\Omega)$ a.e. $t \in (0,T)$ and the following integral conditions hold

$$\int_0^1 \partial_t^{-k} h_{k,n}(t)\phi + \int_0^1 (\bar{Q} + Q_{k,n})^S |(h^m_{k,n})_x|^{p-2} (h^m_{k,n})_x \phi_x = \int_0^1 a\,\phi$$

$$\int_0^1 \xi_{k,n}\psi_x + \int_0^1 (\bar{Q} + Q_{k,n})^S h^{p-1}_{k,n} |(h_{k,n})_x|^{p-1}\,\psi \;=\; \xi_{k,n}(1,t)\psi(1,t)$$

$$\int_0^1 Q_{k,n}\eta_x + \int_0^1 \left[(\bar{Q} + Q_{k,n})^S h^p_{k,n} |(h_{k,n})_x|^p \,+ \gamma\right]\eta \;=\; Q_{k,n}(1,t)\eta(1,t) +$$

$$+\,\mu \int_0^1 (\xi_{k,n})_x\,(\xi_{k,n})^{-1/2}\eta \,+ \lambda \int_0^1 h^{-1}_{k,n}\eta + \int_0^1 b_{k,n}\eta$$

for any vectorial test function $(\phi,\ \psi,\ \eta) \in V_h \times V_\xi \times V_Q$.

In Díaz and Schiavi [12], we present the mathematical analysis of the above system showing the existence of a weak solution of the stationary system. Here we just sketch the basic ideas used to prove the existence result. In what follows, given a function $f : (0,T) \times \Omega \to \mathbb{R}$ we shall write $f \in L^1(\Omega)$ to express that $f(t,\cdot) \in L^1(\Omega)$ for a.e. $t \in (0,T)$.

2.8. Existence of weak solutions via an iterative scheme.

In order to prove the existence of a weak solution of $(\boldsymbol{S}_{k,n})$ it is possible to use an iterative process which allows the system to be uncoupled into three separate problems: $P(h^j_{k,n})$, $P(\xi^j_{k,n})$ and $P(Q^j_{k,n})$ (see the definition below). Then we obtain some a priori estimates which allow us to prove the convergence of such an iterative scheme. The decoupled problem is the following: For each j we shall find three functions $(h^j_{k,n})$, $(\xi^j_{k,n})$ and $(Q^j_{k,n})$ satisfying

$$(\boldsymbol{S}^j_{k,n}) \begin{cases} \partial_t^{-k} h^j_{k,n} - \left[A^{j-1}_{k,n} |\partial_x[(h^j_{k,n})^m]|^{p-2} \partial_x[(h^j_{k,n})^m]\right]_x = a_{k,n} & \text{in } (0,T)\times\Omega, \\ \partial_x \xi^j_{k,n} = A^{j-1}_{k,n} E^j_{k,n} & \text{in } (0,T)\times\Omega, \\ \partial_x Q^j_{k,n} + \beta(Q^j_{k,n}) \ni (\bar{Q} + Q^j_{k,n})^S B^j_{k,n} - C^j_{k,n} + \gamma - D^j_{k,n} & \text{in } (0,T)\times\Omega, \\ h^j_{k,n}(t,1) = h_{D_{k,n}}(t,1) & t \in (0,T), \\ (h^j_{k,n})_x(t,0) = 0, Q^j_{k,n}(t,0) = Q_{D_{k,n}}(t), \xi^j_{k,n}(t,0) = 0 & t \in (0,T), \\ h^j_{k,n}(0,x) = h_0(x) & \text{on } \Omega. \end{cases}$$

where the coefficient $A_{k,n}^{j-1} \doteq (\bar{Q} + Q_{k,n}^{j-1})^S$, $A_{k,n}^0 \doteq \bar{Q}^S$, is assumed to be known, positive and uniformly bounded at each $j-$step of the iterative process, i.e., $A_{k,n}^{j-1} \geq \bar{Q}^S > 0$, $||A_{k,n}^{j-1}||_{L^\infty} \leq C$. By using suitable supersolutions of the multivalued water flux equation it is proved in that these conditions hold uniformly in j, k and n. The main result is as follows

Theorem 1 *Assume the data satisfying (21), (22), (23), (24). Then for any $j \in \mathbb{N}$ there exist three functions $(h_{k,n}^j, \xi_{k,n}^j, Q_{k,n}^j)$ verifying $(\boldsymbol{S}_{k,n}^j)$. Moreover the sequence $(h_{k,n}^j,\ \xi_{k,n}^j,\ Q_{k,n}^j)$ converges to $(h_{k,n},\ \xi_{k,n},\ Q_{k,n})$, solution of $(\boldsymbol{S}_{k,n})$, when $j \to \infty$.* $\square$

On the free boundary

Assuming extra regularity on the solutions of the previous obstacle formulation, some results on the existence and the behavior of the free boundary associated to the Q function (which represents the amount of water flux produced by frictional heating in the basal drainage system) can also be found in Díaz and Schiavi [12]. In this regard it is convenient to consider the following complementary formulation:

Complementary formulation

Let $T > 0$ and $\Omega = (0,1)$. System (S) admits the following complementary formulation: find three functions h, Q y ξ satisfying:

$$
\begin{gathered}
\partial_t h - \partial_x \left[(\bar{Q}+Q)^S h^{R+1} |h_x|^{R-1} h_x \right] = a \qquad \text{in } (0,T)\times\Omega \\
\partial_x Q - (\bar{Q}+Q)^S\, h^{R+1}|h_x|^{R+1} + \mu\, \xi_x\, \xi^{-1/2} - \gamma + \lambda h^{-1} \geq 0, \qquad \text{in } (0,T)\times\Omega, \\
Q \geq 0, \qquad \text{in } (0,T)\times\Omega, \\
\left(\partial_x Q - (\bar{Q}+Q)^S h^{R+1}|h_x|^{R+1} + \mu\, \xi_x\, \xi^{-1/2} - \gamma + \lambda h^{-1} \right) Q = 0, \quad \text{in } (0,T)\times\Omega, \\
\partial_x \xi \ = \ (\bar{Q}+Q)^S h^R |h_x|^R \qquad \text{in } (0,T)\times\Omega.
\end{gathered}
$$

$$
\left.
\begin{array}{lcll}
h(t,1) & = & h_D(t,1) & t \in (0,T) \\
h_x(t,0) & = & 0 & t \in (0,T) \\
Q(t,0) & = & Q_D(t) & t \in (0,T) \\
\xi(t,0) & = & 0 & t \in (0,T) \\
h(0,x) & = & h_0(x) & \text{on } \Omega
\end{array}
\right\}
$$

By means of some comparison properties of the solutions of the water flux equation we can find sufficient conditions to have a moving interface at the base of the ice sheet, separating cold regions ($Q = 0$) from the temperate ones ($Q > 0$). Also it is possible to give sufficient conditions for the backward propagation of this interface and this is consistent with the numerical results obtained in Fowler and Schiavi [24]. The proofs are quite technical and we remit to Díaz and Schiavi [12], for the details.

3. Ice Stream Generation

Ice streams are like enormous glaciers, conveyors of cracked ice and snow stretching across vast portions of Antarctica. Ice streams can switch between slow and fast modes of flow. The example of *Slessor Glacier*, an East Antarctic ice stream, shows in fact a sharply delineated transition from slow sheet flow to rapid streaming flow. The transition from slow (sheet) to fast (stream) flow is spatial as well as temporal and represents a challenging modelling problem. It corresponds to transition from slow (or zero) sliding to fast sliding and can be considered a transition from the non-surging to the surging state.

3.1. The Siple Coast model

In this section we consider sheet flow over a deformable bed, where the width of the flow is much greater than the depth and the flow is fed from a stable catchment. As an application, we choose the Siple Coast region, West Antarctic. The basic equations are those of Fowler and Johnson [20], and the model is based on the Walder and Fowler [37], subglacial drainage theory. The model has been proposed to examine the effect of the possible instability in a laterally extensive flow. This is similar to, but different from, the Hudson model where an instability mechanism is proposed along the longitudinal, main flow direction. Having in mind the generation of ice stream flow similar to that in the Siple Coast we must consider lateral and longitudinal variations in the effective pressure so as to generate a lateral flow of water from regions of low effective pressure (ice streams) to regions of high effective pressure (slow flowing ice). Fowler and Johnson [20], conjectured that a uniform flow is laterally unstable and will break up into fast and slow moving portions and that these are precisely *ice streams*.

Figure 13. Geographical localization of the phenomenon.

3.2. Model equations

We consider a simple, rectangular geometry simulating the Siple Coast flow area, where x is the downstream longitudinal coordinate and y is the cross-stream latitudinal coordinate. Let also the vertical coordinate be z (with $z = 0$ at the base). The model derivation is as follows and can be found in Fowler and Johnson[20]. Let the horizontal ice velocity be u and h be the thickness of ice. The horizontal motion of ice sheets is driven by the shear stresses at depth generated by the surface slope of ice. This flow is thermally activated by the dependence of viscosity on temperature. Assuming $h = h(x)$ (see later discussion of this) the basal shear stress is given (in parametric form) by (3)

$$\tau = -\rho_i g h h_x,$$

where ρ_i is ice density, g is gravity and h is the ice depth (and where $h_x = dh/dx$). The heat flux into the plug flow of ice above the shear layer is given by (6),

$$q = \Delta T \left(\frac{\rho_i c_p k}{\pi}\right)^{1/2} \frac{u}{\xi^{1/2}} + \frac{k \Delta T}{h},$$

where $\Delta T = T_m - T_a$, c_p is the specific heat, k is the thermal conductivity, and (as in the surging model) the accumulated velocity ξ, is given by (7),

$$\xi = \int_0^x u dx.$$

The basal condition is as in the surging model, so when the base reaches the pressure melting point, sliding is initiated and we have that the sliding law is (2),

$$\tau = c u^r N^s,$$

with typical values $r \approx 1/2$, $s \approx 1/2$. To consider a laterally varying water pressure Fowler and Johnson prescribed (following Walder and Fowler[37]) a vector water flux $\vec{Q}_w = (Q_\parallel, Q_\perp)$ in the form:

$$\vec{Q}_w = \frac{\tilde{c}}{N^3} |\tilde{x} + \psi \nabla N| (\tilde{x} + \psi \nabla N), \tag{26}$$

where $\tilde{x}$ is the downslope unit vector and ψ is defined as:

$$\psi = \frac{1}{\rho_i g \sin(\alpha)}$$

where $\sin(\alpha) \approx -h_x$. Since in ice sheets, typically, $|\psi \nabla N| \ll 1$, they approximated the downslope and cross-stream water fluxes as

$$Q_\parallel = \frac{\tilde{c}}{N^3}, \qquad Q_\perp = \frac{\tilde{c}}{N^3} \psi \frac{\partial N}{\partial y}. \tag{27}$$

In steady flow, conservation of water flux is then

$$\frac{\partial Q_\parallel}{\partial x} = \frac{(G + \tau u - q) w_d}{\rho_w L} - \frac{\partial Q_\perp}{\partial y}. \tag{28}$$

The variation of Q with y suggests that we should allow for lateral variation of h also. However as a first approximation to the problem, we assume (see [20]) that $h \approx h(x)$. To consider a sheet flow fed from a stable catchment we need to prescribe an initial ice flux, and conservation of mass is then

$$h \int u dy = M, \tag{29}$$

where M is a prescribed (dimensional) ice flux entering the model catchment.

Non-dimensionalisation

The equations (2), (3), (6), (27), (28) and (29) are scaled by choosing scales

$$\begin{gathered} x \sim l \quad y \sim d, \quad u \sim [u], \quad \tau \sim [\tau], \quad h \sim [h] \\ Q_\parallel \sim [Q_\parallel] \quad Q_\perp \sim [Q_\perp], \quad N \sim [N], \\ |T_m - T_a| \sim \Delta T, \quad q \sim [q] \end{gathered} \tag{30}$$

where we define

$$\begin{gathered} d = (l \psi [N])^{1/2}, \quad [N] = \frac{\tilde{c}^{1/3}}{[Q_\parallel]^{1/3}}, \\ q = \Delta T \left(\frac{\rho_i c_p k}{\pi} \right)^{1/2} \left(\frac{[u]}{l} \right)^{1/2} = [\tau][u], \\ [Q_\parallel] = \frac{l w_d [\tau][u]}{\rho_w L}, \quad [Q_\perp] = \frac{\psi [N][Q_\parallel]}{d}, \end{gathered}$$

$$[\tau] = \rho_i g [h]^2/l = c[u]^r [N]^s. \tag{31}$$

From present conditions on ice stream B where $\tau = 0.15$ bars, $u = 500$ my^{-1}, $Q = 1$ m s^{-1}, $N = 0.4$ bars and $l = 400$ Km, (Bentley [3]), Fowler and Johnson inferred the values of c, $\tilde{c}$ and w_d as:

$$c = \frac{[\tau]}{[u]^r[N]^s}, \quad \tilde{c} = \frac{1}{3}\,\text{bar m s}^{-1/3}, \quad w_d = \frac{[Q_{\|}]\rho_w L}{[\tau][u]l}.$$

For values:

$$k = 2.1\,\text{W m}^{-1}\text{K}^{-1}, \quad \rho_1 = 917\ \text{Kg m}^3, \quad g = 9.8\ \text{m s}^{-2},$$
$$l = 400\,\text{Km} \quad c_p = 2009\,\text{J Kg}^{-1}\text{K}^{-1}, \quad G = 0.05\,\text{W m}^{-2},$$
$$\Delta T = 30\,\text{K} \quad \rho_w = 10^3\text{Kg m}^{-3}, \quad L = 335\,\text{kJ Kg}^{-1},$$

and $r = s = 1/2$ we find:

$$[h] \sim 775\,\text{m}, \quad d \sim 50\,\text{Km}, \quad [u] \sim 500\,\text{m y}^{-1}, \quad [\tau] \sim 0.15\,\text{bars},$$
$$[N] \sim 0.4\,\text{bars}, \quad [Q_{\|}] \sim 1\,\text{m}^3\text{s}^{-1}, \quad [q] \sim 2.5 \times 10^{-6}\,\text{bars ms}^{-1}. \tag{32}$$

These values are approximately representative of fast flowing ice streams, and are relevant to conditions of a temperate bed. The scales $[h]$ and d evolve from specifications within the other scales. Both $[h]$ and d accord with observed orders of magnitude, and suggest the consistency of the scales defined in (31). The corresponding dimensionless model equations are:

$$\begin{gathered}
Q_{\|} = \frac{1}{N^3}, \qquad Q_{\perp} = \frac{1}{N^3}\frac{\partial N}{\partial y}, \\
\frac{\partial Q_{\|}}{\partial x} = (\gamma + \tau u - q) - \frac{\partial Q_{\perp}}{\partial y}, \\
q = \frac{u}{\xi^{1/2}} + \frac{\delta}{h}, \\
\tau = -hh_x, \\
\tau = u^r N^s, \\
\xi = \textstyle\int_0^x u dx, \\
h\int_0^L u dy = M.
\end{gathered} \tag{33}$$

where L is the (scaled) width of the flow area, and M is the scaled mass flux (a positive, constant initial ice flux entering the model catchment). We are assuming a prescribed value of M at $x = 0$ with no accumulation so that the mass flux M is constant in x. The (dimensionless) parameters are defined by

$$\gamma = \frac{G}{[\tau][u]}, \qquad \delta = \left(\frac{\pi k l}{\rho_i c_p [u][h]^2}\right)^{1/2} \tag{34}$$

and using the values defined above, we have:

$$\gamma = 0.2, \qquad \delta = 0.36$$

With:

$$Q_{\|} = \frac{1}{N^3}, \qquad Q_{\perp} = \frac{1}{N^3}\frac{\partial N}{\partial y} = -\frac{1}{3}Q_{\|}^{-1/3}\frac{\partial Q_{\|}}{\partial y}$$

the water flux conservation equation takes the diffusive form (for $Q_{\|} = Q$)

$$\frac{\partial Q}{\partial x} = (\gamma + \tau u - q) + \frac{1}{3}\frac{\partial}{\partial y}\left[Q^{-1/3}\frac{\partial Q}{\partial y}\right]. \tag{35}$$

3.3. The multivalued Siple Coast model

The previous model has to be modified in order to deal, properly, with the cold/temperate transition at the base when $Q \to 0$. Also the vectorial drainage law, $(33)_1$, cannot strictly apply for a small Q, since it implies that $N \to \infty$ as $Q \to 0$. In order to avoid this, we argue as before (see (10)) and we replace Q by $(Q + \bar{Q})$ in the drainage law. Let $Q_{\|} = Q$, be a physically admissible (non negative) solution of the water flux equation (28). Then $h_x < 0$ and considering the sliding law and the drainage law, is easy to see that

$$u = (Q + \bar{Q})^S h^R |h_x|^R, \tag{36}$$

where $S = s/(3r)$ and $R = 1/r$, as before. Let $p = R + 1$. Then, the dissipation term is

$$\tau u = (Q + \bar{Q})^S h^p |h_x|^p.$$

Using (2) and (7) the cooling term q is

$$q = (Q + \bar{Q})^S \xi^{-1/2} h^{p-1} |h_x|^{p-1} + \frac{\delta}{h}.$$

The global heat balance $f(\xi, h, h_x, Q) = \gamma + \tau u - q$ at the bed can be written in form:

$$f(\xi, h, h_x, Q) = \gamma + (Q + \bar{Q})^S h^p |h_x|^p - (Q + \bar{Q})^S \xi^{-1/2} h^{p-1} |h_x|^{p-1} - \frac{\delta}{h}, \tag{37}$$

and the water flux conservation equation (35) is then

$$\frac{\partial Q}{\partial x} = \frac{1}{3}\frac{\partial}{\partial y}\left[(Q + \bar{Q})^{-1/3}\frac{\partial Q}{\partial y}\right] + f(\xi, h, h_x, Q), \tag{38}$$

which is a parabolic p.d.e. with non linear diffusion and non linear absorption/reaction in the source term (37), which, in fact, can be a changing sign function. More precisely, the equation (38), is of the type

$$\frac{\partial v}{\partial t} - \frac{\partial^2}{\partial x^2}(v^m) = f(x, v),$$

where $f(x, v) = g_1(x)v^S + g_2(x)$, with $m = 2/3$ and $0 < S \approx 1/3$, i.e. $0 < S < m < 1$. In this range of parameters, the equation (38) modelizes fast diffusion processes with strong absorption /reaction (see for instance, Borelli and Ughi [7] for a classification of this kind of non linear equations when $f(x, v) \leq 0$). Notice that, despite of the parabolic character of the equation (38), we are dealing with a stationary drainage theory. Nevertheless we shall use the parabolic nature of (38), to solve it numerically (see Muñoz, Schiavi and Kindelán [31], [32]). As in (20), we introduce the maximal monotone graph $\beta(Q)$

$$\beta(r) = \emptyset \quad \text{if} \quad r < 0, \qquad \beta(0) = (-\infty, 0], \qquad \beta(r) = 0 \quad \text{if} \quad r > 0,$$

to obtain the multivalued formulation

$$\frac{\partial Q}{\partial x} - \frac{1}{3}\frac{\partial}{\partial y}\left[(Q + \bar{Q})^{-1/3}\frac{\partial Q}{\partial y}\right] + \beta(Q) \ni f(\xi, h, h_x, Q). \tag{39}$$

A system for the variables $Q = Q(x, y)$, $h = h(x)$ and $\xi = \xi(x, y)$ can be obtained. In such a system, the mass conservation equation can be written in the following form (taking into account the drainage law, the sliding law and the momentum equation)

$$\frac{dh}{dx} = -M^{1/(p-1)} h^{p/(p-1)} \left(\int_0^L (Q + \bar{Q})^S dy\right)^{-1/(p-1)}. \tag{40}$$

Finally, using (7) and (36), the sliding law is:

$$\xi_x = (Q + \bar{Q})^S h^{p-1} |h_x|^{p-1} \tag{41}$$

where typically $S \approx 1/3$ and $p \approx 3$. We have to solve the coupled multivalued system (39),(40),(41) complemented with feasible initial conditions (at the divide) and boundary conditions (along the lateral boundaries). At the divide, we prescribe the water flux $Q(0, y) = Q_0(y)$, the ice thickness $h(0) = h_0$ and the accumulated velocity $\xi(0, y) = \xi_0(y)$. Usually we choose $\xi_0(y) \equiv 0.1$, and for $Q_0(y)$ a wavy profile corresponding to five bumps spaced 125 km apart, whose values oscillate between approximately 0.1 and $0.05\ \mathrm{m}^3\ \mathrm{s}^{-1}$. Along the lateral boundaries, we set Neumann homogeneous conditions (no flux) for Q:

$$Q_y(x, 0) = Q_y(x, L) = 0, \qquad \forall x \in (0, X).$$

Following Muñoz, Schiavi and Kindelán[31][32], the multivalued formulation is as follows: Let $X > 0$, $Y > 0$ be fixed positive real numbers and let Ω be a bounded domain, say $\Omega = (0, Y)$ with $(0, X) \times \Omega$ representing the Siple Coast flow area. Given $Q_0(y)$, h_0, $\xi_0(y)$ and an initial flux M, find three functions $Q(x, y)$, $h(x)$, $\xi(x, y)$ satisfying

$$(MF) = \begin{cases} \dfrac{\partial Q}{\partial x} - \dfrac{1}{3}\dfrac{\partial}{\partial y}\left[(Q + \bar{Q})^{-1/3}\dfrac{\partial Q}{\partial y}\right] + \beta(Q) \ni f(\xi, \xi_x, h, h_x, Q) \text{in} \ \ (0, X) \times \Omega, \\[2ex] \dfrac{\partial \xi}{\partial x} = (h|h_x|)^{p-1}(Q + \bar{Q})^S \text{ in } \quad (0, X) \times \Omega, \\[2ex] \dfrac{dh}{dx} = -M^{1/(p-1)} h^{p/(p-1)} \left(\displaystyle\int_0^L (Q + \bar{Q})^S dy\right)^{-1/(p-1)} \quad \text{in } (0, X) \times \Omega, \\[2ex] Q_y(x, 0) = Q_y(x, L) = 0, \qquad x \in (0, X), \\[2ex] Q(0, y) = Q_0(y), \qquad y \text{ on } \Omega \\[2ex] h(0) = h_0 \in \mathbb{R}^+, \\[2ex] \xi(0, y) = \xi_0(y), \qquad y \text{ on } \Omega. \end{cases}$$

where the heat balance is defined in (37).

3.4. Numerical results

Before considering the idea of introducing a monotone maximal graph in the water flux conservation equation (38), we made some attempts (see [31]) to solve numerically the system of equations (33) by using finite difference methods. In figures 14 and 15, we represent the water flux and the ice velocity fields (figure 14) and their level curves (figure 15). An inspection of the sliding law (36), shows that the variations of the ice velocity across the y-direction depend critically on the cross variations of the water flux in the drainage system and on the water flux. In fact, for small values of Q, near the cold-temperate transition region, the water flux cross variation is amplified by a factor inversely depending on the value of Q. Consequently, the variations of the ice velocity in the y-direction are much greater than the corresponding water flux variations in the y-direction along the streaming ice, and this indicates the importance of the drainage system in the overall movement of the ice sheet.

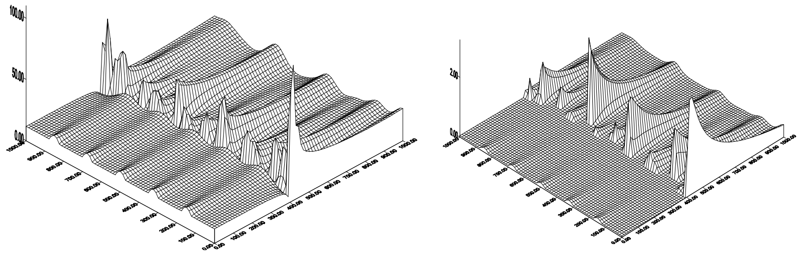

Figure 14. Ice velocity (left) and water flux (right), obtained by a finite difference method.

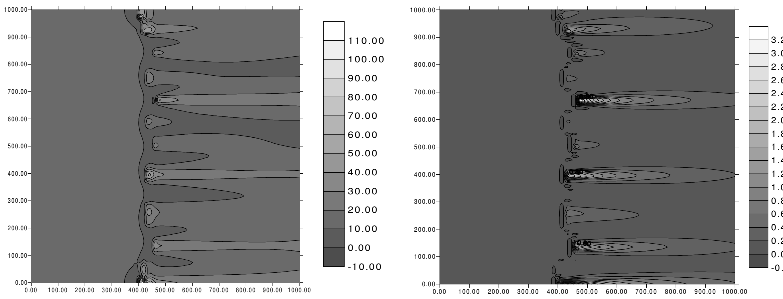

Figure 15. Ice velocity and water flux level curves, obtained by a finite difference method.

These results show the existence of fast flow regions alternated with regions of slow flow, i.e., ice streams. The water flux, prescribed at the divide, decrease initially until km 400 where a sharp transition takes place. The ice velocity increases in the temperate regions and a sustained ice flow is correspondingly generated. This is clearly indicated by the level curves in figure 15. Nevertheless our computations also showed that, for some special parametric values, it was possible to obtain very small, but negative, values for Q which are not physically admissible. This advocates for a formulation of the problem in terms of multivalued operators and the use of suitable numerical methods to deal with the variational inequalities associated to the obstacle problem formulation. Some preliminary numerical results on the multivalued formulation have been obtained in Muñoz, Schiavi and Kindelán [31], [32], where finite differences ([31]) and finite elements methods ([32]) have been employed (see figures 16, 17).

Figures 16 and 17 indicate that with our initial conditions we can generate fast ice streams (velocities are about 500 meters per year) of 50 km width and accord well with satellite data. These fast ice streams are associated with a qualitative change of the normal drainage system and suggest that fast dynamics can occur when a soft, deformable bed is considered.

4. Open problems and future research

There are several points of the above analysis which deserve a deeper investigation. All the basic aspects of the applied mathematics strategy (modelling, analysis and validation) are covered by the previous approach but many mathematical questions remain unsolved. With regards to the system of equations (S), the study of the implicit scheme's convergence (see section 2.8) seems a very delicate task due to the lack of information about the time dependence of the function $Q(x,t)$ and so about the coefficient of the nonlinear diffusion

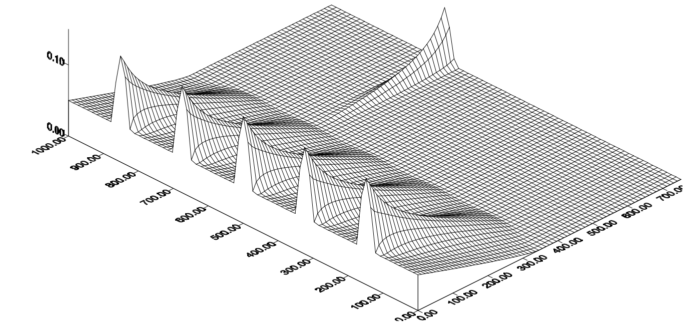

Figure 16. Water flux, obtained by a finite element method.

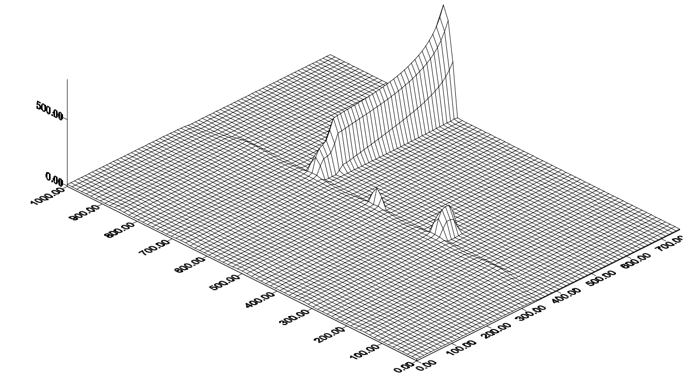

Figure 17. Ice velocity, obtained by a finite element method.

operator. The existence of a weak solution of system (S) is then still open. The same is true for the modified model (19) where a freezing base is considered and the (unknown) depth of frost line below the ice-sediment interface has to be determined. It represents a second free boundary in the model and its numerical approximation is still preliminary. Future work will involve to solve numerically the Hudson model (8) and its multivalued version (see system (S) and (20)) by using finite elements methods together with a duality algorithm for the associated complementary formulation. This kind of techniques have been recently used in the glaciological context by Calvo *et al.* [9] (see also references therein). Regarding the Siple Coast model we have some more mathematical and numerical results but from the physical point of view the modelling exercise is not finished. In fact we would like to get rid of the very special initial conditions prescribed at the divide. However, these fast ice streams are strongly dependent on the initial (divide) conditions for the water flux and this can be considered a weakness of the model. This would imply to consider a fully 2D version of the model which includes spatial (lateral) dependence of h and τ (see Muñoz, Schiavi and Kindelán ([33]) for some preliminary results in this direction and where a fully elliptic version of the Siple Coast model is deduced for the thickness and effective pressure formulation). Finally, some other questions such as: what controls the ice streams width, which is the role of marginal stress or how to model the transition between ice sheet and ice shelf are still unresolved. The treatment of the previous items appeals to a more intimate research and collaboration between glaciologists, oceanographers and mathematicians.

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