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# Topology of 2-D incompressible flows and applications to geophysical fluid dynamics

**Tian Ma and Shouhong Wang**

**Abstract.** This article presents a survey of a new dynamical systems theory for 2D incompressible flows and its applications to geophysical fluid dynamics.

**Topología de flujos incompresibles 2-D y aplicaciones a la dinámica de los fluídos geofísicos**

**Resumen.** Este artículo presenta un estudio de los desarrollos recientes de una nueva teoría de sistemas dinámicos para flujos incompresibles 2-D y sus aplicaciones a la dinámica de los fluídos geofísicos.

## 1. Introduction

The use of topological ideas in physics and fluid mechanics goes back to the very origin of topology as an independent science. In this article, we present a brief survey on a newly developed dynamical systems theory for 2D incompressible flows. This program of study consists of research in two areas: a) the study of the structure and its transitions/evolutions of divergence-free vector fields, and b) the study of the structure and its transitions of velocity fields for 2D incompressible fluid flows governed by the Navier-Stokes equations or the Euler equations.

Mathematically speaking, there are two general methods describing a fluid flow: the Euler representation and the Lagrange representation. For the Euler method, the motion and states of a fluid are described by a set of partial differential equations, such as the Euler equations or the Navier-Stokes equations supplemented with proper boundary conditions, and possibly with additional equations of passive scalars, depending on different physical situations. The Lagrange representation of a fluid flow, on the other hand, amounts to studying the dynamics and trajectories of fluid particles in the (two or three dimensional) *physical space that the fluid occupies*. Of course the velocities of the particles satisfy the PDEs we just mentioned. Our main philosophy is to classify the topological structure and its transitions of the *instantaneous* velocity field, treating the time variable as a parameter. The aforementioned two aspects of study are based directly on this philosophy.

The study in Area a) is more kinematic in nature, and the results and methods developed can naturally be applied to other problems of mathematical physics involving divergence-free vector fields. The main topics in this area include structural classification, structural stability, and structural bifurcation, as well as their applications to fluid dynamics and to geophysical fluid dynamics.

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In fluid dynamics context, the study in Area b) involves specific connections between the solutions of the Navier-Stokes or the Euler equations and flow structure in the physical space. We shall present in particular a new rigorous characterization of boundary layer separations for 2D viscous incompressible flows. As we know, boundary layer separations problem is a long standing problem in fluid mechanics going back to the pioneering work of L. Prandtl [25] in 1904. Basically, in the boundary layer, the shear flow can detach/separate from the boundary, generating bubbles and leading to more complicated turbulent behavior. It is important to characterize the separation. It is clear now that the new dynamical systems theory developed here provides a natural tool for the rigorous study.

The original motivation of this research program was to understand the structure and its stability/transitions of geophysical flow patterns in the physical space. There are two general areas of study in geophysics. One area of research devoted to the study of general circulation models; see [9, 10, 11, 12, 13]. The other area studies specific physically phenomena. The study of these physically related problems involves on the one hand applications of the existing mathematical theory to the understanding of the underlying physical problems, and on the other hand the development of new mathematical theories. The program of research presented in this article can be considered as an attempt to this latter aspect. Namely, it is motivated by the study of geophysical fluid dynamics problems, the new mathematical theory is developed under close links to the physics, and in return the theory is applied to the physical problems although more applications are yet to be explored.

The results presented in this article are based on recent papers including [6, 7, 16, 14, 17, 18, 19, 20, 21] in Area a), and [6, 7, 15] in Area b). This article is organized as follows. First, in Section 2, we present the structural stability theorems for divergence-free vector fields with divergence-free vector field perturbations in both the free boundary and the Dirichlet boundary conditions cases. Some examples are given with schematic pictures, including in particular, the instability and transitions of the dipole structure. In Section 3, we present a theory on structural bifurcation and its applications to boundary layer separations, including an application to the solutions of the quasi-geostrophic equations. Sections 4 and 5 short commenting respectively the study for divergence-free fields on general 2D manifolds and further study on Area b).

## 2. Structural stability of 2D incompressible flows

The study of structural stability has been the main driving force behind much of the development of dynamical systems theory following the program initiated by S. Smale and others (see among others [23, 24, 26, 27, 28, 30, 31]). We are interested in the structural stability of an incompressible vector field with perturbations of incompressible vector fields. We call this notion of structural stability the incompressibly structural stability. We proceed in two cases: a) the free boundary condition case, and b) the Dirichlet boundary condition case.

### 2.1. Flows with free-slip boundary conditions

Let $M$ be a two dimensional differentiable Riemannian manifold with boundary $\partial M$ and with the Riemannian metric $g$. In this article, unless otherwise stated, we always assume that $r \geq 1$ be an integer. Let $C^r_n(TM)$ be the space of all $r$-th differentiable vector fields $v$ on $M$ such that $v|_{\partial M} \in C^r(T\partial M)$, namely the restriction of any $r$-th differentiable vector field $v \in C^r(TM)$ on the boundary $\partial M$ is a $r$-th differentiable vector field of the tangent bundle of $\partial M$.

Consider a vector field $v \in C^r_n(TM)$. A point $p \in M$ is called a singular point of $v$ if $v(p) = 0$; a singular point $p$ of $v$ is called non-degenerate if the Jacobian matrix $Dv(p)$ is invertible; $v$ is called regular if all singular points of $v$ are non-degenerate. For convenience, we set

$$
\begin{aligned}
D^r(TM) &= \{v \in C_n^r(TM) | \ \text{div}\, v = 0\}, \\
B^r(TM) &= \{v \in D^r(TM) | \ \frac{\partial v_\tau}{\partial n}|_{\partial M} = 0\}, \\
B_0^r(TM) &= \{v \in D^r(TM) | \ v|_{\partial M} = 0\}.
\end{aligned}
$$

Let $\Phi(x,t)$ be the orbit passing through $x \in M$ at $t = 0$ of the flow generated by $v$. The $\omega$-limit set $\omega(x)$ and the $\alpha$-limit set $\alpha(x)$ of the trajectory $\Phi(x,t)$ are defined by

$$
\begin{aligned}
\omega(x) &= \{y \in M \mid \text{ there exist } t_n \to \infty \quad \text{such that } \Phi(x,t_n) \to y\}, \\
\alpha(x) &= \{y \in M \mid \text{ there exist } t_n \to -\infty \quad \text{such that } \Phi(x,t_n) \to y\}.
\end{aligned}
$$

An orbit with its end points is called a saddle connection if its $\alpha$ and $\omega$-limit sets are saddle points.

**Definition 1** *Two vector fields $u, v \in D^r(TM)$ are called topologically equivalent if there exists a homeomorphism of $\varphi : M \to M$, which takes the orbits of $u$ to orbits of $v$ and preserves their orientation.*

**Definition 2** *A vector field $v \in X = D^r(TM)$ or $B^r(TM)$ or $B_0^r(TM)$ is called structurally stable in $X$ if there exists a neighborhood $\mathcal{O} \subset X$ of $v$ such that for any $u \in \mathcal{O}$, $u$ and $v$ are topologically equivalent.*

Consider now $v \in D^r(TM)$. Thanks to the divergence-free conditions, the properties of divergence-free vector fields are quite different from those of general vector fields. In particular, it is easy to see that for any $v \in D^r(TM)$, an interior non-degenerate singular point of $v$ can either be a center or a saddle, and a non-degenerate boundary singularity must be a saddle. An interior saddle $p \in \overset{\circ}{M}$ is called *self-connected* if $p$ is connected only to itself, i.e., $p$ occurs in a graph whose topological form is that of the number 8.

The following theorem was proved in [14, 21], providing necessary and sufficient conditions for structural stability of a divergence-free vector field.

**Theorem 1** *A divergence-free vector field $v \in X = D^r(TM)$ or $B^r(TM)$ is structurally stable in $X$ if and only if*

*(1) $v$ is regular;*

*(2) all interior saddles of $v$ are self–connected; and*

*(3) each boundary saddle point is connected to boundary saddle points on the same connected component of the boundary.*

*Moreover, the set of all structurally stable vector fields is open and dense in $X$.* □

This theorem provides necessary and sufficient conditions for structural stability of a divergence-free vector field. Notice that the divergence-free condition changes completely the general features of structurally stable fields as compared to the situation when this condition is not present. The latter case was studied in 2-D by Peixoto [24]. The conditions for structural stability and genericity in Peixoto's theorem are: (i) the field can have only a finite number of singularities and closed orbits (critical elements) which must be hyperbolic; (ii) there are no saddle connections; (iii) the non-wandering set consists of singular points and closed orbits.

The first condition in Theorem 1 above requires only regularity of the field and does not exclude centers; the latter are not hyperbolic and thus are excluded by condition (i) in Peixoto's result. Our theorem's second condition is also of a completely different nature than the corresponding one in the Peixoto theorem. Namely, Peixoto's condition (ii) excludes the possibility of saddle connections altogether, while our condition (2) requires all interior saddles are self-connected!

Moreover, a direct consequence of the Peixoto structural stability theorem and the structural stability theorem we obtained is that no divergence-free vector field is structurally stable under general $C^r$ vector fields perturbations. Such a drastic change in the stable configurations is explained by the fact that divergence-free fields preserve volume and so attractors and sources can never occur for these fields. In particular, this makes it natural the restriction that saddles in the boundary must be connected with saddles in the boundary on the same connected component, in the third condition.

## 2.2. Flows with the Dirichlet Boundary Conditions

For a divergence-free vector field $u \in B_0^r(TM)$ with the Dirichlet boundary conditions $u|_{\partial M} = 0$, all points on the boundary are singular points in the usual sense. To study the structure of $u$, we need to classify these boundary points.

**Definition 3** *Let $u \in B_0^r(TM)(r \geq 2)$.*

1. *A point $p \in \partial M$ is called a $\partial$-regular point of $u$ if $\frac{\partial u_\tau(p)}{\partial n} \neq 0$; otherwise, $p \in \partial M$ is called a $\partial$-singular point of $u$.*

2. *A $\partial$-singular point $p \in \partial M$ of $u$ is called nondegenerate if*

$$\det \begin{pmatrix} \frac{\partial^2 u_\tau(p)}{\partial \tau \partial n} & \frac{\partial^2 u_\tau(p)}{\partial n^2} \\ \frac{\partial^2 u_n(p)}{\partial \tau \partial n} & \frac{\partial^2 u_n(p)}{\partial n^2} \end{pmatrix} \neq 0. \tag{1}$$

   *A non-degenerate $\partial$-singular point of $u$ is also called a $\partial$-saddle point of $u$.*

3. *$u \in B_0^r(TM)$ $(r \geq 2)$ is called $D$-regular if a) $u$ is regular in $\overset{\circ}{M}$, and b) all $\partial$-singular points of $u$ on $\partial M$ are non-degenerate.*

Then it is easy to see that each non-degenerate $\partial$-singular point of $u \in B_0^r(TM)$ is isolated. Therefore if all $\partial$-singular points of $u$ on $\partial M$ are non-degenerate, then the number of all $\partial$-singular points of $u$ is finite.

The following theorem generalizes Theorem 1 to divergence-free vector fields with the Dirichlet boundary conditions.

**Theorem 2** [20] *Let $u \in B_0^r(TM)(r \geq 2)$. Then $u$ is structurally stable in $B_0^r(TM)$ if and only if*

1) *$u$ is $D$-regular;*

2) *all interior saddle points of $u$ are self-connection; and*

3) *each $\partial$-saddle point of $u$ on $\partial M$ is connected to a $\partial$-saddle point on the same connected component of $\partial M$.*

*Moreover, the set of all structurally stable vector fields is open and dense in $B_0^r(TM)$.*

## 2.3. Examples

We examine now a few flow patterns as shown in Figures 1–4 below. By Theorem 1, it is easy to see that both flow patterns given in Figures 1 and 2 are structurally stable.

By Theorems 2, the two flow patterns in Figures 3 and 4 are structurally unstable. The flow pattern given by Figure 3 does not have $\partial$-saddle points. The instability is caused by the saddle connection connecting two interior saddles $p$ and $q$. With arbitrarily small perturbations with tubular divergence-free vector fields

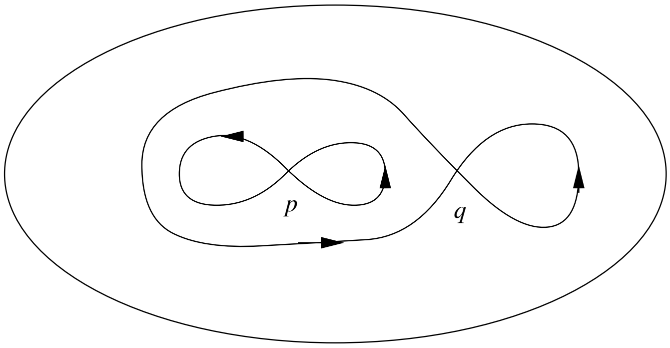


Figure 1.

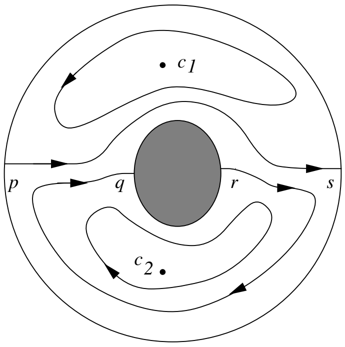


Figure 2.

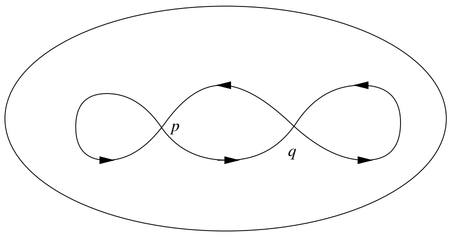


Figure 3.

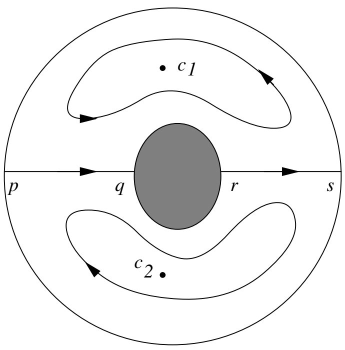


Figure 4.

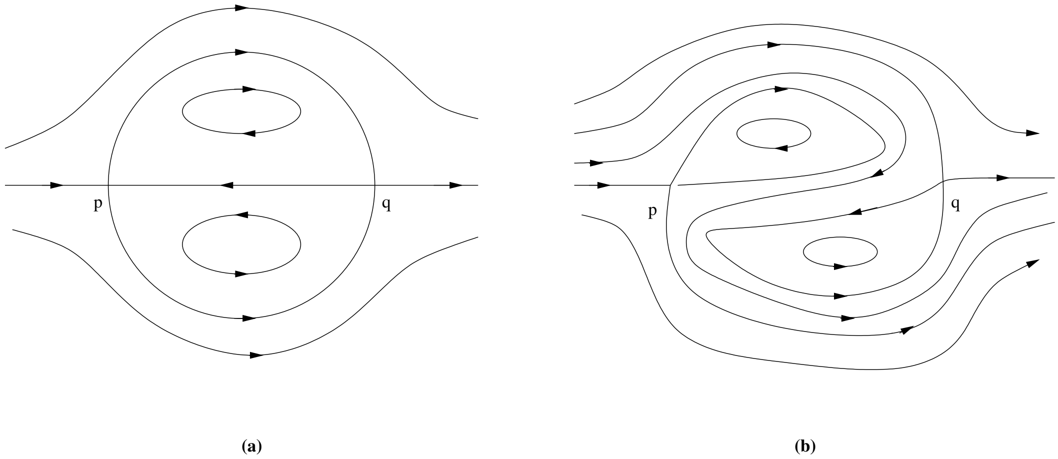


Figure 5. Dipole structure

near the saddle point $p$, the saddle connection will break, and lead to a new stable patterns such as given by Figure 1.

In Figure 4, the flow pattern has four $\partial$-saddles. Here the instability comes from the $\partial$ saddle connections between different connected components of the boundary; i.e. $p$ is connected to $q$, and $r$ to $s$. As in the previous situation, with arbitrarily small perturbations with tubular divergence-free vector fields surrounding the inner island both saddle connections ( $p$ to $q$ and $r$ to $s$) break and lead to a new stable pattern as given by Figure 2.

### 2.4. Dipole structure

The dipole flow pattern refers to the flow pattern as shown in Figure 5. It appears in many fluid mechanics problems, and in particular in some typical weather patterns and ocean circulation patterns. By the structural stability theorem, the flow structure as shown in Figure 5(a) is not stable, as two saddle points are connected. In fact, by any small tubular flow perturbation as constructed in Lemma 4.1 in [21] around the saddle point $p$ in Figure 5(a), the flow pattern as shown in Figure 5(a) becomes the one as shown in Figure 5(b). Hence, from the physical point of view, we would claim that the dipole type of flow patterns observed phyically in fluid flows, and in particular in geophysical flows, should really be the one as shown in Figure 5(b) instead of that in Figure 5(a).

## 3. Boundary layer separations and applications to oceanic boundary currents

The nature of flow's boundary layer separation from the boundary plays a fundamental role in many physical problems, and often determines the nature of the flow in the interior as well. The main objective of this section is to present a rigorous characterization of the boundary layer separations of 2D incompressible fluid flows. This is a long standing problem in fluid mechanics going back to the pioneering work of Prandtl [25] in 1904. The classical theory of boundary layer can be found in [8, 29, 1]. Also, we refer the readers to a recent textbook [22] and articles [3, 4, 32] for the mathematical analysis of the Prandtl equation, an approximation of the Navier-Stokes equations for the boundary layer analysis.

Basically, in the boundary layer, the shear flow can detach/separate from the boundary, generating bubbles and leading to more complicated turbulent behavior. It is important to characterize the separation. It is observed experimentally that the points where the normal derivative of the velocity field vanishes may just be the separation point. No known theorem, which can be applied to determine the separation, is available until the recent work [6, 7, 20], which provides a first rigorous characterization. We address in this section some results obtained in these articles.

## 3.1. Structural bifurcation

For convenience, we assume the boundary $\partial M$ contains a flat part $\Gamma \subset \partial M$, and consider structural bifurcation near a $\partial$–singular point $\bar{x} \in \Gamma$. For simplicity, we take a coordinate system $(x_1, x_2)$ with $\bar{x}$ at the origin and with $\Gamma$ given by

$$\Gamma = \{(x_1, 0) \,\big|\, |x_1| \leq \delta_0\}, \qquad \bar{x} = 0,$$

for some $\delta_0 > 0$. Obviously, the tangent and normal vectors on $\Gamma$ are the unit vectors in $x_1$ and $x_2$ directions respectively. Let $u \in C^1([0,T], B_0^r(TM))$ $(r \geq 2)$ be a one-parameter family of divergence-free vector fields with the homogeneous Dirichlet boundary condition. In a neighborhood $U \subset M$ of $\bar{x} \in \Gamma$, $u(x,t)$ can be expressed by

$$u(x,t) = x_2 v(x,t). \tag{2}$$

To proceed, we consider the Taylor expansions of both $u(x,t)$ and $v(x,t)$ at $t_0$ $(0 < t_0 < t)$

$$\begin{cases} u(x,t) = u^0(x) + (t-t_0)u^1(x) + o(|t-t_0|^2) \\ u^0(x) = u(x,t_0), \\ u^1(x) = \dfrac{\partial u(x,t_0)}{\partial t}, \end{cases} \tag{3}$$

$$\begin{cases} v(x,t) = v^0(x) + (t-t_0)v^1(x) = o(|t-t_0|)^2, \\ v^0(x) = v(x,t_0), \\ v^1(x) = \dfrac{\partial v(x,t_0)}{\partial t}. \end{cases} \tag{4}$$

We proceed with the follwing defintions.

**Definition 4** *Let $u \in C^1([0,T], X)$. The vector field $u_0 = u(\cdot, t_0)$ $(0 < t_0 < T)$ is called a bifurcation point of $u$ at time $t_0$ if, for any $t^- < t_0$ and $t_0 < t^+$ with $t^-$ and $t^+$ sufficiently close to $t_0$, the vector field $u(\cdot; t^-)$ is not topologically equivalent to $u(\cdot; t^+)$. In this case, we say that $u(x,t)$ has a bifurcation at $t_0$ in its global structure.*

**Definition 5** *Let $u \in C^1([0,T], X)$. We say that $u(x,t)$ has a bifurcation in its local structure in a neighborhood $U \subset M$ of $x_0$ at $t_0$ $(0 < t_0 < T)$ if, for any $t^- < t_0$ and $t_0 < t^+$ with $t^-$ and $t^+$ sufficiently close to $t_0$, the vector fields $u(\,\cdot\,; t^-)$ and $u(\,\cdot\,; t^+)$ are not topologically equivalent locally in $U \subset M$.*

Let $u^i = (u_1^i, u_2^i)$, $v^i = (v_1^i, v_2^i)$, $i = 0, 1$. We start with the following conditions for the structural bifurcation.

ASSUMPTION (H). Let $u^0 \in C^{k+1}$ near $\bar{x} \in \Gamma$ for some $k \geq 2$, and $\bar{x} = 0$ be an isolated degenerate $\partial$-singular point of $u^0(x)$ such that

$$\frac{\partial u^0(0)}{\partial n} = 0, \tag{5}$$

$$\text{ind}(v^0, 0) \neq -\frac{1}{2}, \tag{6}$$

$$\frac{\partial^{k+1} u_1^0(0)}{\partial^k \tau \partial n} \neq 0, \tag{7}$$

$$\frac{\partial u^1(0)}{\partial n} \neq 0. \tag{8}$$

The main results on structural bifurcations of $u(x,t)$ near a degenerate $\partial$-singular point on flat boundary are given in the following theorems. The detailed proofs of these theorems are given in [7].

**Theorem 3** [7] *Let $u \in C^1([0,T], B_0^r(TM))$ $(r \geq 2)$ satisfy Assumtion (H). Then there exists a neighborhood $\Gamma_0 \subset \Gamma$ of $\bar{x}$ and an $\varepsilon_0 > 0$ such that all $\partial$-singular points of $u(x, t_0 \pm \varepsilon)$ are nondegenerate for any $0 < \varepsilon \leq \varepsilon_0$. Furthermore,*

1. *if the index $\text{ind}(v^0, 0)$ is an integer, then one of $u(x, t_0 \pm \varepsilon)$ has two $\partial$–singular points on $\Gamma_0$, and the other has no $\partial$-singular point on $\Gamma_0$; and*
2. *if the index $\text{ind}(v^0, 0)$ is not an integer, then each of $u(x, t_0 \pm \varepsilon)$ has exactly one $\partial$-singular point on $\Gamma_0$.*

**Theorem 4** *(Structural Bifurcation Theorem, [7]) Let $u \in C^1([0,T], B_0^r(TM))$ $(r \geq 2)$ satisfy Assumption (H). Then*

1. *$u(x,t)$ has a bifurcation in its local structure at $(\bar{x}, t_0)$.*
2. *if $\bar{x} \in \partial M$ is a unique singular point which has the same index as $\text{ind}(v^0, 0)$ on $\partial M$, then $u(x,t)$ has a bifurcation in its global structure at $t = t_0$.*

Technically speaking, the proof of the results are highly nontrivial. Basically, the results are achieved with delicate analysis of the flow structure near the boundary for both the free boundary and the Dirichlet boundary conditions. The first step is to classify the flow structure and its transitions near the boundary for flows with only no normal flow boundary conditions; see [6]. Second, we analyze in [20] detailed flow structure in the boundary layer for flows with the Dirichlet boundary condition. Third, we make connections in [7] between the structure of the original velocity fields and the structure of the normal derivative of the velocity field. Here we only make a few remarks on Assimption (H), and will discuss the applications of these two theorems to boundary layer separations in next section.

**Remark 1** Condition (5) says that $\bar{x} = 0 \in \Gamma$ is a $\partial$-singular point of $u^0(x)$, which is equivalent to the leading order vorticity vanishes at $\bar{x}$. This is socalled Prandtl condition, which was suggested by Prandtl to identify possible boundary layer separation points of incompressible flows. ■

**Remark 2** Since $u^0 = x_2 v^0$, condition (6) is equivalent to that the index of $v^0$ at $\bar{x} = 0$ is different from $-1$. Hence, let

$$\text{ind}\,(v^0, 0) = -\frac{n}{2} \qquad (n \neq 1).$$

Then there are exactly $n \neq 1$ interior orbits of $u^0$ connected to $\bar{x} \in \Gamma$. This shows that $p \in \Gamma$ $(x = 0)$ is a degenerate $\partial$-singular point of $u^0(x)$, which is necessary for structural bifurcation due to the structural stability theorem. ■

**Remark 3** Condition (8) amounts to saying that the first order term $u^1$ of the Taylor expansion for the normal derivative of $u$ is different from zero. Also, this is necessary; otherwise, we need to work on higher order Taylor expansion, and the corresponding results proved in this article will be true as well. In view of fluid mechanics applications, Condition (8) is equivalent to nonzero vorticity for $u^1$. F is is a necessary condition for the bifurcation. In addition, it is easy to see that

$$\frac{\partial u_1^1(0)}{\partial x_2} = \frac{\partial u_1^1(0)}{\partial n} \neq 0,$$

which shows that the acceleration of fluid in tangent direction at $p$ near the boundary layer is nonzero. ■

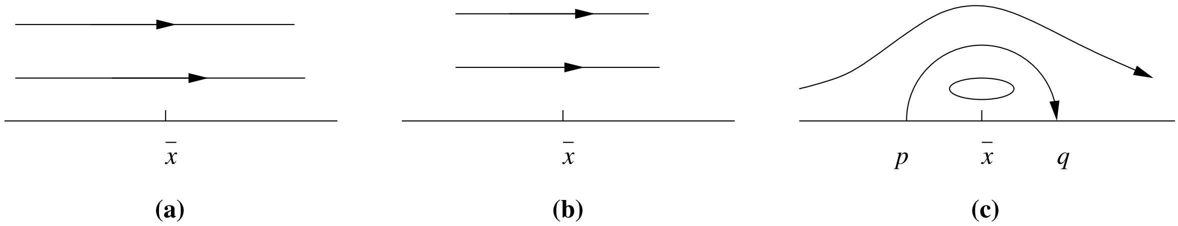


Figure 6. Boundary layer separation of shear flow

**Remark 4** Condition (7) is a technical condition, and amounts to saying that the tangential component $u_1^0$ of the leading order term is Taylor expandable. Furthermore, let $k$ be the smallest integer satisfying condition (7). It is easy to show that $k \geq 2$. In fact, $u^0(x)$ has the Taylor expansion at $x = 0$

$$u^0(x) = \begin{cases} cx_2 + 2ax_1x_2 + bx_2^2 + x_2h_1(x), \\ \qquad\qquad - ax_2^2 + x_2h_2(x), \end{cases} \tag{9}$$

with $h_i(x) = o(|x|)$ $(i = 1, 2)$. Since $p \in \Gamma$ $(x = 0)$ is a degenerate $\partial$–singular point of $u^0(x)$, it follows that $c = 0$, $a = 0$, which implies that $k \geq 2$. ■

Finally we remark that the same results hold true as well for structural bifurcation near a curved boundary. In that situation, we only need to replace Condition (6) by the geometrical condition: There are $n$ $(n \neq 1)$ interior orbits of $u^0$ connected to $\bar{x}$. We refer the interested readers to [7] for details.

## 3.2. Boundary layer separations

Boundary layer separation is a long standing problem in fluid mechanics going back to the pioneering work of L. Prandtl [25] in 1904. In the boundary layer, the shear flow can detach/separate from the boundary, generating bubbles and leading to more complicated turbulent behavior. The streamlines breaking away from the boundary is called boundary layer separation. Prandtl suggests to identify the point of separation $\bar{x}$ where the circulation vanishes: $\partial u/\partial n|_{\bar{x}} = \omega(\bar{x}) = 0$. This is the condition 5.

From the mathematical point of view, the above structural bifurcation theory provides a nature tool for characterizing boundary layer separations of incompressible flows. More precisely, in the case where ind $(v^0, 0) = 0$, the above structural bifurcation theorem corresponds to boundary layer separation of fluid flows as we mentioned before. Schematically, the structural bifurcation theorem in this case can be illustrated schematically in Figure 6. The velocity field at the bifurcation instant, i.e. $u^0$, is given by Figure 6(b). For any small $\varepsilon > 0$, $u(x, t_0 - \varepsilon)$ given by Figure 6(a) has no singular points near $\bar{x}$, representing a typical shear flow. On the other hand, after the bifurcation/separation, $u(\bar{x}, t_0 + \varepsilon)$ given by Figure 6(c) has two singular points near $\bar{x}$ on the boundary and one center near $\bar{x}$ in the interior, all of which are non-degenerate.

We now apply the above results to the boundary layer separation problem for a quasi-geostrophic model, which reads

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p + fk \times u - \nu \triangle u = W, \\ \text{div}\, u = 0, \\ u = 0\,, \qquad \text{on} \quad \partial M. \end{cases} \tag{10}$$

Here $u = (u_1, u_2)$ is the (horizontal) velocity field, $p$ the surface pressure, $W$ the wind stress forcing, $k$ is the unit vector in the vertical direction so that $k \times u = (-u_2, u_1)$, $f = f_0 + \beta x_2$ is the Coriolis parameter with $f_0$ and $\beta$ constants.

Regarding to the context of wind-driven, double-gyre oceanic behavior with respect to boundary layer separation, we denote a section of the vertical line $x_1 = 0$ as a boundary section $\Gamma$, with its normal direction going leftward; see Figure 7.

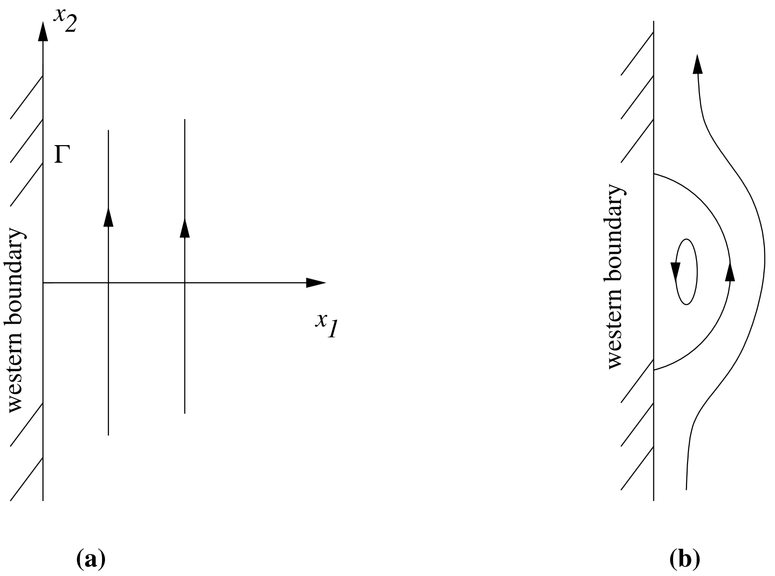


Figure 7. Structural bifurcation for ind$(v^0, 0) = 0$

Let $(u, p)$ be the solution of the quasi-geostrophic equation (10) such that Assumption (H) holds true at $(\bar{x}, t_0)$ with $\bar{x} \in \Gamma$ and $t_0 > 0$. Moreover, for $t \leq t_0$ and near $t_0$, the vorticity $\omega = -\partial u_1/\partial x_2 + \partial u_2/\partial x_1 \geq 0$. Namely, $u$ is an upward shear flow for $t \leq t_0$ as shown in Figure 7(a). Then the following results hold true:

1. The velocity field $u$ has structural bifurcation in its local structure and boundary layer separation at the boundary point $\bar{x}$ as $t$ crosses $t_0$ with flow structure for $t > t_0$ given as shown in Figure 7(b).

2. There is an adverse pressure gradient in the tangential direction present at $(\bar{x}, t_0)$, see [5].

We end this section with a few remarks. First, an important issue is to estimate the time $t_0$ and the location $x_0$ in terms of the Reynolds number, the forcing or the outer flow, as well as the initial velocity profile. Very interesting things can be obtained, and will be addressed by the authors in a forthcoming paper.

Second, it is not hard to see potential applications of the above theory to the Gulf stream separation from the North American coastline at Cape Hatteras.

Third, our study opens a door to classify the structural transitions in the interior points. For the interior flow separation/spin-off, one extra degree of freedom allows more patterns mathematically, which in general do not appear in the physical situation. This project is to develop a rigorous theory to identify physically interesting patterns and their transitions, and to apply the theory to typical oceanic flows. Another important example we shall study is to classify the structure and its transitions of the Jupiter's Red Sport as shown in `http://heritage.stsci.edu/public/aug5/jupgrsbig.html`, which shows clearly the formation and separation of bubbles from the zonal flow.

Fourth, in both Figures 6 and 7, we see reattachment of the stream lines to the boundary, although the real particle line may not be. One could think the real particle moves into the interior of the domain as the bubbles (of the stream lines) amplifies as the time evolves. This fact is proved in [6, 7], and supported by numerical simulations for the driven cavity flow in [5]. Also, the reattachment is in consistant with experiments as shown in the von Dyke's book [2].

## 4. Periodic structure and block stability

For incompressible flows defined on 2D tori or on nonzero genus 2D manifolds, quite different senerio appears. However, it is still possible to derive a complete classification on the structure and stability of divergence-free vector fields on general 2D orientable compact manifolds, including the torus corresponding to the periodic boundary conditions in the Euler representation of fluid flows.

The main technical method is based on a complete understanding of the $\omega$ and $\alpha$ limit sets of general divergence free vector fields. It is well known that flows on a torus may be non-trivially recurrent, and the $\omega$ and $\alpha$ limit sets can be very complicated sets. For instance, the $\omega$ and $\alpha$ limit sets of the Cherry flow can have the structure of Cantor sets. Thanks to the incompressibility conditions, the divergence-free vector fields have better properties. First, we prove in [42] a Limit Set Theorem, a version of Poincare-Bendixson theorem. It shows that the $\omega$ (resp. $\alpha$) limit of a regular point of a regular divergence-free vector field is either a saddle, or a non-limiting closed orbit, or an ergodic set which is a closed domain with boundaries consisting of saddle connections of finite length. The structure of the Cherry flow shows that the same result is not true for general vector fields, and the divergence-free condition is crucial for the ergodic set being a closed domain. Furthermore, the detailed structure of the ergodic set is fully characterized with its Euler characteristic explicitly calculated.

With the structure of limit sets at our disposal, the following results are then quite nature.

1. No divergence-free vector fields is structurally stable with divergence-free vector fields perturbations.

2. For a complete classification, we need to beyond this instability result. To this end, we can introduce two new concepts: block structure and block stability. We call a divergence-free vector field a basic vector field if its phase diagram has a block structure. Namely, the phase diagram is decomposed into a finite number of flow-invariant retractable blocks and ergodic sets such that the restrictions of the vector field on the retractable blocks are self-connected. We prove that (i). all basic vector fields form an open and dense set of all divergence-free vector fields, (ii). the block structure is stable, (iii). the flow is either periodic or non-trivially recurrent on the ergodic sets, and (iv). the structural instability is due completely to the ergodicity and/or periodicity on the ergodic sets.

3. In addition, the periodic structure defined by the Taylor vortices can also be fully classified. More precisely, we classify in [17] the structure and its transitions/evolution of the Taylor vortices under small perturbations of either the Hamiltonian vector fields or divergence-free vector fields on the two-dimensional torus. In particular, we show that there is only one stable block structure near the Taylor vortices with Hamiltonian perturbations, and there are exactly five block stable structures near the Taylor field with general divergence-free vector field perturbations.

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Tian Ma
Department of Mathematics
Sichuan University
Chengdu, Sichuan, P. R. China
tima@indiana.edu

Shouhnog Wang
Department of Mathematics
Indiana University
Bloomington, IN 47405, USA
showang@indiana.edu