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Estimates based on scale separation for geophysical flows

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Abstract. The objective of this work is to obtain theoretical estimates on the large and small scales for geophysical flows. Firstly, we consider the shallow water problem in the one-dimensional case, then in the two-dimensional case. Finally we consider geophysical flows under the hydrostatic hypothesis and Boussinesq approximation. Scale separation is based on Fourier series, with N modes in each spatial direction, and the choice of a cut-off level $N_1 < N$ to define large and small scales. We establish that, for a given (quite high) cut-off level, and for (quite regular) initial conditions, the small scales (and their time derivative) are small in energy norm by comparison with the large scales (and their time derivative).

Ecuaciones basadas en separación de escalas para flujos geofísicos

Resumen. El objetivo de este trabajo es obtener estimaciones teóricas sobre escalas grandes y pequeñas para flujos geofísicos. En primer lugar, consideramos flujos geofísicos bajo la hipótesis hidrostática y la aproximación de Boussinesq. La separación de escalas está basada en series de Fourier, con N modos en cada dirección espacial en la elección de un conjunto de nivel de corte $N_1 < N$ para definir las escalas grandes y pequeñas. Mostramos que, para un nivel de corte (bastante grande) y para condiciones iniciales (bastante regulares), las escalas pequeñas (y sus derivadas con respecto al tiempo) son pequeñas con la norma de de la energía en comparación con escalas grandes (y sus derivadas con respecto al tiempo).

1. Introduction and motivation

In meteorology problems, the stability constraint on numerical schemes are quite strong. For example, for the one-dimensional shallow water problem, the stability constraint for an explicit scheme (Leap-Frog scheme) can be written (see *e.g.* [40]):

$$\left| U \pm \sqrt{gH + f^2 \frac{\Delta x^2}{(\sin(k\Delta x))^2}} \right| \frac{\Delta t}{\Delta x} \sin(k\Delta x) \leq 1 \,, \tag{1}$$

with U a constant advecting velocity in the x direction, H the mean height, f the Coriolis parameter associated with the rotating systems (considered as constant here), and g the gravitational constant. The discretisation parameters are Δt (time step), Δx (mesh size) and k (wavenumber).

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From the stability condition (1) we can define three stability conditions. One stability condition comes from the explicit treatment of convective terms (CFL condition), setting $f = g = 0$:

$$|U| \frac{\Delta t}{\Delta x} \leq 1 \,. \tag{2}$$

Another stability constraint comes from the explicit treatment of the terms associated with rotation, *i.e.* setting $U = g = 0$:

$$f \Delta t \leq 1 \,. \tag{3}$$

Finally, the third stability condition is due to the explicit treatment of the gravity terms, namely setting $U = f = 0$:

$$\sqrt{gH} \frac{\Delta t}{\Delta x} \sin(k \Delta x) \leq 1 \,. \tag{4}$$

For $k\Delta x$ sufficiently small, we have $\sin(k\Delta x) \simeq k\Delta x$. Moreover, Δx can be expressed in terms of the largest wavenumber k_{max} in the following manner: $\Delta x \simeq \frac{1}{k_{max}}$. So (4) can be rewritten:

$$\sqrt{gH} \frac{k}{k_{max}} \frac{\Delta t}{\Delta x} \leq 1 \tag{5}$$

Numerically we have (see [40] and [28]): $U \simeq 10\, m/s$ (common wind) to $100\, m/s$ (Jet-Stream) for atmospheric type flows, $H \simeq 10^4\, m$ (troposphere), $g \simeq 9.81\, m/s^2$ and $f \simeq 10^{-4}\, s^{-1}$. So the more restrictive stability condition in (1) is (5), requiring $\Delta t \leq 300 \Delta x \frac{k_{max}}{k}$. Moreover, the stability constraint (5), associated with an explicit scheme is wavenumber dependent : it is stronger for the small scales (associated with high wavenumbers) than for the large scales (associated with small wavenumbers).

In order to improve the previous stability constraint, we can use semi-implicit schemes. Since (5) is due to the explicit treatment of the gravity terms, it is usual to use an implicit scheme (Crank-Nicholson) to integrate these terms and an explicit scheme (Leap-Frog) to integrate the convective and rotation terms. The stability condition associated with this semi-implicit approximation can be written (see [40]):

$$\left| U \pm \sqrt{gH \left(\cos(\nu \Delta t)\right)^2 + f^2 \frac{\Delta x^2}{\left(\sin(k\Delta x)\right)^2}} \right| \frac{\Delta t}{\Delta x} \sin(k\Delta x) \leq 1 \,, \tag{6}$$

where ν is the frequency associated with the wave propagation. The factor $\left(\cos(\nu\Delta t)\right)^2$ multiplying the gravity term gH reduces its value. So the stability constraint due to the implicit treatment of the gravity terms is weaker than (4):

$$\sqrt{gH}\, |\cos(\nu \Delta t)| \frac{\Delta t}{\Delta x} \sin(k\Delta x) \leq 1 \,. \tag{7}$$

However, the drawback of the semi-implicit scheme is that it modifies the propagation speed (dispersion error) of the gravity waves. But the relative error induced on the propagation speed is smaller for the small scales than for the large ones, because propagation speed is higher for the small scales than for the large scales.

Furthermore, it is important to compute accurately the slow waves (*i.e.* the large scales). Indeed the Coriolis force has a more important effect on the slow waves and, in the atmosphere, the rotation effect is important (quasi-geostrophic flow). For more details, see [5].

So, since the stability constraint is stronger for high wavenumbers than for small wavenumbers, and since accuracy is more important for the large scales than for the small scales, we want to compute differently the small and large scales. The aim of this paper is to establish some estimates to compare the small and large scales and their time derivative, in order to deduce new schemes to compute them. Previously, theoretical estimates have been established on the Navier-Stokes equations by Foias, Manley and Temam in [13], and theoretical and numerical studies of schemes, based on scale separation, have been done in [12], [36], [18], [24], [35], [3] and [9] for example.

In this work, firstly we consider the shallow water problem in the one-dimensional case and then in the two-dimensional case. We then consider the more general case of geophysical flows, under the Boussinesq approximation and hydrostatic hypothesis.

We consider linearized problems. Indeed, in many cases, the study of linear shallow water problem gives quantitatively correct results for orders of magnitude, that is the flows are often not too far from linear in their instanteneous behaviour (see [39]); on Figure 1, we can see that the nonlinear terms are indeed not dominant.

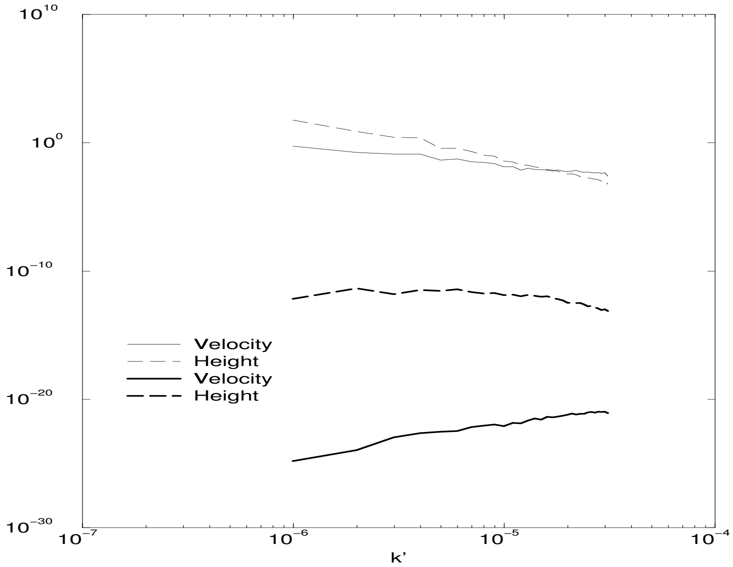

Figure 1. Spectra of velocity (energy spectrum) and height: thin lines. Spectra of nonlinear terms for velocity and height equations: thick lines.

In all cases, we consider a spatial discretisation of the equations using a Fourier spectral Galerkin method. The scale separation between the large and small scales in the physical space (global separation) is done using the wavenumbers in the spectral space. We consider the semi-discretized equations, since it is on these equations that time schemes are applied for numerical computation (see [10]). From the theoretical estimates obtained on the large and small scales, we shall propose numerical time schemes accurate on the large scales, and saving computational time (better stability properties). This numerical work is in progress; it necessitates extensive implementation work, it will be presented elsewhere (see [10]).

2. The shallow water problem: the one-dimensional case

The shallow water problem is deduced from homogeneous and frictionless geophysical flows, integrating the incompressibility constraint over the entire fluid height. For more details, see, for example, [5], [15]. Let us consider the linearized shallow water problem (see [40]):

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - fv + g\frac{\partial h}{\partial x} = 0 \,, \tag{8}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + fu = 0 \,, \tag{9}$$

$$\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0 \,, \tag{10}$$

where (u, v) denote the velocity components in the (x, y) directions, h is the height of the surface of the water above (or below) some mean height H, and U is a constant advecting velocity in the x direction. The Coriolis parameter f is considered as constant here. We have assumed in (8)-(10) that the independent variables u, v and h are uniform in the y direction (one-dimensional case).

For explicit examples, the numerical values of the physical parameters are chosen as follows (see previous section): $U \simeq 10$ to $100\ m/s$, $H \simeq 10^4\ m$, $g \simeq 9.81\ m/s^2$ and $f \simeq 10^{-4}\ s^{-1}$.

Let us consider a solution $\mathbf{U}^T = (u, v, h)$ for (8)-(10) of the following form (spectral method):

$$\mathbf{U}(x,t) = \sum_{k=1-N/2}^{N/2} \widehat{\mathbf{U}}_k(t) \exp(\mathbf{i}\, kx) = \sum_{k=1-N/2}^{N/2} \begin{pmatrix} \hat{u}_k(t) \\ \hat{v}_k(t) \\ \hat{h}_k(t) \end{pmatrix} \exp(\mathbf{i}\, kx) \,, \tag{11}$$

where N is a given parameter. The choice of this parameter depends in particular on the regularity of the initial condition. We suppose that $\mathbf{U}$ is periodic in the spatial direction x (periodic boundary conditions) and with zero average, so that $\langle \mathbf{U}(t) \rangle_\Omega = \widehat{\mathbf{U}}_0 = \mathbf{0}$, at each time $t \geq 0$, where $\Omega = (0, 2\pi)$. This will be the case if the initial condition has zero average.

Using (11), we can rewrite (8)-(10) in the following manner:

$$\frac{d\widehat{\mathbf{U}}_k}{dt} + A_k \widehat{\mathbf{U}}_k = 0 \,, \tag{12}$$

with:

$$\widehat{\mathbf{U}}_k = \begin{pmatrix} \hat{u}_k \\ \hat{v}_k \\ \hat{h}_k \end{pmatrix} \,,$$

and the matrix A_k is defined by:

$$A_k = \begin{pmatrix} \mathbf{i}\, Uk & -f & \mathbf{i}\, gk \\ f & \mathbf{i}\, Uk & 0 \\ \mathbf{i}\, Hk & 0 & \mathbf{i}\, Uk \end{pmatrix} . \tag{13}$$

The eigenvalues of A_k are:

$$\begin{cases} \lambda_{1,k} = \mathbf{i}\, Uk \,, \\ \lambda_{2,k} = \mathbf{i}\, Uk + \mathbf{i}\, \sqrt{f^2 + gHk^2} \,, \\ \lambda_{3,k} = \mathbf{i}\, Uk - \mathbf{i}\, \sqrt{f^2 + gHk^2} \,. \end{cases} \tag{14}$$

For each $k \in [1-N/2, N/2]$, we have $|\lambda_{1,k}| < |\lambda_{2,k}|$ and $|\lambda_{1,k}| < |\lambda_{3,k}|$ as we can see on Figure 2 .

If we consider the basis made of the eigenvectors of A_k, we obtain:

$$\frac{d\hat{\widetilde{\mathbf{U}}}_k}{dt} + \begin{pmatrix} \lambda_{1,k} & 0 & 0 \\ 0 & \lambda_{2,k} & 0 \\ 0 & 0 & \lambda_{3,k} \end{pmatrix} \hat{\widetilde{\mathbf{U}}}_k = 0 \, , \tag{15}$$

where $\widetilde{\mathbf{U}}$ denotes the vector $\mathbf{U}$ written in the eigenvectors basis. Time integration of (15) gives:

$$\begin{cases} \hat{\tilde{u}}_k(t) = \hat{\tilde{u}}_k(0) \exp(-\lambda_{1,k} t) \, , \\ \hat{\tilde{v}}_k(t) = \hat{\tilde{v}}_k(0) \exp(-\lambda_{2,k} t) \, , \\ \hat{\tilde{h}}_k(t) = \hat{\tilde{h}}_k(0) \exp(-\lambda_{3,k} t) \, , \end{cases} \tag{16}$$

which can be written also:

$$\hat{\widetilde{\mathbf{U}}}_k(t) = \hat{\widetilde{\mathbf{U}}}_k(0) \exp(-D_k t) \, , \tag{17}$$

where D_k is a diagonal matrix, $D_k = \text{Diag}\,(\lambda_{i,k})_{i=1,2,3}$. We deduce from (16) that $\left|\hat{\tilde{u}}_k(t)\right|$, $\left|\hat{\tilde{v}}_k(t)\right|$ and $\left|\hat{\tilde{h}}_k(t)\right|$ are time independent: $||\hat{\widetilde{\mathbf{U}}}_k(t)||_2 = ||\hat{\widetilde{\mathbf{U}}}_k(0)||_2$ for each value of $t > 0$, denoting by $||\cdot||_2$ the Euclidean norm of a vector.

Let us consider:

$$\begin{cases} \omega_{1,k} = -Uk \, , \\ \omega_{2,k} = -\left(Uk + \sqrt{f^2 + gHk^2}\right) \, , \\ \omega_{3,k} = -\left(Uk - \sqrt{f^2 + gHk^2}\right) \, ; \end{cases} \tag{18}$$

we obtain, using the spectral expansion (11) for the eigenvector basis and (16):

$$\begin{cases} \tilde{u}(x,t) = \displaystyle\sum_{k=1-N/2}^{N/2} \hat{\tilde{u}}_k(0) \exp\left(\mathbf{i}\,(kx + \omega_{1,k} t)\right) \, , \\ \tilde{v}(x,t) = \displaystyle\sum_{k=1-N/2}^{N/2} \hat{\tilde{v}}_k(0) \exp\left(\mathbf{i}\,(kx + \omega_{2,k} t)\right) \, , \\ \tilde{h}(x,t) = \displaystyle\sum_{k=1-N/2}^{N/2} \hat{\tilde{h}}_k(0) \exp\left(\mathbf{i}\,(kx + \omega_{3,k} t)\right) \, , \end{cases} \tag{19}$$

where $\omega_{1,k}$, $\omega_{2,k}$ and $\omega_{3,k}$ are associated with wave propagation (Rossby waves and inertial-gravity waves). As we can notice on Figure 2, $|\omega_{1,k}|$ is smaller than $|\omega_{2,k}|$ and $|\omega_{3,k}|$. Moreover, $|\omega_{i,k}|$ increases with $|k|$, for $i = 1, 2, 3$. So, the wave propagation is faster for high wavenumbers (small scales) than for small wavenumbers (large scales); spatial scale separation induces temporal scale separation.

Let us denote by P_k, $k \neq 0$, the modal matrix defined by:

$$\begin{pmatrix} \hat{u}_k \\ \hat{v}_k \\ \hat{h}_k \end{pmatrix} = P_k \begin{pmatrix} \hat{\tilde{u}}_k \\ \hat{\tilde{v}}_k \\ \hat{\tilde{h}}_k \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \hat{\tilde{u}}_k \\ \hat{\tilde{v}}_k \\ \hat{\tilde{h}}_k \end{pmatrix} = P_k^{-1} \begin{pmatrix} \hat{u}_k \\ \hat{v}_k \\ \hat{h}_k \end{pmatrix} \, . \tag{20}$$

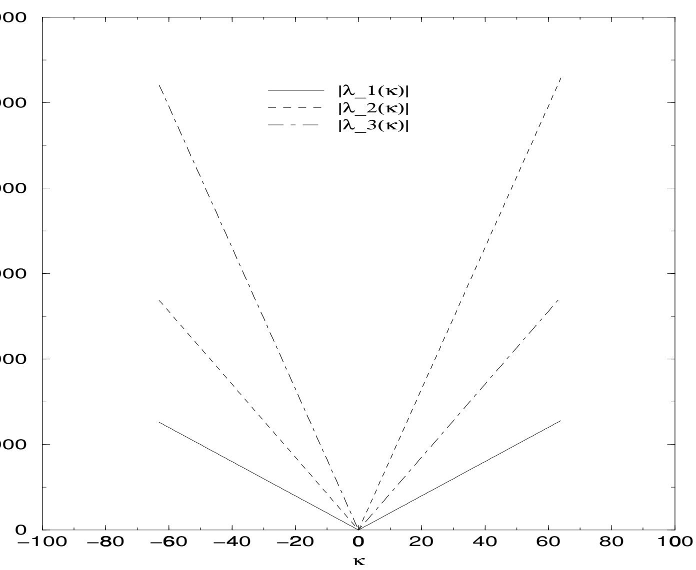

Figure 2. Representation of $|\lambda_{1,k}|$, $|\lambda_{2,k}|$ and $|\lambda_{3,k}|$ as functions of k.

We can verify that, for $k \neq 0$, the matrix P_k and P_k^{-1} are as follows:

$$P_k = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -\mathbf{i}\, f/\sqrt{\alpha_k} & \mathbf{i}\, f/\sqrt{\alpha_k} \\ -\mathbf{i}\, f/gk & Hk/\sqrt{\alpha_k} & -Hk/\sqrt{\alpha_k} \end{pmatrix}, \tag{21}$$

and:

$$P_k^{-1} = \begin{pmatrix} 0 & gHk^2/\alpha_k & \mathbf{i}\, fgk/\alpha_k \\ 1/2 & \mathbf{i}\, f/2\sqrt{\alpha_k} & gk/2\sqrt{\alpha_k} \\ 1/2 & -\mathbf{i}\, f/2\sqrt{\alpha_k} & -gk/2\sqrt{\alpha_k} \end{pmatrix}, \tag{22}$$

where $\alpha_k = f^2 + gHk^2$. Since $\left|\hat{\hat{u}}_k(t)\right|$, $\left|\hat{\hat{v}}_k(t)\right|$ and $\left|\hat{\hat{h}}_k(t)\right|$ are constant for all t, we have:

$$\begin{aligned}
&||\widehat{\widehat{\mathbf{U}}}_k(t)||_2^2 = ||\widehat{\widehat{\mathbf{U}}}_k(0)||_2^2 = ||P_k^{-1}\widehat{\mathbf{U}}_k(0)||_2^2 \\
&= \left|\frac{gHk^2}{\alpha_k}\hat{v}_k(0) + \frac{fgk}{\alpha_k}\mathbf{i}\,\hat{h}_k(0)\right|^2 + \frac{1}{2}\left|\hat{u}_k(0)\right|^2 + \frac{1}{2}\left|\frac{\mathbf{i}\, f}{\sqrt{\alpha_k}}\hat{v}_k(0) + \frac{gk}{\sqrt{\alpha_k}}\hat{h}_k(0)\right|^2 \\
&\leq \frac{g^2H^2k^4}{\alpha_k^2}\left|\hat{v}_k(0)\right|^2 + \frac{f^2g^2k^2}{\alpha_k^2}\left|\hat{h}_k(0)\right|^2 + \left|\hat{u}_k(0)\right|^2 + \frac{f^2}{\alpha_k}\left|\hat{v}_k(0)\right|^2 + \frac{g^2k^2}{\alpha_k}\left|\hat{h}_k(0)\right|^2 \\
&+2\frac{fg^2H\left|k\right|^3}{\alpha_k^2}\left|\hat{v}_k(0)\right| \times \left|\hat{h}_k(0)\right| + 2\frac{fg\left|k\right|}{\alpha_k}\left|\hat{v}_k(0)\right| \times \left|\hat{h}_k(0)\right| .
\end{aligned}$$

Using the inequality $2ab \leq a^2 + b^2, \forall a \geq 0, \forall b \geq 0$, we obtain:

$$\begin{aligned}
||P_k^{-1}\widehat{\mathbf{U}}_k(0)||_2^2 \leq \; & \frac{g^2H^2k^4}{\alpha_k^2}\left|\hat{v}_k(0)\right|^2 + \frac{f^2g^2k^2}{\alpha_k^2}\left|\hat{h}_k(0)\right|^2 + \left|\hat{u}_k(0)\right|^2 \\
& + \frac{f^2}{\alpha_k}\left|\hat{v}_k(0)\right|^2 + \frac{g^2k^2}{\alpha_k}\left|\hat{h}_k(0)\right|^2 + \frac{fg\,|k|}{\alpha_k}\left[\left|\hat{v}_k(0)\right|^2 + \left|\hat{h}_k(0)\right|^2\right] \\
& + \frac{fg^2H\,|k|^3}{\alpha_k^2}\left[\left|\hat{v}_k(0)\right|^2 + \left|\hat{h}_k(0)\right|^2\right] \\
& = \left|\hat{u}_k(0)\right|^2 + \beta_k^+\left|\hat{v}_k(0)\right|^2 + \gamma_k^+\left|\hat{h}_k(0)\right|^2 \;,
\end{aligned}$$

with:

$$\beta_k^+ = \left(\frac{g^2H^2k^4}{\alpha_k^2} + \frac{f^2}{\alpha_k} + \frac{fg\,|k|}{\alpha_k} + \frac{fg^2H\,|k|^3}{\alpha_k^2}\right) \;,$$

and:

$$\gamma_k^+ = \left(\frac{f^2g^2k^2}{\alpha_k^2} + \frac{g^2k^2}{\alpha_k} + \frac{fg\,|k|}{\alpha_k} + \frac{fg^2H\,|k|^3}{\alpha_k^2}\right) \;.$$

Finally, we have:

$$||\widetilde{\widehat{\mathbf{U}}}_k(t)||_2^2 \leq \max\left(1, \beta_k^+, \gamma_k^+\right)||\widehat{\mathbf{U}}_k(0)||_2^2 \;.$$

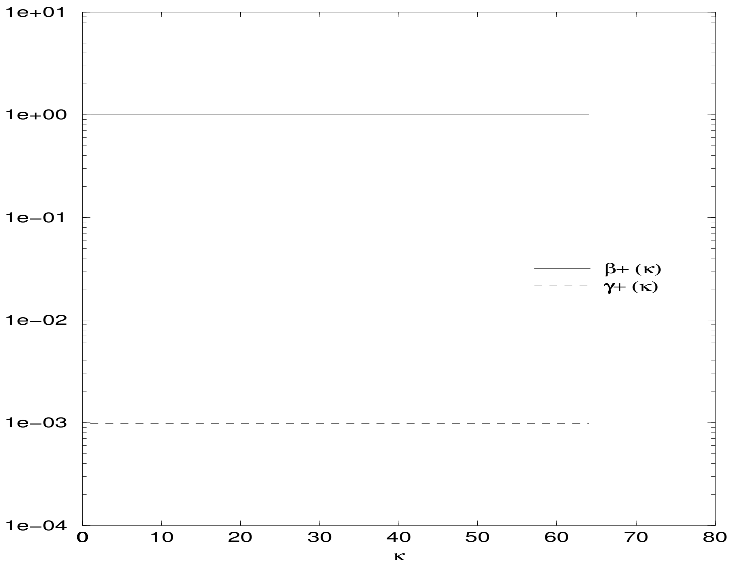

Figure 3. Representation of β_k^+ and γ_k^+ as functions of k.

On Figure 3, we have represented β_k^+ and γ_k^+ as functions of k. So we have established the following result:

Proposition 1

$$||\widehat{\widetilde{\mathbf{U}}}_k(t)||_2 \leq ||\widehat{\mathbf{U}}_k(0)||_2 \ , \ \forall t \geq 0 \ . \tag{23}$$

More precisely, it follows from (23) that $\widetilde{\mathbf{U}}$ *has the same regularity as the initial condition (same decrease of the Fourier coefficients when* $|k|$ *increases). So, under the hypothesis that the initial condition is regular, we can deduce that,* $\forall t \geq 0$*, the waves associated with fast propagation (high wavenumbers, i.e. the small scales, see Figure 2) have small spatial oscillations: when the propagation speed increases, the height of the oscillations decreases.* □

Let us now consider the velocity vector in the eigenvector basis. Writing (11) in the eigenvector basis, we have:

$$\widetilde{\mathbf{U}}(x,t) = \sum_{k=1-N/2}^{N/2} \widehat{\widetilde{\mathbf{U}}}_k(t) \exp(\mathbf{i}\, kx) \ . \tag{24}$$

Let $N_1 < N$; using (24), we define as follows a scale separation based on the given cut-off level N_1:

$$\widetilde{\mathbf{U}} = \widetilde{\mathbf{Y}} + \widetilde{\mathbf{Z}} \ , \tag{25}$$

with:

$$\widetilde{\mathbf{Y}}(x,t) = \sum_{k \in I_{N_1}} \widehat{\widetilde{\mathbf{U}}}_k(t) \exp(\mathbf{i}\, kx) \ , \tag{26}$$

and:

$$\widetilde{\mathbf{Z}}(x,t) = \sum_{k \in I_N \setminus I_{N_1}} \widehat{\widetilde{\mathbf{U}}}_k(t) \exp(\mathbf{i}\, kx) \ , \tag{27}$$

where we have denoted $I_{N_1} = [1 - N_1/2, N_1/2]$ and $I_N \setminus I_{N_1} = [1 - N/2, N_1/2] \bigcup [N_1/2 + 1, N/2]$.

As it has been said previously, we denote by $\Omega = (0, 2\pi)$ the domain associated with (8)-(10). We want to compare $||\widetilde{\mathbf{Y}}||_{L^2(\Omega)^3}$ with $||\widetilde{\mathbf{Z}}||_{L^2(\Omega)^3}$. We have, using Parseval's equality (see [4]):

$$\begin{aligned} ||\widetilde{\mathbf{Y}}(t)||^2_{L^2(\Omega)^3} &= ||\sum_{k \in I_{N_1}} \widehat{\widetilde{\mathbf{U}}}_k(t) \exp(\mathbf{i}\, kx)||^2_{L^2(\Omega)^3} = \sum_{k \in I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_k(t)||_2^2 \\ &= \sum_{k \in I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_k(0)||_2^2 = ||\widetilde{\mathbf{Y}}(0)||^2_{L^2(\Omega)^3}, \ \forall t > 0 \ . \end{aligned}$$

In the same manner, we obtain that:

$$||\widetilde{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3} = ||\widetilde{\mathbf{Z}}(0)||^2_{L^2(\Omega)^3}, \ \forall t > 0 \ .$$

So we have:

$$\frac{||\widetilde{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||^2_{L^2(\Omega)^3}} = \frac{||\widetilde{\mathbf{Z}}(0)||^2_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(0)||^2_{L^2(\Omega)^3}} \ . \tag{28}$$

In order to establish a majoration (resp. a minoration) of the previous ratio, we look for a majoration (resp. a minoration) of $||\widetilde{\mathbf{Z}}(0)||^2_{L^2(\Omega)^3}$ (resp. $||\widetilde{\mathbf{Y}}(0)||^2_{L^2(\Omega)^3}$). Using Parseval's equality and (23) we have:

$$||\widetilde{\mathbf{Z}}(0)||^2_{L^2(\Omega)^3} = \sum_{k \in I_N \setminus I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_k(0)||_2^2 \leq \sum_{k \in I_N \setminus I_{N_1}} ||\widehat{\mathbf{U}}_k(0)||_2^2 = ||\mathbf{Z}(0)||^2_{L^2(\Omega)^3} \ . \tag{29}$$

Now, considering $\widetilde{\mathbf{Y}}(0)$ we have:

$$||\widetilde{\mathbf{Y}}(0)||^2_{L^2(\Omega)^3} = \sum_{k\in I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_k(0)||_2^2 = \sum_{k\in I_{N_1}} ||P_k^{-1}\widehat{\mathbf{U}}_k(0)||_2^2 \, .$$

Yet:

$$||P_k^{-1}\widehat{\mathbf{U}}_k(0)||_2^2 = \left|\frac{gHk^2}{\alpha_k}\hat{v}_k(0) + \frac{fgk}{\alpha_k}\mathbf{i}\,\hat{h}_k(0)\right|^2 + \frac{1}{2}\left|\hat{u}_k(0)\right|^2 + \frac{1}{2}\left|\frac{\mathbf{i}\,f}{\sqrt{\alpha_k}}\hat{v}_k(0) + \frac{gk}{\sqrt{\alpha_k}}\hat{h}_k(0)\right|^2 \, .$$

Using the inequalities $||z| - |z'|| \leq |z + z'| \leq |z| + |z'|, \forall z$ and $z' \in \mathbb{C}$, we obtain the following minoration:

$$\begin{aligned} ||P_k^{-1}\widehat{\mathbf{U}}_k(0)||_2^2 &\geq \left|\frac{gHk^2}{\alpha_k}\left|\hat{v}_k(0)\right| - \frac{fg\,|k|}{\alpha_k}\left|\hat{h}_k(0)\right|\right|^2 + \frac{1}{2}\left|\hat{u}_k(0)\right|^2 + \frac{1}{2}\left|\frac{f}{\sqrt{\alpha_k}}\left|\hat{v}_k(0)\right| - \frac{g\,|k|}{\sqrt{\alpha_k}}\left|\hat{h}_k(0)\right|\right|^2 \\ &= \frac{g^2H^2k^4}{\alpha_k^2}\left|\hat{v}_k(0)\right|^2 + \frac{f^2g^2k^2}{\alpha_k^2}\left|\hat{h}_k(0)\right|^2 - 2\frac{fg^2H\,|k|^3}{\alpha_k^2}\left|\hat{v}_k(0)\right| \times \left|\hat{h}_k(0)\right| \\ &+ \frac{1}{2}\left|\hat{u}_k(0)\right|^2 + \frac{f^2}{2\alpha_k}\left|\hat{v}_k(0)\right|^2 + \frac{g^2k^2}{2\alpha_k}\left|\hat{h}_k(0)\right|^2 - \frac{fg\,|k|}{\alpha_k}\left|\hat{v}_k(0)\right| \times \left|\hat{h}_k(0)\right| \, . \end{aligned}$$

As previously, using the fact that $2ab \leq a^2 + b^2, \forall a \geq 0$ and $\forall b \geq 0$, we obtain:

$$\begin{aligned} ||P_k^{-1}\widehat{\mathbf{U}}_k(0)||_2^2 &\geq \left(\frac{g^2H^2k^4}{\alpha_k^2} + \frac{f^2}{2\alpha_k}\right)\left|\hat{v}_k(0)\right|^2 + \frac{1}{2}\left|\hat{u}_k(0)\right|^2 + \left(\frac{f^2g^2k^2}{\alpha_k^2} + \frac{g^2k^2}{2\alpha_k}\right)\left|\hat{h}_k(0)\right|^2 \\ &- \frac{fg^2H\,|k|^3}{\alpha_k^2}\left(\left|\hat{v}_k(0)\right|^2 + \left|\hat{h}_k(0)\right|^2\right) - \frac{fg\,|k|}{2\alpha_k}\left(\left|\hat{v}_k(0)\right|^2 + \left|\hat{h}_k(0)\right|^2\right) \, . \end{aligned}$$

Finally:

$$\begin{aligned} ||P_k^{-1}\widehat{\mathbf{U}}_k(0)||_2^2 &\geq \frac{1}{2}\left|\hat{u}_k(0)\right|^2 + \left(\frac{g^2H^2k^4}{\alpha_k^2} + \frac{f^2}{2\alpha_k} - \frac{fg^2H\,|k|^3}{\alpha_k^2} - \frac{fg\,|k|}{2\alpha_k}\right)\left|\hat{v}_k(0)\right|^2 \\ &+ \left(\frac{f^2g^2k^2}{\alpha_k^2} + \frac{g^2k^2}{2\alpha_k} - \frac{fg^2H\,|k|^3}{\alpha_k^2} - \frac{fg\,|k|}{2\alpha_k}\right)\left|\hat{h}_k(0)\right|^2 \, . \end{aligned}$$

Let us set

$$\beta_k^- = \left(\frac{g^2H^2k^4}{\alpha_k^2} + \frac{f^2}{2\alpha_k} - \frac{fg\,|k|}{2\alpha_k} - \frac{fg^2H\,|k|^3}{\alpha_k^2}\right) \, ,$$

and

$$\gamma_k^- = \left(\frac{f^2g^2k^2}{\alpha_k^2} + \frac{g^2k^2}{2\alpha_k} - \frac{fg\,|k|}{2\alpha_k} - \frac{fg^2H\,|k|^3}{\alpha_k^2}\right) \, .$$

We can prove and check on Figure 4, that β_k^- and γ_k^- are positive for every k. Consequently, we can write:

$$||P_k^{-1}\widehat{\mathbf{U}}_k(0)||_2^2 \geq \min\left(\frac{1}{2}, \beta_k^-, \gamma_k^-\right)||\widehat{\mathbf{U}}_k(0)||_2^2 \, . \tag{30}$$

If we set $m_k^- = \min\left(\frac{1}{2}, \beta_k^-, \gamma_k^-\right)$, we have obtained that:

$$||\widetilde{\mathbf{Y}}(0)||^2_{L^2(\Omega)^3} \geq \sum_{k\in I_{N_1}} m_k^-||\widehat{\mathbf{U}}_k(0)||_2^2 \simeq m^-||\mathbf{Y}(0)||^2_{L^2(\Omega)^3} \, , \tag{31}$$

since, as we can notice on Figure 4, $m_k^- \simeq m^- \simeq 10^{-3}$ for every k. So we have proved the following result:

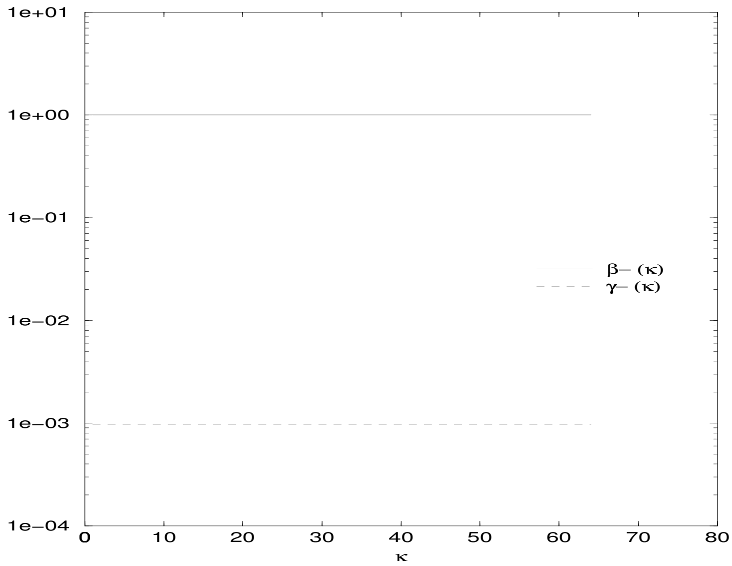

Figure 4. Representation of β_k^- and γ_k^- as functions of k.

Proposition 2

$$\frac{||\widetilde{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} = \frac{||\widetilde{\mathbf{Z}}(0)||_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(0)||_{L^2(\Omega)^3}} \leq \frac{1}{\sqrt{m^-}} \frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \ , \ \forall t > 0 \ , \tag{32}$$

with $m^- \simeq 10^{-3}$*. It follows that if the initial condition is regular, we have* $\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \ll 1$*, so* $\frac{||\widetilde{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \ll 1,\ \forall t > 0$*, for a given cut-off level* N_1 *quite high (we will come back on the choice of the level* N_1 *later).* □

Now we consider the velocity vector in the canonical basis, instead of the eigenvector basis as previously. We have (see (11)):

$$\mathbf{U}(x,t) = \sum_{k=1-N/2}^{N/2} \widehat{\mathbf{U}}_k(t) \exp(\mathbf{i}\, kx) \ . \tag{33}$$

As previously, we define a scale separation based on the given cut-off level $N_1 < N$:

$$\mathbf{Y}(x,t) = \sum_{k \in I_{N_1}} \widehat{\mathbf{U}}_k(t) \exp(\mathbf{i}\, kx) \ , \tag{34}$$

and:

$$\mathbf{Z}(x,t) = \sum_{k \in I_N \setminus I_{N_1}} \widehat{\mathbf{U}}_k(t) \exp(\mathbf{i}\, kx) \ . \tag{35}$$

We want to compare $\mathbf{Y}$ and $\mathbf{Z}$. In order to establish a majoration of the ratio $\dfrac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}}$, we look for a minoration of $\dfrac{||\widetilde{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}}$ in function of $\dfrac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}}$. We have (Parseval's equality):

$$||\widetilde{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3} = \sum_{k\in I_N\setminus I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_k(t)||_2^2 = \sum_{k\in I_N\setminus I_{N_1}} ||P_k^{-1}\widehat{\mathbf{U}}_k(t)||_2^2 \,.$$

From (30) which is valid at each time $t \geq 0$, we deduce that:

$$||P_k^{-1}\widehat{\mathbf{U}}_k(t)||_2^2 \geq m^- ||\widehat{\mathbf{U}}_k(t)||_2^2 \,. \tag{36}$$

Hence

$$||\widetilde{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3} \geq m^- \sum_{k\in I_N\setminus I_{N_1}} ||\widehat{\mathbf{U}}_k(t)||_2^2 = m^- ||\mathbf{Z}(t)||^2_{L^2(\Omega)^3} \,.$$

In the same way, proceeding as for (29), it follows that

$$||\widetilde{\mathbf{Y}}(t)||^2_{L^2(\Omega)^3} = \sum_{k\in I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_k(t)||_2^2 = \sum_{k\in I_{N_1}} ||P_k^{-1}\widehat{\mathbf{U}}_k(t)||_2^2 \leq \sum_{k\in I_{N_1}} ||\widehat{\mathbf{U}}_k(t)||_2^2 = ||\mathbf{Y}(t)||^2_{L^2(\Omega)^3} \,. \tag{37}$$

Finally, we can write:

$$\frac{||\widetilde{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||^2_{L^2(\Omega)^3}} \geq m^- \frac{||\mathbf{Z}(t)||^2_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||^2_{L^2(\Omega)^3}} \,,$$

and, using (32) we obtain the following result:

Proposition 3

$$\frac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}} \leq \frac{1}{m^-}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \,,\ \forall t > 0 \,, \tag{38}$$

with $m^- \simeq 10^{-3}$. *If the initial condition is regular, we have* $\dfrac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \ll 1$, *so* $\dfrac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}} \ll 1$, $\forall t > 0$, *for a given cut-off level* N_1 *quite high.* □

Remark 1 More precisely, if we assume that the initial data $\mathbf{U}(0)$ lies in $\left(H_p^m(\Omega)\right)^3$, with $H_p^m(\Omega) = \left\{u \in H^m(\Omega),\ u^{(j)} \text{ periodic for } j = 0,\dots,m\right\}$, the decrease of the velocity spectrum is like $|k|^{-m}$ (see [4]). So:

$$\frac{||\mathbf{Z}(0)||^2_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||^2_{L^2(\Omega)^3}} = \frac{\displaystyle\sum_{k\in I_N\setminus I_{N_1}} ||\widehat{\mathbf{U}}_k||_2^2}{\displaystyle\sum_{k\in I_{N_1}} ||\widehat{\mathbf{U}}_k||_2^2} \simeq \frac{\displaystyle\sum_{k\in I_N\setminus I_{N_1}} \frac{c}{k^{2m}}}{\displaystyle\sum_{k\in I_{N_1-\{0\}}} \frac{c}{k^{2m}}}$$

where c is a positive constant depending on the initial condition.

For $N = 1024$, we have represented, on Figure 5, the ratio $\dfrac{1}{m^-}\dfrac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}$ appearing in (38), for different choices of the cut-off level $N_1 = \alpha N$, $\alpha \in\ ]0,1[$, and for different values of the regularity parameter m $(m = 1,2,3)$. ■

We are now interested in comparing the time derivatives of the quantities associated with the large and small scales. In the eigenvectors basis, we have (see (19) and (24)):

$$\widetilde{\mathbf{U}}(x,t) = \sum_{k\in I_N} \widehat{\widetilde{\mathbf{U}}}_k(t)\exp(\mathbf{i}\,kx) = \sum_{k\in I_N} \begin{pmatrix} \hat{\tilde{u}}_k(0)\exp(\mathbf{i}\,\omega_{1,k}t) \\ \hat{\tilde{v}}_k(0)\exp(\mathbf{i}\,\omega_{2,k}t) \\ \hat{\tilde{h}}_k(0)\exp(\mathbf{i}\,\omega_{3,k}t) \end{pmatrix} \exp(\mathbf{i}\,kx)\ .$$

The time derivative $\dot{\widetilde{\mathbf{U}}}(x,t)$ of $\widetilde{\mathbf{U}}(x,t)$ is written:

$$\dot{\widetilde{\mathbf{U}}}(x,t) = \sum_{k\in I_N} \begin{pmatrix} \omega_{1,k}\hat{\tilde{u}}_k(0)\exp(\mathbf{i}\,\omega_{1,k}t) \\ \omega_{2,k}\hat{\tilde{v}}_k(0)\exp(\mathbf{i}\,\omega_{2,k}t) \\ \omega_{3,k}\hat{\tilde{h}}_k(0)\exp(\mathbf{i}\,\omega_{3,k}t) \end{pmatrix} \exp(\mathbf{i}\,kx) = \sum_{k\in I_N} \widehat{\dot{\widetilde{\mathbf{U}}}}_k(t)\exp(\mathbf{i}\,kx)\ ,$$

with:

$$\widehat{\dot{\widetilde{\mathbf{U}}}}_k(t) = \begin{pmatrix} \omega_{1,k}\hat{\tilde{u}}_k(0)\exp(\mathbf{i}\,\omega_{1,k}t) \\ \omega_{2,k}\hat{\tilde{v}}_k(0)\exp(\mathbf{i}\,\omega_{2,k}t) \\ \omega_{3,k}\hat{\tilde{h}}_k(0)\exp(\mathbf{i}\,\omega_{3,k}t) \end{pmatrix}\ . \tag{39}$$

Scale decomposition for the time derivative is obtained as previously (see (25)):

$$\begin{aligned} \dot{\widetilde{\mathbf{U}}}(x,t) &= \sum_{k\in I_{N_1}} \widehat{\dot{\widetilde{\mathbf{U}}}}_k(t)\exp(\mathbf{i}\,kx) + \sum_{k\in I_N\setminus I_{N_1}} \widehat{\dot{\widetilde{\mathbf{U}}}}_k(t)\exp(\mathbf{i}\,kx) \\ &= \dot{\widetilde{\mathbf{Y}}}(x,t) + \dot{\widetilde{\mathbf{Z}}}(x,t) \end{aligned}$$

Using Parseval's equality and (39) it follows that:

$$\begin{aligned} ||\dot{\widetilde{\mathbf{Z}}}(t)||^2_{L^2(\Omega)^3} &= \sum_{k\in I_N\setminus I_{N_1}} ||\widehat{\dot{\widetilde{\mathbf{U}}}}_k(t)||_2^2 = \sum_{k\in I_N\setminus I_{N_1}} ||\widehat{\dot{\widetilde{\mathbf{U}}}}_k(0)||_2^2 \\ &\leq \sum_{k\in I_N\setminus I_{N_1}} \left|\omega_{2,\,|k|}\right|^2 ||\widehat{\widetilde{\mathbf{U}}}_k(0)||_2^2 \qquad \text{(see Figure 2)} \\ &\leq \left|\omega_{2,N/2}\right|^2 \sum_{k\in I_N\setminus I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_k(0)||_2^2 \\ &= \left|\omega_{2,N/2}\right|^2 ||\widetilde{\mathbf{Z}}(0)||^2_{L^2(\Omega)^3} \\ &\leq \left|\omega_{2,N/2}\right|^2 ||\mathbf{Z}(0)||^2_{L^2(\Omega)^3}\ . \qquad \text{(see (29))} \end{aligned}$$

For the large scales, proceeding as before, we obtain:

$$\begin{aligned} ||\dot{\widetilde{\mathbf{Y}}}(t)||^2_{L^2(\Omega)^3} &= \sum_{k\in I_{N_1}} ||\widehat{\dot{\widetilde{\mathbf{U}}}}_k(t)||_2^2 = \sum_{k\in I_{N_1}} ||\widehat{\dot{\widetilde{\mathbf{U}}}}_k(0)||_2^2 \\ &\geq \sum_{k\in I_{N_1}} |\omega_{1,k}|^2 ||\widehat{\widetilde{\mathbf{U}}}_k(0)||_2^2 \qquad \text{(see Figure 2)} \\ &\geq |\omega_{1,1}|^2 \sum_{k\in I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_k(0)||_2^2 \qquad \text{(by hypothesis, U is zero average)} \\ &= U^2 ||\widetilde{\mathbf{Y}}(0)||^2_{L^2(\Omega)^3} \\ &\geq U^2 m^- ||\mathbf{Y}(0)||^2_{L^2(\Omega)^3}\ . \qquad \text{(see (31))} \end{aligned}$$

Finally, we can write:

Proposition 4

$$\frac{||\dot{\tilde{\mathbf{Z}}}(t)||_{L^2(\Omega)^3}}{||\dot{\tilde{\mathbf{Y}}}(t)||_{L^2(\Omega)^3}} \leq \frac{\left|\omega_{2,N/2}\right|}{U\sqrt{m^-}} \frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \;,\; \forall t \geq 0 \;, \tag{40}$$

with $m^- \simeq 10^{-3}$, $U \simeq 10$ *to* $100\ m/s$ *and* $\omega_{2,N/2} \simeq 25 \times 10^3$ *for* $N = 128$ *(see Figure 2). So, if the initial condition is quite regular, we have* $\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \ll 1$, *and thus* $\frac{||\dot{\tilde{\mathbf{Z}}}(t)||_{L^2(\Omega)^3}}{||\dot{\tilde{\mathbf{Y}}}(t)||_{L^2(\Omega)^3}} \ll 1$, $\forall t > 0$, *for a suitable choice of the cut-off level* N_1. □

Proposition 4 gives a comparison of the large and small scales in the eigenvector basis. Now we look for a comparison in the canonical basis, *i.e.* we want to compare $\dot{\mathbf{Y}}$ and $\dot{\mathbf{Z}}$. We have:

$$\begin{aligned} ||\dot{\tilde{\mathbf{Z}}}(t)||^2_{L^2(\Omega)^3} &= \sum_{k \in I_N \setminus I_{N_1}} ||\widehat{\dot{\tilde{\mathbf{U}}}}_k(t)||_2^2 = \sum_{k \in I_N \setminus I_{N_1}} ||P_k^{-1} \widehat{\dot{\mathbf{U}}}_k(t)||_2^2 \\ &\geq m^- \sum_{k \in I_N \setminus I_{N_1}} ||\widehat{\dot{\mathbf{U}}}_k(t)||_2^2 \qquad \text{(see (36))} \\ &= m^- ||\dot{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3} \end{aligned}$$

and:

$$\begin{aligned} ||\dot{\tilde{\mathbf{Y}}}(t)||^2_{L^2(\Omega)^3} &= \sum_{k \in I_{N_1}} ||\widehat{\dot{\tilde{\mathbf{U}}}}_k(t)||_2^2 = \sum_{k \in I_{N_1}} ||P_k^{-1} \widehat{\dot{\mathbf{U}}}_k(t)||_2^2 \\ &\leq \sum_{k \in I_{N_1}} ||\widehat{\dot{\mathbf{U}}}_k(t)||_2^2 \qquad \text{(see (37))} \\ &= ||\dot{\mathbf{Y}}(t)||^2_{L^2(\Omega)^3} \end{aligned}$$

So we have obtained:

$$\frac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \leq \frac{1}{\sqrt{m^-}} \frac{||\dot{\tilde{\mathbf{Z}}}(t)||_{L^2(\Omega)^3}}{||\dot{\tilde{\mathbf{Y}}}(t)||_{L^2(\Omega)^3}} \;,\; \forall t \geq 0 \;.$$

The following result is infered from (40):

Proposition 5

$$\frac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \leq \frac{\left|\omega_{2,N/2}\right|}{U m^-} \frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \;,\; \forall t > 0 \tag{41}$$

If the initial condition is regular, we have $\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \ll 1$, *so* $\frac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \ll 1$, $\forall t > 0$, *for a given cut-off level* N_1 *quite high.* □

Remark 2 If, as for Proposition 3, we consider that the initial condition lies in $H_p^m(\Omega)$, the decrease of the velocity spectrum is like $|k|^{-m}$. Similarly with Figure 5, we have represented, on Figure 6, the majoration of $\frac{\left|\omega_{2,N/2}\right|}{U m^-} \frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}$ for $N = 1024$ and for different choices of the cut-off level $N_1 < N$ and of the regularity parameter m. ■

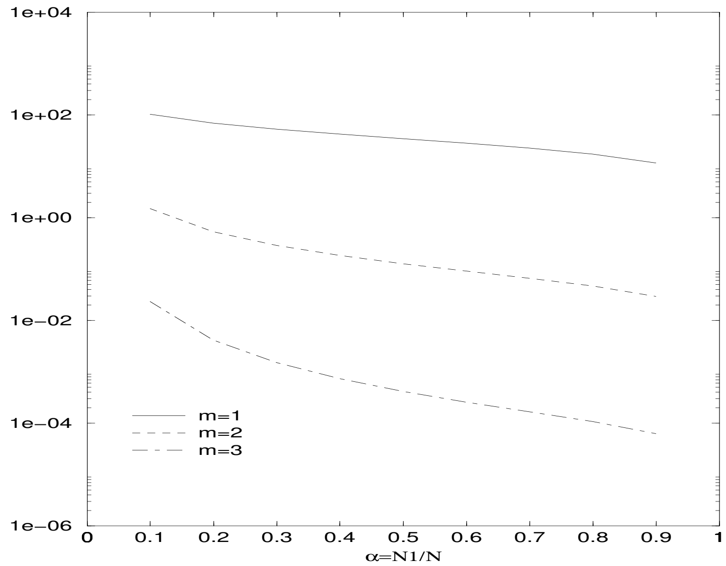

Figure 5. Representation of $\frac{1}{m^-}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}$ as a function of N_1/N, for $N = 1024$.

3. The shallow water problem: the two-dimensional case

In the two-dimensional case, the linearized shallow water equations are written:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} , \qquad (42)$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} , \qquad (43)$$

$$\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 . \qquad (44)$$

Here u, v, h, H, f and U have the same meaning as in the one-dimensional case, and V is a constant advecting velocity in the y direction: U and V are typically of order of 10 to 100 m/s.

Let us consider a solution $\mathbf{U}^T = (u, v, h)$ for (42)-(44) of the following form (see (11) for the one-dimensional case):

$$\mathbf{U}(\mathbf{x}, t) = \sum_{\mathbf{k}\in I_N} \widehat{\mathbf{U}}_{\mathbf{k}}(t) \exp(\mathbf{i}\,\mathbf{k}\cdot\mathbf{x}) = \sum_{\mathbf{k}\in I_N} \begin{pmatrix} \hat{u}_{\mathbf{k}} \\ \hat{v}_{\mathbf{k}} \\ \hat{h}_{\mathbf{k}} \end{pmatrix} (t) \exp(\mathbf{i}\,\mathbf{k}\cdot\mathbf{x}) , \qquad (45)$$

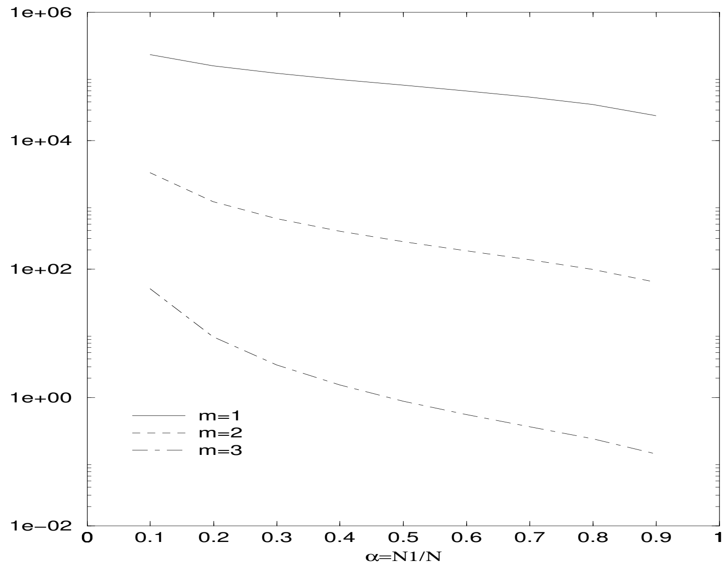

Figure 6. Representation of $\frac{|\omega_{2,N/2}|}{Um^{-}} \frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}$ as a function of N_1/N, for $N = 1024$.

where $I_N = [1 - N/2, N/2]^2$, $\mathbf{k}^T = (k_1, k_2)$ is the wavenumber vector, and $\mathbf{k} \cdot \mathbf{x} = k_1 x + k_2 y$ is the Euclidean scalar product. We suppose periodic boundary conditions in the x and y directions and that $\langle \mathbf{U} \rangle_\Omega = \widehat{\mathbf{U}}_0 = \mathbf{0}$ ($\mathbf{U}$ has zero mean value), at each time $t \geq 0$; here $\Omega = (0, 2\pi)^2$. This will be the case if the initial conditions have zero average.

Using (45), we can rewrite (42)-(44) in the following manner:

$$\frac{d\widehat{\mathbf{U}}_{\mathbf{k}}}{dt} + A_{\mathbf{k}} \widehat{\mathbf{U}}_{\mathbf{k}} = \mathbf{0} \; , \tag{46}$$

with $A_{\mathbf{k}}$ the matrix defined by:

$$A_{\mathbf{k}} = \begin{pmatrix} \mathbf{i}\,\mathcal{U} \cdot \mathbf{k} & -f & g\mathbf{i}\,k_1 \\ f & \mathbf{i}\,\mathcal{U} \cdot \mathbf{k} & g\mathbf{i}\,k_2 \\ \mathbf{i}\,Hk_1 & \mathbf{i}\,Hk_2 & \mathbf{i}\,\mathcal{U} \cdot \mathbf{k} \end{pmatrix} , \tag{47}$$

where $\mathcal{U} \cdot \mathbf{k} = Uk_1 + Vk_2$, $\mathcal{U}^T = (U, V)$ being the constant advecting velocity vector. The eigenvalues of the matrix $A_{\mathbf{k}}$ are:

$$\begin{cases} \lambda_{1,\mathbf{k}} = \mathbf{i}\,\mathcal{U} \cdot \mathbf{k} \; , \\ \lambda_{2,\mathbf{k}} = \mathbf{i}\,\mathcal{U} \cdot \mathbf{k} + \mathbf{i}\,\sqrt{f^2 + gH\,|\mathbf{k}|^2} \; , \\ \lambda_{3,\mathbf{k}} = \mathbf{i}\,\mathcal{U} \cdot \mathbf{k} - \mathbf{i}\,\sqrt{f^2 + gH\,|\mathbf{k}|^2} \; . \end{cases} \tag{48}$$

In the eigenvector basis, (46) gives:

$$\frac{d\widehat{\widehat{\mathbf{U}}}_{\mathbf{k}}}{dt} + D_{\mathbf{k}}\widehat{\widehat{\mathbf{U}}}_{\mathbf{k}} = \mathbf{0}\ , \tag{49}$$

where $D_{\mathbf{k}}$ denotes the diagonal matrix Diag $(\lambda_{i,\mathbf{k}})_{i=1,2,3}$. Results similar to (17), (18) and (19) can be deduced for the two-dimensional case. The matrix $P_{\mathbf{k}}$ defined by (20) is as follows:

$$P_{\mathbf{k}} = \begin{pmatrix} -k_2 & \mathbf{i}\, f k_2 + \sqrt{\alpha_k} k_1 & \mathbf{i}\, f k_2 - \sqrt{\alpha_k} k_1 \\ k_1 & -\mathbf{i}\, f k_1 + \sqrt{\alpha_k} k_2 & -\mathbf{i}\, f k_1 - \sqrt{\alpha_k} k_2 \\ \dfrac{-\mathbf{i}\, f}{g} & H\,|\mathbf{k}|^2 & H\,|\mathbf{k}|^2 \end{pmatrix}\ , \tag{50}$$

with $\alpha_k = f^2 + gHk^2$, and $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$. For $\mathbf{k} \neq \mathbf{0}$, the inverse matrix $P_{\mathbf{k}}^{-1}$ is expressed in the following manner:

$$P_{\mathbf{k}}^{-1} = \begin{pmatrix} -\dfrac{gHk_2}{\alpha_k} & \dfrac{gHk_1}{\alpha_k} & \dfrac{\mathbf{i}\, fg}{\alpha_k} \\ \dfrac{k_1\sqrt{\alpha_k} - \mathbf{i}\, f k_2}{2\alpha_k\,|\mathbf{k}|^2} & \dfrac{k_2\sqrt{\alpha_k} + \mathbf{i}\, f k_1}{2\alpha_k\,|\mathbf{k}|^2} & \dfrac{g}{2\alpha_k} \\ -\dfrac{k_1\sqrt{\alpha_k} + \mathbf{i}\, f k_2}{2\alpha_k\,|\mathbf{k}|^2} & \dfrac{-k_2\sqrt{\alpha_k} + \mathbf{i}\, f k_1}{2\alpha_k\,|\mathbf{k}|^2} & \dfrac{g}{2\alpha_k} \end{pmatrix}\ . \tag{51}$$

In order to establish, for the two-dimensional case, results similar to (38), we look for a majoration and a minoration of $||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2$, $\mathbf{k} \neq \mathbf{0}$. Firstly, consider a majoration. We have:

$$\begin{aligned} ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 = \; & \left|\frac{gH}{\alpha_k}\left(-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right) + \frac{\mathbf{i}\, fg}{\alpha_k}\hat{h}_{\mathbf{k}}\right|^2 + \\ & \left|\frac{\sqrt{\alpha_k}}{2\alpha_k\,|\mathbf{k}|^2}\left(k_1\hat{u}_{\mathbf{k}} + k_2\hat{v}_{\mathbf{k}}\right) + \frac{\mathbf{i}\, f}{2\alpha_k\,|\mathbf{k}|^2}\left(-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right) + \frac{g}{2\alpha_k}\hat{h}_{\mathbf{k}}\right|^2 + \\ & \left|-\frac{\sqrt{\alpha_k}}{2\alpha_k\,|\mathbf{k}|^2}\left(k_1\hat{u}_{\mathbf{k}} + k_2\hat{v}_{\mathbf{k}}\right) + \frac{\mathbf{i}\, f}{2\alpha_k\,|\mathbf{k}|^2}\left(-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right) + \frac{g}{2\alpha_k}\hat{h}_{\mathbf{k}}\right|^2 \\ = \; & \left|\frac{gH}{\alpha_k}\left(-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right) + \frac{\mathbf{i}\, fg}{\alpha_k}\hat{h}_{\mathbf{k}}\right|^2 + \\ & 2\left|\frac{\sqrt{\alpha_k}}{2\alpha_k\,|\mathbf{k}|^2}\left(k_1\hat{u}_{\mathbf{k}} + k_2\hat{v}_{\mathbf{k}}\right)\right|^2 + 2\left|\frac{\mathbf{i}\, f}{2\alpha_k\,|\mathbf{k}|^2}\left(-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right) + \frac{g}{2\alpha_k}\hat{h}_{\mathbf{k}}\right|^2 . \end{aligned}$$

So we find:

$$\begin{aligned} ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \leq \; & \left|\frac{gH}{\alpha_k}\,|\hat{\omega}_{\mathbf{k}}| + \frac{fg}{\alpha_k}\left|\hat{h}_{\mathbf{k}}\right|\right|^2 + 2\left|\frac{\sqrt{\alpha_k}}{2\alpha_k\,|\mathbf{k}|^2}\left|\hat{\delta}_{\mathbf{k}}\right|\right|^2 + \\ & 2\left|\frac{f}{2\alpha_k\,|\mathbf{k}|^2}\,|\hat{\omega}_{\mathbf{k}}| + \frac{g}{2\alpha_k}\left|\hat{h}_{\mathbf{k}}\right|\right|^2 . \end{aligned}$$

Finally we obtain:

$$\begin{aligned} ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \; & \leq \frac{g^2H^2}{\alpha_k^2}\,|\hat{\omega}_{\mathbf{k}}|^2 + \frac{f^2g^2}{\alpha_k^2}\left|\hat{h}_{\mathbf{k}}\right|^2 + \frac{2fg^2H}{\alpha_k^2}\,|\hat{\omega}_{\mathbf{k}}| \times \left|\hat{h}_{\mathbf{k}}\right| + \frac{1}{2\alpha_k\,|\mathbf{k}|^4}\left|\hat{\delta}_{\mathbf{k}}\right|^2 \\ & + \frac{f^2}{2\alpha_k^2\,|\mathbf{k}|^4}\,|\hat{\omega}_{\mathbf{k}}|^2 + \frac{g^2}{2\alpha_k^2}\left|\hat{h}_{\mathbf{k}}\right|^2 + \frac{fg}{\alpha_k^2\,|\mathbf{k}|^2}\,|\hat{\omega}_{\mathbf{k}}| \times \left|\hat{h}_{\mathbf{k}}\right|\ , \end{aligned}$$

where we have denoted by $\omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ the vorticity, and by $\delta = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$ the divergence. These scalar variables ω and δ are well appropriate for the approximation of the shallow water problem on a sphere, as opposed to vector velocity components which are multi-valued at the pole (pole problem). For more details see, for example, [40].

Using the inequality $2ab \leq a^2 + b^2$, $\forall a \geq 0, \forall b \geq 0$, we find:

$$
\begin{aligned}
||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \leq \; & \left(\frac{g^2H^2}{\alpha_k^2} + \frac{fg^2H}{\alpha_k^2} + \frac{f^2}{2\alpha_k^2\,|\mathbf{k}|^4} + \frac{fg}{2\alpha_k^2\,|\mathbf{k}|^2}\right)|\hat{\omega}_{\mathbf{k}}|^2 + \frac{1}{2\alpha_k\,|\mathbf{k}|^4}\left|\hat{\delta}_{\mathbf{k}}\right|^2 + \\
& \left(\frac{f^2g^2}{\alpha_k^2} + \frac{fg^2H}{\alpha_k^2} + \frac{g^2}{2\alpha_k^2} + \frac{fg}{2\alpha_k^2\,|\mathbf{k}|^2}\right)\left|\hat{h}_{\mathbf{k}}\right|^2 .
\end{aligned}
$$

Finally, we have obtained:

$$
\begin{aligned}
||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 & \leq \beta_k^+\,|\hat{\omega}_{\mathbf{k}}|^2 + \gamma_k^+\left|\hat{\delta}_{\mathbf{k}}\right|^2 + \zeta_k^+\left|\hat{h}_{\mathbf{k}}\right|^2 \\
& \leq m_k^+\,||\widehat{\mathbf{U}}_{\text{div},\mathbf{k}}||_2^2 \, ,
\end{aligned}
\tag{52}
$$

with:

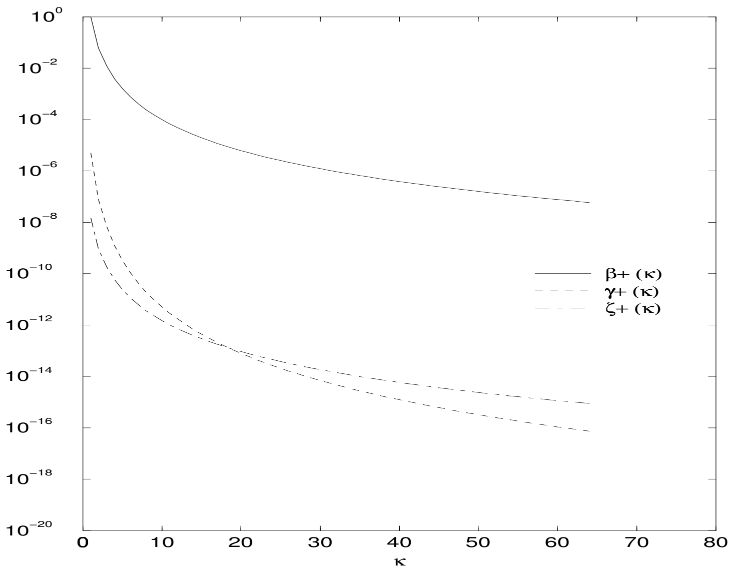

Figure 7. Representation of β_k^+, γ_k^+ and ζ_k^+ as functions of k.

$$\begin{aligned}
\beta_k^+ &= \frac{g^2H^2}{\alpha_k^2} + \frac{fg^2H}{\alpha_k^2} + \frac{f^2}{2\alpha_k^2\,|\mathbf{k}|^4} + \frac{fg}{2\alpha_k^2\,|\mathbf{k}|^2}\,, \\
\gamma_k^+ &= \frac{1}{2\alpha_k\,|\mathbf{k}|^4}\,, \\
\zeta_k^+ &= \frac{f^2g^2}{\alpha_k^2} + \frac{fg^2H}{\alpha_k^2} + \frac{g^2}{2\alpha_k^2} + \frac{fg}{2\alpha_k^2\,|\mathbf{k}|^2}\,, \\
m_k^+ &= \max\left(\beta_k^+, \gamma_k^+, \zeta_k^+\right)\,,
\end{aligned}$$

and:

$$\mathbf{U}_{\text{div}} = \begin{pmatrix} \omega \\ \delta \\ h \end{pmatrix}\,.$$

The graphs of β_k^+, γ_k^+ and ζ_k^+ as functions of k are shown on Figure 7.

Now we aim to derive a lower bound. We have:

$$\begin{aligned}
||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 = & \left|\frac{gH}{\alpha_k}\left(-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right) + \frac{\mathbf{i}\,fg}{\alpha_k}\hat{h}_{\mathbf{k}}\right|^2 + \\
& 2\left|\frac{\sqrt{\alpha_k}}{2\alpha_k\,|\mathbf{k}|^2}\left(k_1\hat{u}_{\mathbf{k}} + k_2\hat{v}_{\mathbf{k}}\right)\right|^2 + 2\left|\frac{f\mathbf{i}}{2\alpha_k\,|\mathbf{k}|^2}\left(-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right) + \frac{g}{2\alpha_k}\hat{h}_{\mathbf{k}}\right|^2 \\
\geq & \left|\frac{gH}{\alpha_k}\left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right| - \frac{fg}{\alpha_k}\left|\hat{h}_{\mathbf{k}}\right|\right|^2 + \frac{1}{2\alpha_k\,|\mathbf{k}|^4}\left|k_1\hat{u}_{\mathbf{k}} + k_2\hat{v}_{\mathbf{k}}\right|^2 + \\
& 2\left|\frac{f}{2\alpha_k\,|\mathbf{k}|^2}\left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right| - \frac{g}{2\alpha_k}\left|\hat{h}_{\mathbf{k}}\right|\right|^2\,.
\end{aligned}$$

So:

$$\begin{aligned}
||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \geq & \frac{g^2H^2}{\alpha_k^2}\left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right|^2 + \frac{f^2g^2}{\alpha_k^2}\left|\hat{h}_{\mathbf{k}}\right|^2 - \frac{2fg^2H}{\alpha_k^2}\left|\hat{h}_{\mathbf{k}}\right| \times \left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right| + \\
& \frac{1}{2\alpha_k\,|\mathbf{k}|^4}\left|k_1\hat{u}_{\mathbf{k}} + k_2\hat{v}_{\mathbf{k}}\right|^2 + \frac{f^2}{2\alpha_k^2\,|\mathbf{k}|^4}\left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right|^2 + \frac{g^2}{2\alpha_k^2}\left|\hat{h}_{\mathbf{k}}\right|^2 - \\
& \frac{fg}{\alpha_k^2\,|\mathbf{k}|^2}\left|\hat{h}_{\mathbf{k}}\right| \times \left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right| \\
\geq & \frac{g^2H^2}{\alpha_k^2}\left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right|^2 + \frac{f^2g^2}{\alpha_k^2}\left|\hat{h}_{\mathbf{k}}\right|^2 - \frac{fg^2H}{\alpha_k^2}\left(\left|\hat{h}_{\mathbf{k}}\right|^2 + \left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right|^2\right) + \\
& \frac{1}{2\alpha_k\,|\mathbf{k}|^4}\left|k_1\hat{u}_{\mathbf{k}} + k_2\hat{v}_{\mathbf{k}}\right|^2 + \frac{f^2}{2\alpha_k^2\,|\mathbf{k}|^4}\left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right|^2 + \frac{g^2}{2\alpha_k^2}\left|\hat{h}_{\mathbf{k}}\right|^2 - \\
& \frac{fg}{2\alpha_k^2\,|\mathbf{k}|^2}\left(\left|\hat{h}_{\mathbf{k}}\right|^2 + \left|-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right|^2\right) \\
= & \left(\frac{g^2H^2}{\alpha_k^2} - \frac{fg^2H}{\alpha_k^2} + \frac{f^2}{2\alpha_k^2\,|\mathbf{k}|^4} - \frac{fg}{2\alpha_k^2\,|\mathbf{k}|^2}\right)\left|\hat{\omega}_{\mathbf{k}}\right|^2 + \\
& \frac{1}{2\alpha_k\,|\mathbf{k}|^4}\left|\hat{\delta}_{\mathbf{k}}\right|^2 + \left(\frac{f^2g^2}{\alpha_k^2} - \frac{fg^2H}{\alpha_k^2} + \frac{g^2}{2\alpha_k^2} - \frac{fg}{2\alpha_k^2\,|\mathbf{k}|^2}\right)\left|\hat{h}_{\mathbf{k}}\right|^2\,.
\end{aligned}$$

Finally, we have established the following inequality:

$$||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \geq m_k^- ||\widehat{\mathbf{U}}_{\mathrm{div},\mathbf{k}}||_2^2 \,, \tag{53}$$

with:

$$\begin{aligned}
\beta_k^- &= \frac{g^2H^2}{\alpha_k^2} - \frac{fg^2H}{\alpha_k^2} + \frac{f^2}{2\alpha_k^2\,|\mathbf{k}|^4} - \frac{fg}{2\alpha_k^2\,|\mathbf{k}|^2} \,, \\
\gamma_k^- &= \frac{1}{2\alpha_k\,|\mathbf{k}|^4} \,, \\
\zeta_k^- &= \frac{f^2g^2}{\alpha_k^2} - \frac{fg^2H}{\alpha_k^2} + \frac{g^2}{2\alpha_k^2} - \frac{fg}{2\alpha_k^2\,|\mathbf{k}|^2} \,, \\
m_k^- &= \min \quad \left(\beta_k^-, \gamma_k^-, \zeta_k^-\right) \,.
\end{aligned}$$

The graphs of β_k^-, γ_k^- and ζ_k^- as functions of k are given on Figure 8.

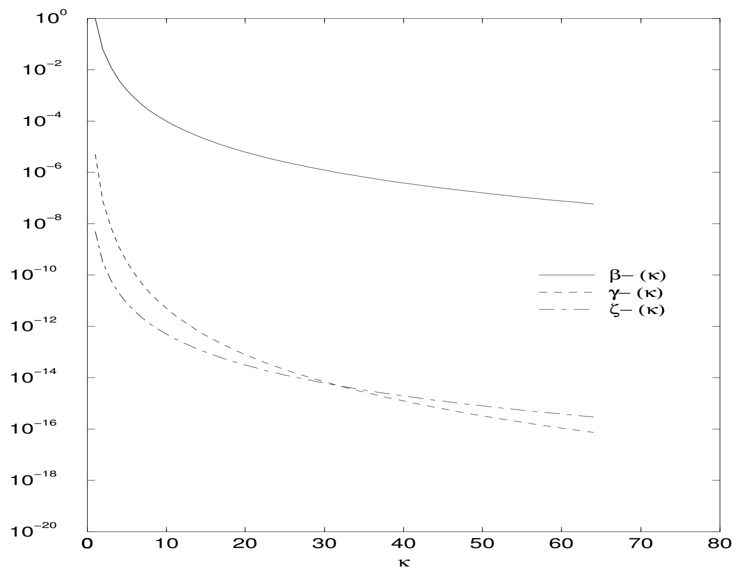

Figure 8. Representation of β_k^-, γ_k^- and ζ_k^- as functions of k.

As for the one-dimensional case, we have (see (28)):

$$\frac{||\widetilde{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||^2_{L^2(\Omega)^3}} = \frac{||\widetilde{\mathbf{Z}}(0)||^2_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(0)||^2_{L^2(\Omega)^3}} = \frac{\sum_{\mathbf{k}\in I_N\setminus I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_{\mathbf{k}}(0)||_2^2}{\sum_{\mathbf{k}\in I_{N_1}} ||\widehat{\widetilde{\mathbf{U}}}_{\mathbf{k}}(0)||_2^2}\,, \tag{54}$$

with $\Omega = (0,2\pi)^2$, $I_{N_1} = [1 - N_1/2, N_1/2]^2$. Using the fact that $\widehat{\widetilde{\mathbf{U}}}_{\mathbf{k}} = P_k^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}$ (see (20)), and since m_k^+ and m_k^- decrease when k increases (see Figures 7 and 8), we deduce the following bounds:

$$\frac{m^-_{N/2}}{m^+_1}\frac{||\mathbf{Z}_{\text{div}}(t)||^2_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(t)||^2_{L^2(\Omega)^3}} \le \frac{||\widetilde{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||^2_{L^2(\Omega)^3}} \le \frac{m^+_{N_1/2}}{m^-_{N_1/2}}\frac{||\mathbf{Z}_{\text{div}}(0)||^2_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(0)||^2_{L^2(\Omega)^3}}\,. \tag{55}$$

We have denoted $\mathbf{U}_{\text{div}} = \mathbf{Y}_{\text{div}} + \mathbf{Z}_{\text{div}}$, with $\mathbf{U}^T_{\text{div}} = (\omega, \delta, h)$. From (55) we conclude the following

Proposition 6

$$\frac{||\mathbf{Z}_{\text{div}}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(t)||_{L^2(\Omega)^3}} \le \sqrt{\frac{m^+_1 m^+_{N_1/2}}{m^-_{N_1/2}m^-_{N/2}}\frac{||\mathbf{Z}_{\text{div}}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(0)||_{L^2(\Omega)^3}}}\,. \tag{56}$$

If the initial condition $\mathbf{U}_{\text{div}}(0)$ *is regular, we have* $\frac{||\mathbf{Z}_{\text{div}}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(0)||_{L^2(\Omega)^3}} \ll 1$*, so* $\frac{||\mathbf{Z}_{\text{div}}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(t)||_{L^2(\Omega)^3}} \ll 1$, $\forall t > 0$*, for a given cut-off level* N_1 *quite high. Moreover, the ratio* $\frac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}}$ *will be smaller than* $\frac{||\mathbf{Z}_{\text{div}}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(t)||_{L^2(\Omega)^3}}$*, since* $\mathbf{U}$ *has more spatial regularity than* $\mathbf{U}_{\text{div}}$. □

Remark 3 If we choose an initial condition such that $\delta(t = 0) \simeq 0$ (which implies that convergence in one horizontal direction must be neutralized by a divergence in the other horizontal direction, see [5]) and $h(t = 0) \simeq 0$, we can see on Figures 7 and 8 that $m^+_{N_1/2} = m^-_{N_1/2}$. It follows from (56) that, $\forall t > 0$, $\frac{||\mathbf{Z}_{\text{div}}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(t)||_{L^2(\Omega)^3}}$ decreases, when N_1 increases, as $\sqrt{\frac{m^+_1}{m^-_{N/2}}\frac{||\mathbf{Z}_{\text{div}}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(0)||_{L^2(\Omega)^3}}}$. In particular, if the initial condition is associated with a two-dimensional turbulent flow (developed turbulence), the decrease of the kinetic-energy spectrum is like k^{-3} (see [22]), and the kinetic energy is essentially contained in the large scales (see [27]); so the ratio $\frac{||\mathbf{Z}_{\text{div}}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(0)||_{L^2(\Omega)^3}}$ will be small. ■

Now we compare the time derivatives of the quantities associated with the large and small scales. We proceed as for the one-dimensional case. We have:

$$||\dot{\widehat{\widetilde{\mathbf{U}}}}_{\mathbf{k}}||_2^2 \le \omega^2_{2,k}||\widehat{\widetilde{\mathbf{U}}}_{\mathbf{k}}||_2^2 = \omega^2_{2,k}||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2\,,$$

with $\omega_{2,k} = ||\mathcal{U}||_2 k + \sqrt{f^2 + gHk^2}$ and $k = |\mathbf{k}|$. Using (52) we obtain:

$$||\dot{\widehat{\widetilde{\mathbf{U}}}}_{\mathbf{k}}||_2^2 \le \omega^2_{2,k} m_k^+ ||\widehat{\mathbf{U}}_{\text{div},\mathbf{k}}||_2^2\,. \tag{57}$$

In the same manner, using the fact that $|\omega_{1,\mathbf{k}}| \le |\omega_{i,\mathbf{k}}|$, $i = 2, 3$, (since, for the characteristic values retained here for H, U and V, we have: $|\omega_{i,\mathbf{k}}| - |\omega_{1,\mathbf{k}}| \ge \sqrt{f^2 + gHk^2} - |\mathcal{U}\cdot\mathbf{k}| - |\omega_{1,\mathbf{k}}| \ge \sqrt{f^2 + gHk^2} - 2k||\mathcal{U}||_2 \ge 0$, for all k), we obtain:

$$||\dot{\widehat{\widetilde{\mathbf{U}}}}_{\mathbf{k}}||_2^2 \ge |\omega_{1,\mathbf{k}}|^2 \times ||\widehat{\widetilde{\mathbf{U}}}_{\mathbf{k}}||_2^2 = |\omega_{1,\mathbf{k}}|^2 \times ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2\,.$$

So, using (53), we deduce that:

$$||\hat{\dot{\widehat{\mathbf{U}}}}_{\mathbf{k}}||_2^2 \geq |\omega_{1,\mathbf{k}}|^2 \, m_k^- \, ||\widehat{\mathbf{U}}_{\text{div},\mathbf{k}}||_2^2 \,, \tag{58}$$

with $\omega_{1,\mathbf{k}} = -\mathcal{U} \cdot \mathbf{k}$. For $\mathcal{U}^T = (U, V)$ with $U \simeq V \simeq 10$ to $100\; m/s$, we have $\omega_{1,\mathbf{k}} \simeq U\,(k_1 + k_2)$, so $|\omega_{1,\mathbf{k}}| \simeq U\,|k_1 + k_2| \geq U$, if $k_1 + k_2 \neq 0$. So, if we choose an initial condition of zero average, we will have $\widehat{\mathbf{U}}_{\text{div},\mathbf{k}}(t = 0) = \mathbf{0}$ for $k_1 + k_2 = 0$. Using the first inequality in (55) for the time derivative, (57) and (58), we arrive at the following result:

Proposition 7

$$\frac{||\dot{\mathbf{Z}}_{\text{div}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}_{\text{div}}(t)||_{L^2(\Omega)^3}} \leq \frac{\omega_{2,N/2}}{U} \sqrt{\frac{m_1^+ m_{N_1/2}^+}{m_{N_1/2}^- m_{N/2}^-}} \, \frac{||\mathbf{Z}_{\text{div}}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(0)||_{L^2(\Omega)^3}} \,. \tag{59}$$

Hence, if the initial condition $\mathbf{U}_{\text{div}}(0)$ *is regular, we have* $\dfrac{||\mathbf{Z}_{\text{div}}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}_{\text{div}}(0)||_{L^2(\Omega)^3}} \ll 1$, *so that* $\dfrac{||\dot{\mathbf{Z}}_{\text{div}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}_{\text{div}}(t)||_{L^2(\Omega)^3}} \ll 1$, $\forall t > 0$, *for a given quite high cut-off level* N_1 *. Moreover, the ratio* $\dfrac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}}$ *will be smaller than* $\dfrac{||\dot{\mathbf{Z}}_{\text{div}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}_{\text{div}}(t)||_{L^2(\Omega)^3}}$, *since* $\dot{\mathbf{U}}$ *has more spatial regularity than* $\dot{\mathbf{U}}_{\text{div}}$. □

4. Geophysical flows

The geophysical flow equations, under the hydrostatic hypothesis and Boussinesq approximation, are expressed as follows (see [5], [15] for example):

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} + W\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}\,, \quad \text{(conservation of momentum in the } x \text{ direction)} \tag{60}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + W\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}\,, \quad \text{(conservation of momentum in the } y \text{ direction)} \tag{61}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g\,, \quad \text{(hydrostatic equation)} \tag{62}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,, \quad \text{(continuity equation)} \tag{63}$$

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} + V\frac{\partial \rho}{\partial y} + W\frac{\partial \rho}{\partial z} = \kappa\frac{\partial^2 \rho}{\partial z^2}\,, \quad \text{(density equation)} \tag{64}$$

where $\mathcal{U}^T = (U, V, W)$, with U, V and $W \simeq 10$ to $100\; m/s$, is a constant advecting velocity vector. The vector (u, v, w) denotes the velocity components in the (x, y, z) directions, p is the hydrostatic pressure and ρ is the density variations around a mean value ρ_0 (for example, $\rho_0 = 1028 kg/m^3$ for sea water, see [5]). The constants f and g are the Coriolis and gravity parameters; ν and κ represent diffusion. More precisely, ν is associated with molecular diffusion, or eddy diffusivity if an eddy viscosity model is used for turbulent diffusion; κ is function of thermal diffusivity. Generally $\kappa \simeq 10^{-2} m^2/s$ (see [5]). If ν is a molecular diffusion, we will have $\kappa \gg \nu$. For example, for water $\nu = 10^{-6} m^2/s$. If ν is an eddy viscosity model, we will have $\nu \simeq \kappa$. Equations (60), (61), and (64) are linearization, around the mean velocity $\mathcal{U}^T = (U, V, W)$, of a general nonlinear geophysical flow.

We suppose that $\mathbf{U}_w^T = (u, v, w, \rho)$ is periodic in the spatial directions (x, y, z) (periodic boundary conditions), with zero average: $\langle \mathbf{U}_w \rangle_\Omega = \widehat{\mathbf{U}}_{w,\mathbf{0}} = 0$, at each time $t \geq 0$, where $\Omega = (0, 2\pi)^3$. This will be the case if the initial condition has zero average.

Let us consider a solution $\mathbf{U}_w$ for (60)-(64) of the following form:

$$\mathbf{U}_w(\mathbf{x}, t) = \sum_{\mathbf{k} \in I_N} \widehat{\mathbf{U}}_{w,\mathbf{k}}(t) \exp(\mathbf{i}\, \mathbf{k} \cdot \mathbf{x}) \ , \tag{65}$$

where $I_N = [1 - N/2, N/2]^3$, $\mathbf{k} = (k_1, k_2, k_3)$ is the wavenumber, and $\mathbf{k} \cdot \mathbf{x} = k_1 x + k_2 y + k_3 z$ is the Euclidean scalar product. Using (65), we can rewrite (60)-(64) in the following manner:

$$\left\{ \begin{array}{rcl} \frac{d\hat{u}_k}{dt} + \mathbf{i}\, \mathcal{U} \cdot \mathbf{k}\hat{u}_\mathbf{k} - f\hat{v}_\mathbf{k} & = & -\frac{1}{\rho_0}\mathbf{i}\, k_1 \hat{p}_\mathbf{k} - \nu k_3^2 \hat{u}_\mathbf{k} \ , \\ \frac{d\hat{v}_k}{dt} + \mathbf{i}\, \mathcal{U} \cdot \mathbf{k}\hat{v}_\mathbf{k} + f\hat{u}_\mathbf{k} & = & -\frac{1}{\rho_0}\mathbf{i}\, k_2 \hat{p}_\mathbf{k} - \nu k_3^2 \hat{v}_\mathbf{k} \ , \\ 0 = -\mathbf{i}\, k_3 \hat{p}_\mathbf{k} - g\hat{\rho}_\mathbf{k} & \Longrightarrow & \hat{p}_\mathbf{k} = -\frac{g}{\mathbf{i}\, k_3}\hat{\rho}_\mathbf{k} \ (k_3 \neq 0) \ , \\ k_1\hat{u}_\mathbf{k} + k_2\hat{v}_\mathbf{k} + k_3\hat{w}_\mathbf{k} & = & 0 \ , \\ \frac{d\hat{\rho}_\mathbf{k}}{dt} + \mathbf{i}\, \mathcal{U} \cdot \mathbf{k}\hat{\rho}_\mathbf{k} & = & -\kappa k_3^2 \hat{\rho}_\mathbf{k} \ . \end{array} \right. \tag{66}$$

For $k_3 \neq 0$, the system of equations (66) can be rewritten as follows:

$$\left\{ \begin{array}{rcl} \frac{d\hat{u}_\mathbf{k}}{dt} + \mathbf{i}\, \mathcal{U} \cdot \mathbf{k}\hat{u}_\mathbf{k} - f\hat{v}_\mathbf{k} & = & \frac{g}{\rho_0}\frac{k_1}{k_3}\hat{\rho}_\mathbf{k} - \nu k_3^2 \hat{u}_\mathbf{k} \ , \\ \frac{d\hat{v}_\mathbf{k}}{dt} + \mathbf{i}\, \mathcal{U} \cdot \mathbf{k}\hat{v}_\mathbf{k} + f\hat{u}_\mathbf{k} & = & \frac{g}{\rho_0}\frac{k_2}{k_3}\hat{\rho}_\mathbf{k} - \nu k_3^2 \hat{v}_\mathbf{k} \ , \\ \frac{d\hat{\rho}_\mathbf{k}}{dt} + \mathbf{i}\, \mathcal{U} \cdot \mathbf{k}\hat{\rho}_\mathbf{k} & = & -\kappa k_3^2 \hat{\rho}_\mathbf{k} \ , \end{array} \right. \tag{67}$$

$\hat{w}_\mathbf{k}$ being then determined, for $k_3 \neq 0$, by using the continuity equation. Finally, (67) can be rewritten in the following manner:

$$\frac{d\widehat{\mathbf{U}}_\mathbf{k}}{dt} + A_\mathbf{k}\widehat{\mathbf{U}}_\mathbf{k} = \mathbf{0} \ , \tag{68}$$

with $\mathbf{U}^T = (u, v, \rho)$ and $A_\mathbf{k}$ is the matrix defined, for $k_3 \neq 0$, by:

$$A_\mathbf{k} = \begin{pmatrix} \mathbf{i}\, \mathcal{U} \cdot \mathbf{k} + \nu k_3^2 & -f & -\frac{g}{\rho_0}\frac{k_1}{k_3} \\ f & \mathbf{i}\, \mathcal{U} \cdot \mathbf{k} + \nu k_3^2 & -\frac{g}{\rho_0}\frac{k_2}{k_3} \\ 0 & 0 & \mathbf{i}\, \mathcal{U} \cdot \mathbf{k} + \kappa k_3^2 \end{pmatrix} . \tag{69}$$

The eigenvalues of the matrix $A_\mathbf{k}$ are:

$$\left\{ \begin{array}{l} \lambda_{1,\mathbf{k}} = \mathbf{i}\, \mathcal{U} \cdot \mathbf{k} + \kappa k_3^2 \ , \\ \lambda_{2,\mathbf{k}} = \mathbf{i}\, \mathcal{U} \cdot \mathbf{k} + \nu k_3^2 + \mathbf{i}\, f \ , \\ \lambda_{3,\mathbf{k}} = \mathbf{i}\, \mathcal{U} \cdot \mathbf{k} + \nu k_3^2 - \mathbf{i}\, f \ . \end{array} \right. \tag{70}$$

In the eigenvector basis, equation (67) gives:

$$\frac{d\widehat{\widehat{\mathbf{U}}}_\mathbf{k}}{dt} + D_\mathbf{k}\widehat{\widehat{\mathbf{U}}}_\mathbf{k} = \mathbf{0} \ , \tag{71}$$

with $D_{\mathbf{k}}$ the diagonal matrix $D_{\mathbf{k}} = \text{Diag}\,(\lambda_{i,\mathbf{k}})_{i=1,2,3}$. Time integration of (71) gives us:

$$\hat{\hat{\mathbf{U}}}_{\mathbf{k}}(t) = \hat{\hat{\mathbf{U}}}_{\mathbf{k}}(0)\exp\left(-D_{\mathbf{k}}t\right) \;, \tag{72}$$

showing that $\left|\hat{\hat{u}}_{\mathbf{k}}(t)\right|$, $\left|\hat{\hat{v}}_{\mathbf{k}}(t)\right|$ and $\left|\hat{\hat{\rho}}_{\mathbf{k}}(t)\right|$ decrease when t increases (dissipation due to ν and κ). If we denote by $P_{\mathbf{k}}$, $k_3 \neq 0$, the non singular matrix defined, as previously, by:

$$\begin{pmatrix} \hat{u}_{\mathbf{k}} \\ \hat{v}_{\mathbf{k}} \\ \hat{\rho}_{\mathbf{k}} \end{pmatrix} = P_{\mathbf{k}} \begin{pmatrix} \hat{\hat{u}}_{\mathbf{k}} \\ \hat{\hat{v}}_{\mathbf{k}} \\ \hat{\hat{\rho}}_{\mathbf{k}} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \hat{\hat{u}}_{\mathbf{k}} \\ \hat{\hat{v}}_{\mathbf{k}} \\ \hat{\hat{\rho}}_{\mathbf{k}} \end{pmatrix} = P_{\mathbf{k}}^{-1} \begin{pmatrix} \hat{u}_{\mathbf{k}} \\ \hat{v}_{\mathbf{k}} \\ \hat{\rho}_{\mathbf{k}} \end{pmatrix} \;, \tag{73}$$

we have:

$$P_{\mathbf{k}} = \begin{pmatrix} -\dfrac{g}{\rho_0 k_3}\left(f k_2 + (\nu-\kappa)k_3^2 k_1\right) & \mathbf{i} & \mathbf{i} \\ -\dfrac{g}{\rho_0 k_3}\left(-f k_1 + (\nu-\kappa)k_3^2 k_2\right) & 1 & -1 \\ -\left(f^2 + k_3^4(\nu-\kappa)^2\right) & 0 & 0 \end{pmatrix} \;, \tag{74}$$

and:

$$P_{\mathbf{k}}^{-1} = -\frac{1}{2\mathbf{i}\,\left(f^2 + k_3^4(\nu-\kappa)^2\right)} \times$$

$$\times \begin{pmatrix} 0 & 0 & 2\mathbf{i} \\ -\left(f^2 + k_3^4(\nu-\kappa)^2\right) & -\mathbf{i}\,\left(f^2 + k_3^4(\nu-\kappa)^2\right) & \dfrac{g}{\rho_0 k_3}\left((k_1 + \mathbf{i}\,k_2)((\nu-\kappa)k_3^2 - \mathbf{i}\,f)\right) \\ -\left(f^2 + k_3^4(\nu-\kappa)^2\right) & \mathbf{i}\,\left(f^2 + k_3^4(\nu-\kappa)^2\right) & -\dfrac{g}{\rho_0 k_3}\left((\mathbf{i}\,k_2 - k_1)(\mathbf{i}\,f + (\nu-\kappa)k_3^2)\right) \end{pmatrix} . \tag{75}$$

If we set:

$$\begin{cases} \omega_{1,\mathbf{k}} = -\mathcal{U}\cdot\mathbf{k} \;, \\ \omega_{2,\mathbf{k}} = -\left(\mathcal{U}\cdot\mathbf{k} + f\right) \;, \\ \omega_{3,\mathbf{k}} = -\left(\mathcal{U}\cdot\mathbf{k} - f\right) \;, \end{cases} \tag{76}$$

we obtain, using (65) and (72):

$$\begin{cases} \tilde{u}(x,t) = \displaystyle\sum_{\mathbf{k}\in I_N} \hat{\hat{u}}_{\mathbf{k}}(0)\exp\left(-\kappa k_3^2 t\right)\exp\left(\mathbf{i}\left(\mathbf{k}\cdot\mathbf{x} + \omega_{1,\mathbf{k}}t\right)\right) \;, \\ \tilde{v}(x,t) = \displaystyle\sum_{\mathbf{k}\in I_N} \hat{\hat{v}}_{\mathbf{k}}(0)\exp\left(-\nu k_3^2 t\right)\exp\left(\mathbf{i}\left(\mathbf{k}\cdot\mathbf{x} + \omega_{2,\mathbf{k}}t\right)\right) \;, \\ \tilde{\rho}(x,t) = \displaystyle\sum_{\mathbf{k}\in I_N} \hat{\hat{\rho}}_{\mathbf{k}}(0)\exp\left(-\nu k_3^2 t\right)\exp\left(\mathbf{i}\left(\mathbf{k}\cdot\mathbf{x} + \omega_{3,\mathbf{k}}t\right)\right) \;. \end{cases} \tag{77}$$

Now we look for a majoration and a minoration of $||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2$. We set $\alpha_{k_3} = \left(f^2 + k_3^4(\nu-\kappa)^2\right)$. We have, for $k_3 \neq 0$:

$$||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 = \left|-\frac{1}{\alpha_{k_3}}\hat{\rho}_{\mathbf{k}}\right|^2 + \left|-\frac{\mathbf{i}}{2}\hat{u}_{\mathbf{k}} + \frac{1}{2}\hat{v}_{\mathbf{k}} + \frac{\mathbf{i}\,g}{2\rho_0 k_3 \alpha_{k_3}}\left((k_1 + \mathbf{i}\,k_2)((\nu-\kappa)k_3^2 - \mathbf{i}\,f)\right)\hat{\rho}_{\mathbf{k}}\right|^2 + \left|-\frac{\mathbf{i}}{2}\hat{u}_{\mathbf{k}} - \frac{1}{2}\hat{v}_{\mathbf{k}} - \frac{\mathbf{i}\,g}{2\rho_0 k_3 \alpha_{k_3}}\left((\mathbf{i}\,k_2 - k_1)(\mathbf{i}\,f + (\nu-\kappa)k_3^2)\right)\hat{\rho}_{\mathbf{k}}\right|^2$$

$$= \frac{1}{\alpha_{k_3}^2}|\hat{\rho}_{\mathbf{k}}|^2 + \left|-\frac{\mathbf{i}}{2}\hat{u}_{\mathbf{k}} + \frac{1}{2}\hat{v}_{\mathbf{k}} + \frac{\mathbf{i}\,g}{2\rho_0 k_3 \alpha_{k_3}}\left((\nu-\kappa)k_3^2 k_1 - \mathbf{i}\,k_1 f + \mathbf{i}\,(\nu-\kappa)k_2 k_3^2 + f k_2\right)\hat{\rho}_{\mathbf{k}}\right|^2 + \left|-\frac{\mathbf{i}}{2}\hat{u}_{\mathbf{k}} - \frac{1}{2}\hat{v}_{\mathbf{k}} - \frac{\mathbf{i}\,g}{2\rho_0 k_3 \alpha_{k_3}}\left(-k_2 f + \mathbf{i}\,(\nu-\kappa)k_2 k_3^2 - \mathbf{i}\,k_1 f - (\nu-\kappa)k_1 k_3^2\right)\hat{\rho}_{\mathbf{k}}\right|^2,$$

$$||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 = \frac{1}{\alpha_{k_3}^2}|\hat{\rho}_{\mathbf{k}}|^2 + \left|-\frac{\mathbf{i}}{2}\hat{u}_{\mathbf{k}} + \frac{\mathbf{i}\,g}{2\rho_0 k_3 \alpha_{k_3}}\left((\nu-\kappa)k_1 k_3^2 + f k_2\right)\hat{\rho}_{\mathbf{k}} + \frac{1}{2}\hat{v}_{\mathbf{k}} + \frac{g}{2\rho_0 k_3 \alpha_{k_3}}\left(k_1 f - (\nu-\kappa)k_2 k_3^2\right)\hat{\rho}_{\mathbf{k}}\right|^2 + \left|-\frac{\mathbf{i}}{2}\hat{u}_{\mathbf{k}} + \frac{\mathbf{i}\,g}{2\rho_0 k_3 \alpha_{k_3}}\left((\nu-\kappa)k_1 k_3^2 + f k_2)\right)\hat{\rho}_{\mathbf{k}} - \frac{1}{2}\hat{v}_{\mathbf{k}} - \frac{g}{2\rho_0 k_3 \alpha_{k_3}}\left(k_1 f - (\nu-\kappa)k_2 k_3^2\right)\hat{\rho}_{\mathbf{k}}\right|^2$$

$$= \frac{1}{\alpha_{k_3}^2}|\hat{\rho}_{\mathbf{k}}|^2 + \frac{1}{2}\left|-\mathbf{i}\,\hat{u}_{\mathbf{k}} + \frac{\mathbf{i}\,g}{\rho_0 k_3 \alpha_{k_3}}\left((\nu-\kappa)k_1 k_3^2 + f k_2\right)\hat{\rho}_{\mathbf{k}}\right|^2 + \frac{1}{2}\left|\hat{v}_{\mathbf{k}} + \frac{g}{\rho_0 k_3 \alpha_{k_3}}\left(k_1 f - (\nu-\kappa)k_2 k_3^2\right)\hat{\rho}_{\mathbf{k}}\right|^2,$$

$$||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 = \frac{1}{\alpha_{k_3}^2}|\hat{\rho}_{\mathbf{k}}|^2 + \frac{1}{2}|\hat{u}_{\mathbf{k}}|^2 + \frac{g^2}{2\rho_0^2 k_3^2 \alpha_{k_3}^2}\left((\nu-\kappa)k_1 k_3^2 + f k_2\right)^2|\hat{\rho}_{\mathbf{k}}|^2 - \hat{u}_{\mathbf{k}}\frac{g}{2\rho_0 k_3 \alpha_{k_3}}\left((\nu-\kappa)k_1 k_3^2 + f k_2\right)\overline{\hat{\rho}}_{\mathbf{k}} - \overline{\hat{u}}_{\mathbf{k}}\frac{g}{2\rho_0 k_3 \alpha_{k_3}}\left((\nu-\kappa)k_1 k_3^2 + f k_2\right)\hat{\rho}_{\mathbf{k}} + \frac{1}{2}|\hat{v}_{\mathbf{k}}|^2 + \frac{g^2}{2\rho_0^2 k_3^2 \alpha_{k_3}^2}\left(k_1 f - (\nu-\kappa)k_2 k_3^2\right)^2|\hat{\rho}_{\mathbf{k}}|^2 + \overline{\hat{v}}_{\mathbf{k}}\frac{g}{2\rho_0 k_3 \alpha_{k_3}}\left(k_1 f - (\nu-\kappa)k_2 k_3^2\right)\hat{\rho}_{\mathbf{k}} + \hat{v}_{\mathbf{k}}\frac{g}{2\rho_0 k_3 \alpha_{k_3}}\left(k_1 f - (\nu-\kappa)k_2 k_3^2\right)\overline{\hat{\rho}}_{\mathbf{k}}\,.$$

Finally, we obtain:

$$||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 = \frac{1}{2}|\hat{u}_{\mathbf{k}}|^2 + \frac{1}{2}|\hat{v}_{\mathbf{k}}|^2 + \frac{1}{\alpha_{k_3}^2}|\hat{\rho}_{\mathbf{k}}|^2 + \frac{g^2}{2\rho_0^2 k_3^2 \alpha_{k_3}^2}\left(\left((\nu-\kappa)k_1 k_3^2 + f k_2\right)^2 + \left(k_1 f - (\nu-\kappa)k_2 k_3^2\right)^2\right)|\hat{\rho}_{\mathbf{k}}|^2 + \frac{g}{\rho_0 k_3 \alpha_{k_3}} Re\left(-\hat{u}_{\mathbf{k}}\left((\nu-\kappa)k_1 k_3^2 + f k_2\right)\overline{\hat{\rho}}_{\mathbf{k}} + \hat{v}_{\mathbf{k}}\left(k_1 f - (\nu-\kappa)k_2 k_3^2\right)\overline{\hat{\rho}}_{\mathbf{k}}\right),$$

$Re(z)$ denoting the real part of $z \in \mathbb{C}$. Now we consider two distinct useful cases: $\nu \simeq \kappa$ and $\nu \ll \kappa$.

4.1. Case of a Prandtl number of order one ($\nu \simeq \kappa$)

Firstly, we consider the case $\nu \simeq \kappa$ (eddy viscosity). We obtain, for $k_3 \neq 0$:

$$||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \simeq \frac{1}{2}|\hat{u}_{\mathbf{k}}|^2 + \frac{1}{2}|\hat{v}_{\mathbf{k}}|^2 + \frac{1}{\alpha_{k_3}^2}|\hat{\rho}_{\mathbf{k}}|^2 + \frac{g^2 f^2}{2\rho_0^2 k_3^2 \alpha_{k_3}^2}\left(k_1^2 + k_2^2\right)|\hat{\rho}_{\mathbf{k}}|^2 + \frac{g}{\rho_0 k_3 \alpha_{k_3}} Re\left(f\overline{\hat{\rho}}_{\mathbf{k}}\left(-k_2\hat{u}_{\mathbf{k}} + k_1\hat{v}_{\mathbf{k}}\right)\right) .$$

We denote by $\omega_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ the vorticity component in the z direction (vertical direction), and since $\alpha_{k_3} \simeq f^2$, we obtain:

$$||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \simeq \frac{1}{2}|\hat{u}_{\mathbf{k}}|^2 + \frac{1}{2}|\hat{v}_{\mathbf{k}}|^2 + \left(\frac{1}{f^4} + \frac{g^2 f^2 (k_1^2 + k_2^2)}{2\rho_0^2 k_3^2 f^4}\right)|\hat{\rho}_{\mathbf{k}}|^2 + \frac{g}{\rho_0 k_3 f^2} Re\left(f\overline{\hat{\rho}}_{\mathbf{k}}\hat{\omega}_{z,\mathbf{k}}\right) .$$

If we consider a flow such that $Re\left(f\overline{\hat{\rho}}_{\mathbf{k}}\hat{\omega}_{z,\mathbf{k}}\right) \simeq 0$, $\forall t \geq 0$ and $\forall \mathbf{k} \in I_N$, (for example small density variations around the mean value ρ_0, *i.e.* essentially incompressible flow), we can deduce that:

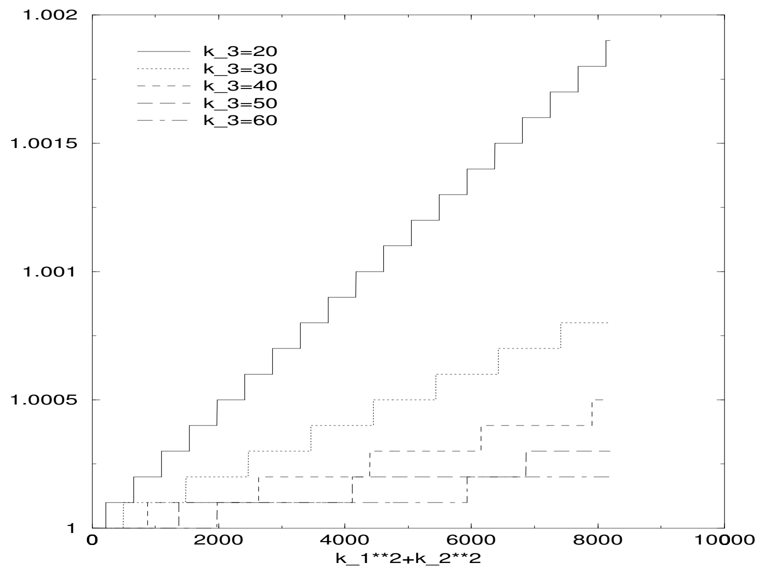

Figure 9. Representation of the ratio $\left(\frac{1}{f^4} + \frac{g^2(k_1^2 + k_2^2)}{2\rho_0^2 k_3^2 f^2}\right) / \left(\frac{1}{f^4}\right)$ as a function of (k_1, k_2) and k_3.

$$||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \simeq \frac{1}{2}\left|\hat{u}_{\mathbf{k}}\right|^2 + \frac{1}{2}\left|\hat{v}_{\mathbf{k}}\right|^2 + \left(\frac{1}{f^4} + \frac{g^2(k_1^2+k_2^2)}{2\rho_0^2 k_3^2 f^2}\right)\left|\hat{\rho}_{\mathbf{k}}\right|^2 \ . \tag{78}$$

Finally, we have established the following upper and lower bounds for $||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2$, $k_3 \neq 0$:

$$m_{\mathbf{k}}^{-}||\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \leq ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \leq m_{\mathbf{k}}^{+}||\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \ , \tag{79}$$

with:

$$\begin{cases} m_{\mathbf{k}}^{-} &= \min\left(\frac{1}{2}, \frac{1}{f^4} + \frac{g^2(k_1^2+k_2^2)}{2\rho_0^2 k_3^2 f^2}\right) \ , \\ m_{\mathbf{k}}^{+} &= \max\left(\frac{1}{2}, \frac{1}{f^4} + \frac{g^2(k_1^2+k_2^2)}{2\rho_0^2 k_3^2 f^2}\right) \ . \end{cases} \tag{80}$$

On Figure 9 we can display the representation of the following expression:

$$\left(\frac{1}{f^4} + \frac{g^2(k_1^2+k_2^2)}{2\rho_0^2 k_3^2 f^2}\right) / \left(\frac{1}{f^4}\right) \ , \tag{81}$$

as a function of (k_1, k_2) and $k_3(\neq 0)$. We can see that $m_{\mathbf{k}}^{-} \simeq m^{-} = \frac{1}{2}$ and $m_{\mathbf{k}}^{+} \simeq m^{+} = \frac{1}{f^4}$ for every $\mathbf{k}$.

As previously, we define a scale separation based on a given cut-off level $N_1 < N$ in the following manner

$$\widetilde{\mathbf{U}} = \widetilde{\mathbf{Y}} + \widetilde{\mathbf{Z}} \ , \tag{82}$$

with

$$\widetilde{\mathbf{Y}}(x,t) = \sum_{\mathbf{k}\in I_{N_1}} \widehat{\widehat{\mathbf{U}}}_{\mathbf{k}}(t)\exp(\mathbf{i}\,\mathbf{k}\cdot\mathbf{x}) \ , \tag{83}$$

and

$$\widetilde{\mathbf{Z}}(x,t) = \sum_{\mathbf{k}\in I_N\setminus I_{N_1}} \widehat{\widehat{\mathbf{U}}}_{\mathbf{k}}(t)\exp(\mathbf{i}\,\mathbf{k}\cdot\mathbf{x}) \ , \tag{84}$$

where $I_{N_1} = [1 - N_1/2, N_1/2]^3$. Moreover, we deduce from (66) that, if $\hat{\omega}_{z,\mathbf{k}} = 0$ for $k_3 = 0$ at the initial time, then $\hat{\rho}_{\mathbf{k}} = 0$ (z-momentum equation), $\hat{\delta}_{\mathbf{k}} = 0$ (continuity equation) and $\hat{\omega}_{z,\mathbf{k}} = 0$ (x and y-momentum equations), $\forall t > 0$, where δ is the plane divergence and ω_z the vertical component of the vorticity. So $\widehat{\mathbf{U}}_{\mathbf{k}} = \mathbf{0}$, $\forall t > 0$, for $k_3 = 0$. Using Parseval's equality we can write, if we denote by $\Omega = (0, 2\pi)^3$:

$$||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}^2 = ||\sum_{\mathbf{k}\in I_{N_1}} \widehat{\widehat{\mathbf{U}}}_{\mathbf{k}}(t)\exp(\mathbf{i}\,\mathbf{k}\cdot\mathbf{x})||_{L^2(\Omega)^3}^2 = \sum_{\mathbf{k}\in I_{N_1}} ||\widehat{\widehat{\mathbf{U}}}_{\mathbf{k}}(t)||_2^2 \ . \tag{85}$$

According to (72) and (77) we have, using the hypothesis $\nu \simeq \kappa$:

$$||\widehat{\widehat{\mathbf{U}}}_{\mathbf{k}}(t)||_2^2 \simeq \exp\left(-2\nu k_3^2 t\right)||\widehat{\widehat{\mathbf{U}}}_{\mathbf{k}}(0)||_2^2 = \exp\left(-2\nu k_3^2 t\right)||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}(0)||_2^2 \ . \tag{86}$$

With (79), we can obtain from (86) that:

$$m^{-}\exp\left(-2\nu k_3^2 t\right)||\widehat{\mathbf{U}}_{\mathbf{k}}(0)||_2^2 \leq ||\widehat{\widehat{\mathbf{U}}}_{\mathbf{k}}(t)||_2^2 \leq m^{+}\exp\left(-2\nu k_3^2 t\right)||\widehat{\mathbf{U}}_{\mathbf{k}}(0)||_2^2 \ . \tag{87}$$

From (85) and (87) we obtain, using the fact that $\widehat{\mathbf{U}}_{\mathbf{k}} = \mathbf{0}$, $\forall t \geq 0$, for $k_3 = 0$:

$$m^{-}\exp\left(-2\nu\left(\frac{N_1}{2}\right)^2 t\right)||\mathbf{Y}(0)||_{L^2(\Omega)^3}^2 \leq ||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}^2 \leq m^{+}\exp\left(-2\nu t\right)||\mathbf{Y}(0)||_{L^2(\Omega)^3}^2 \ . \tag{88}$$

In the same manner, for $\widetilde{\mathbf{Z}}(t)$ we obtain:

$$m^- \exp\left(-2\nu\left(\frac{N}{2}\right)^2 t\right)||\mathbf{Z}(0)||^2_{L^2(\Omega)^3} \leq ||\widetilde{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3} \leq m^+ \exp\left(-2\nu\left(\frac{N_1}{2}\right)^2 t\right)||\mathbf{Z}(0)||^2_{L^2(\Omega)^3}\,. \tag{89}$$

Finally we can write that:

$$\frac{||\widetilde{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \leq \sqrt{\frac{m^+}{m^-}}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \simeq \frac{1}{f^2}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}\,. \tag{90}$$

Now, we want to bound from below the previous ratio $\dfrac{||\widetilde{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}}$, using the ratio $\dfrac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}}$. Using (79) and (85) we have:

$$m^- \sum_{\mathbf{k}\in I_{N_1}} ||\widehat{\mathbf{U}}_{\mathbf{k}}(t)||_2^2 \leq ||\widetilde{\mathbf{Y}}(t)||^2_{L^2(\Omega)^3} = \sum_{\mathbf{k}\in I_{N_1}} ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}(t)||_2^2 \leq m^+ \sum_{\mathbf{k}\in I_{N_1}} ||\widehat{\mathbf{U}}_{\mathbf{k}}(t)||_2^2\,,$$

i.e.:

$$m^-||\mathbf{Y}(t)||^2_{L^2(\Omega)^3} \leq ||\widetilde{\mathbf{Y}}(t)||^2_{L^2(\Omega)^3} \leq m^+||\mathbf{Y}(t)||^2_{L^2(\Omega)^3}\,. \tag{91}$$

In the same manner, we have:

$$m^-||\mathbf{Z}(t)||^2_{L^2(\Omega)^3} \leq ||\widetilde{\mathbf{Z}}(t)||^2_{L^2(\Omega)^3} \leq m^+||\mathbf{Z}(t)||^2_{L^2(\Omega)^3}\,. \tag{92}$$

We deduce from (91) and (92) that:

$$\frac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}} \leq \sqrt{\frac{m^+}{m^-}}\frac{||\widetilde{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}}\,. \tag{93}$$

Finally, with (90) and (93) we have established the following result:

Proposition 8

$$\frac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}} \leq \frac{m^+}{m^-}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \simeq \frac{1}{f^4}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}\,. \tag{94}$$

If the initial condition is regular, we have $\dfrac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \ll 1$, *so* $\dfrac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}} \ll 1$, *for a given cut-off level N_1 quite high.* □

More precisely, after a transient period, we will have that $\rho(t)$ will be small since $\rho(t)$ decreases in energy norm (see (66), density equation). It follows from (78) and (79) that $m_{\mathbf{k}}^- = m_{\mathbf{k}}^+ = \frac{1}{2}$, $\forall \mathbf{k} \in I_N$. We deduce from (94) that, for $t > 0$ sufficiently large, $\dfrac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}}$ decreases, when N_1 increases, as $\dfrac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}$. In particular, if the initial condition is associated with a three-dimensional turbulence flow, the decrease of the kinetic energy spectrum is like $k^{-5/3}$ (see [19], [20], [21]) and the kinetic energy is essentially contained in the large scales (see [38], [17]); so the ratio $\dfrac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}$ will be small.

Remark 4 Choosing an initial condition with $\rho(t = 0)$ small allows to reduce the transient period after which (94) is valid with $m^+ \simeq m^- \simeq \frac{1}{2}$. Moreover, if $\rho(t = 0) = 0$, then $\rho(t)$ will be equal to zero, $\forall t > 0$ (see (67), density equation). ■

Now, we compare the time derivatives of the quantities associated with the large and small scales. From (76), (77), and using the fact that:

$$\left|-\kappa k_3^2 + \mathbf{i}\,\omega_{j,\mathbf{k}}\right|^2 = \kappa^2 k_3^4 + \omega_{j,\mathbf{k}}^2 \simeq \kappa^2 k_3^4 + |\mathcal{U}\cdot\mathbf{k}|^2 \ ,$$

for $j = 1, 2, 3$, we deduce the following result:

$$||\dot{\widehat{\widetilde{\mathbf{U}}}}_{\mathbf{k}}||_2^2 \simeq \exp\left(-2\kappa k_3^2 t\right)\left(\kappa^2 k_3^4 + |\mathcal{U}\cdot\mathbf{k}|^2\right)||\widehat{\widetilde{\mathbf{U}}}_{\mathbf{k}}||_2^2 \ . \tag{95}$$

So, using $\omega_{2,k} = \kappa^2 k^4 + ||\mathcal{U}||_2^2 k^2$ with $k = |\mathbf{k}| = \left(k_1^2 + k_2^2 + k_3^2\right)^{1/2}$, we have the following majoration:

$$||\dot{\widehat{\widetilde{\mathbf{U}}}}_{\mathbf{k}}||_2^2 \leq \exp\left(-2\kappa k_3^2 t\right)\omega_{2,k}||\widehat{\widetilde{\mathbf{U}}}_{\mathbf{k}}||_2^2 \ . \tag{96}$$

For the minoration, from (95) we can write for $\mathcal{U}^T = (U, V, W)$, with $U \simeq V \simeq W \simeq 10$ to $100\ m/s$:

$$||\dot{\widehat{\widetilde{\mathbf{U}}}}_{\mathbf{k}}||_2^2 \geq \exp\left(-2\kappa k_3^2 t\right)\left(\kappa^2 k_3^4\right)||\widehat{\widetilde{\mathbf{U}}}_{\mathbf{k}}||_2^2 \ . \tag{97}$$

So, using Parseval's equality and (97) we deduce:

$$||\dot{\widetilde{\mathbf{Y}}}(t)||_{L^2(\Omega)^3}^2 \geq \exp\left(-2\kappa\left(\frac{N_1}{2}\right)^2 t\right)\kappa^2||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}^2 \ . \tag{98}$$

In the same manner, for $\widetilde{\mathbf{Z}}(t)$ we have, using (96):

$$||\dot{\widetilde{\mathbf{Z}}}(t)||_{L^2(\Omega)^3}^2 \leq \exp\left(-2\kappa\left(\frac{N_1}{2}\right)^2 t\right)\omega_{2,N/2}||\widetilde{\mathbf{Z}}(t)||_{L^2(\Omega)^3}^2 \ . \tag{99}$$

With (98) and (99) we obtain:

$$\frac{||\dot{\widetilde{\mathbf{Z}}}(t)||_{L^2(\Omega)^3}}{||\dot{\widetilde{\mathbf{Y}}}(t)||_{L^2(\Omega)^3}} \leq \sqrt{\frac{\omega_{2,N/2}}{\kappa^2}}\frac{||\widetilde{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\widetilde{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \ . \tag{100}$$

Finally, using (93) for time derivative, (90) and (100), the following result comes:

Proposition 9

$$\frac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \leq \frac{m^+}{m^-}\sqrt{\frac{\omega_{2,N/2}}{\kappa^2}}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \simeq \frac{\omega_{2,N/2}^{1/2}}{\kappa}\frac{1}{f^4}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \ , \tag{101}$$

showing that, if the initial condition is regular, we have $\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}} \ll 1$, *so* $\frac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \ll 1$, *for a given cut-off level* N_1 *quite high.* □

More precisely, after a transient period, we will have $\frac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \leq \frac{\omega_{2,N/2}^{1/2}}{\kappa}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}$.

4.2. Case of a small Prandtl number ($\nu \ll \kappa$)

Now we consider the case $\nu \ll \kappa$ (molecular diffusion). We obtain, for $k_3 \neq 0$:

$$\begin{aligned} ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \simeq\ & \frac{1}{2}\,|\hat{u}_{\mathbf{k}}|^2 + \frac{1}{2}\,|\hat{v}_{\mathbf{k}}|^2 + \frac{1}{\alpha_{k_3}^2}\,|\hat{\rho}_{\mathbf{k}}|^2 + \\ & \frac{g^2}{2\rho_0^2 k_3^2 \alpha_{k_3}^2}\left((-\kappa k_3^2 k_1 + f k_2)^2 + (f k_1 + \kappa k_2 k_3^2)^2\right)|\hat{\rho}_{\mathbf{k}}|^2 + \\ & \frac{g}{\rho_0 k_3 \alpha_{k_3}} Re\left(-\hat{u}_{\mathbf{k}}(-\kappa k_3^2 k_1 + f k_2)\overline{\hat{\rho}}_{\mathbf{k}} + \hat{v}_{\mathbf{k}}(f k_1 + \kappa k_2 k_3^2)\overline{\hat{\rho}}_{\mathbf{k}}\right)\ , \end{aligned}$$

$$\begin{aligned} ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \simeq\ & \frac{1}{2}\,|\hat{u}_{\mathbf{k}}|^2 + \frac{1}{2}\,|\hat{v}_{\mathbf{k}}|^2 + \frac{1}{\alpha_{k_3}^2}\,|\hat{\rho}_{\mathbf{k}}|^2 + \\ & \frac{g^2}{2\rho_0^2 k_3^2 \alpha_{k_3}^2}\left(\kappa^2 k_3^4 (k_1^2 + k_2^2) + f^2 (k_1^2 + k_2^2)\right)|\hat{\rho}_{\mathbf{k}}|^2 + \\ & \frac{g}{\rho_0 k_3 \alpha_{k_3}} Re\left(\kappa k_3^2 (k_1 \hat{u}_{\mathbf{k}} + k_2 \hat{v}_{\mathbf{k}})\overline{\hat{\rho}}_{\mathbf{k}} + f(-k_2 \hat{u}_{\mathbf{k}} + k_1 \hat{v}_{\mathbf{k}})\overline{\hat{\rho}}_{\mathbf{k}}\right) \\ \simeq\ & \frac{1}{2}\,|\hat{u}_{\mathbf{k}}|^2 + \frac{1}{2}\,|\hat{v}_{\mathbf{k}}|^2 + \left(\frac{1}{\alpha_{k_3}^2} + \frac{g^2 \kappa^2 k_3^2 (k_1^2 + k_2^2)}{2\rho_0^2 \alpha_{k_3}^2}\right)|\hat{\rho}_{\mathbf{k}}|^2 + \\ & \frac{g}{\rho_0 k_3 \alpha_{k_3}} Re\left(\kappa k_3^2 \hat{\delta}_{\mathbf{k}} \overline{\hat{\rho}}_{\mathbf{k}} + f \hat{\omega}_{z,\mathbf{k}} \overline{\hat{\rho}}_{\mathbf{k}}\right)\ , \end{aligned}$$

with $\delta = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$ the plane divergence, and ω_z the vorticity component in the z direction.

We consider flow such that:

$$Re\left(\kappa k_3^2 \hat{\delta}_{\mathbf{k}} \overline{\hat{\rho}}_{\mathbf{k}} + f \hat{\omega}_{z,\mathbf{k}} \overline{\hat{\rho}}_{\mathbf{k}}\right) \simeq 0\ .$$

This will be the case if $\rho(t)$ is small, like fully incompressible flow. As it has been said previously, $\rho(t)$ decreases in L^2-norm when t increases. Moreover, since $\nu \ll \kappa$ by hypothesis, we will have that, in energy norm, $\rho(t)$ decreases faster than velocity. So, after a transient period, the hypothesis will be valid. To reduce or eliminate the transient period, we can consider an initial condition for the density, such that $\rho(t=0)$ is small or equal to zero.
We can deduce that:

$$||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \simeq \frac{1}{2}\,|\hat{u}_{\mathbf{k}}|^2 + \frac{1}{2}\,|\hat{v}_{\mathbf{k}}|^2 + \left(\frac{1}{\alpha_{k_3}^2} + \frac{g^2 \kappa^2 k_3^2 (k_1^2 + k_2^2)}{2\rho_0^2 \alpha_{k_3}^2}\right)|\hat{\rho}_{\mathbf{k}}|^2\ .$$

With the hypothesis $\nu \ll \kappa$, we have $\alpha_{k_3} \simeq f^2 + \kappa^2 k_3^4 \simeq \kappa^2 k_3^4$ ($k_3 \neq 0$). So we can write:

$$\begin{aligned} ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \simeq\ & \frac{1}{2}\,|\hat{u}_{\mathbf{k}}|^2 + \frac{1}{2}\,|\hat{v}_{\mathbf{k}}|^2 + \left(\frac{1}{(\kappa^2 k_3^4)^2} + \frac{g^2 (k_1^2 + k_2^2)}{2\rho_0^2 k_3^6 \kappa^2}\right)|\hat{\rho}_{\mathbf{k}}|^2 \\ \simeq\ & \frac{1}{2}\,|\hat{u}_{\mathbf{k}}|^2 + \frac{1}{2}\,|\hat{v}_{\mathbf{k}}|^2 + \left(\frac{2\rho_0^2 + \kappa^2 g^2 (k_1^2 + k_2^2) k_3^2}{2\rho_0^2 k_3^8 \kappa^4}\right)|\hat{\rho}_{\mathbf{k}}|^2\ . \end{aligned} \tag{102}$$

With $\rho_0 = 1028\ kg/m^3$ (sea water), as it has been said previously, we obtain the following majoration and minoration for $||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2$, $k_3 \neq 0$:

$$m_{\mathbf{k}}^- ||\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \leq ||P_{\mathbf{k}}^{-1}\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2 \leq m_{\mathbf{k}}^+ ||\widehat{\mathbf{U}}_{\mathbf{k}}||_2^2\ , \tag{103}$$

with:

$$\left\{ \begin{aligned} m_{\mathbf{k}}^- &= \min\left(\frac{1}{2}, \frac{2\rho_0^2 + \kappa^2 g^2 (k_1^2 + k_2^2) k_3^2}{2\rho_0^2 k_3^8 \kappa^4}\right)\ , \\ m_{\mathbf{k}}^+ &= \max\left(\frac{1}{2}, \frac{2\rho_0^2 + \kappa^2 g^2 (k_1^2 + k_2^2) k_3^2}{2\rho_0^2 k_3^8 \kappa^4}\right)\ . \end{aligned} \right. \tag{104}$$

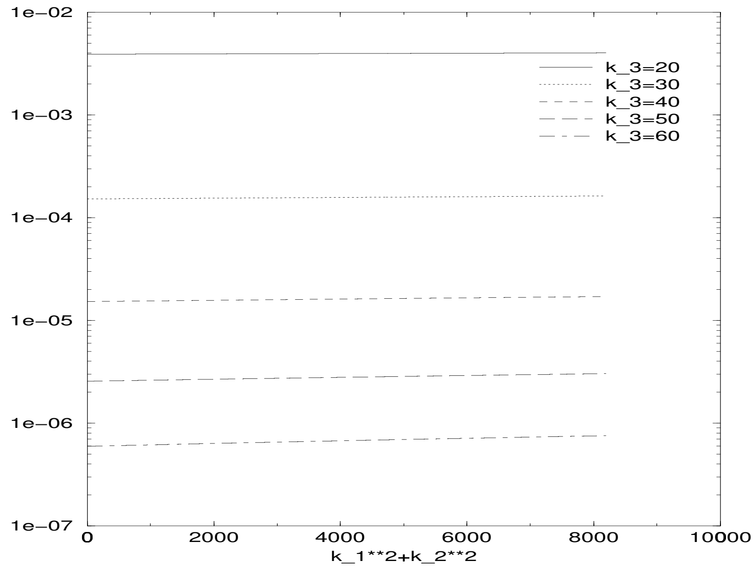

Figure 10. Representation of the ratio $\frac{(2\rho_0^2 + \kappa^2 g^2 (k_1^2 + k_2^2) k_3^2)}{2\rho_0^2 k_3^8 \kappa^4}$ as a function of (k_1, k_2) and k_3.

On Figure 10, we see the representation of the following ratio:

$$\frac{\left(2\rho_0^2 + \kappa^2 g^2 (k_1^2 + k_2^2) k_3^2\right)}{2\rho_0^2 k_3^8 \kappa^4}\,, \tag{105}$$

as a function of (k_1, k_2) and $k_3 (\neq 0)$. Moreover, as it has been said previously, we choose an initial condition such that $\hat{\omega}_{z,\mathbf{k}} = 0$, $\forall t > 0$, for $k_3 = 0$.

We have, $\forall \mathbf{k} \in I_N$, $k_3 \neq 0$:

$$\frac{\rho_0^2}{\rho_0^2 (N/2)^8 \kappa^4} \leq \frac{2\rho_0^2 + \kappa^2 g^2 (k_1^2 + k_2^2) k_3^2}{2\rho_0^2 k_3^8 \kappa^4} \leq \frac{2\rho_0^2 + \kappa^2 g^2 (N^2/2)(N/2)^2}{2\rho_0^2 \kappa^4}\,.$$

So, denoting $m^- = \frac{1}{\kappa^4 (N/2)^8}$ and $m^+ = \frac{2\rho_0^2 + \kappa^2 g^2 (N^2/2)(N/2)^2}{2\rho_0^2 \kappa^4}$, proceeding in the same manner as in Section 4.1, we establish result similar with Proposition 8:

Proposition 10

$$\frac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}} \leq \exp\left(\kappa \left(\frac{N_1}{2}\right)^2 t\right) \frac{m^+}{m^-} \frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}\,, \tag{106}$$

(the exponential being due to $\rho(t)$) showing that if the initial condition is quite regular, the ratio $\frac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}}$ will be small, for a given cut-off level N_1 quite high. $\square$

More precisely, after a transient period, we will have that $\rho(t)$ will be small, since $\kappa \gg \nu$. So, using (102) and (103), it comes that $m^-_{\mathbf{k}} = m^+_{\mathbf{k}} = \frac{1}{2}$, $\forall \mathbf{k} \in I_N$. It follows from (106) that, for $t > 0$ sufficiently large (in order to neglect $\rho(t)$), $\frac{||\mathbf{Z}(t)||_{L^2(\Omega)^3}}{||\mathbf{Y}(t)||_{L^2(\Omega)^3}}$ decreases, when N_1 increases, as $\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}$.

As for the comparison of the time derivatives, we establish a result similar with Proposition 9:

Proposition 11

$$\frac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \leq \exp\left(\kappa\left(\frac{N_1}{2}\right)^2 t\right)\frac{m^+}{m^-}\sqrt{\frac{\omega_{2,N/2}}{\kappa^2}}\,\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}\,, \tag{107}$$

showing that, if the initial condition is quite regular, and for a given cut-off level N_1 quite high, we will have $\frac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \ll 1$ quite small. □

More precisely, after a transient period, $\rho(t)$ will be small and we will have

$$\frac{||\dot{\mathbf{Z}}(t)||_{L^2(\Omega)^3}}{||\dot{\mathbf{Y}}(t)||_{L^2(\Omega)^3}} \leq \frac{\omega^{1/2}_{2,N/2}}{\kappa}\frac{||\mathbf{Z}(0)||_{L^2(\Omega)^3}}{||\mathbf{Y}(0)||_{L^2(\Omega)^3}}.$$

Remark 5 We can improve the estimates in Propositions 10 and 11 using the fact that, for $\mathbf{Y}$, we have $\mathbf{k} \in I_{N_1}$. Hence, for $k_3 \neq 0$:

$$\frac{1}{\kappa^4(N_1/2)^8} \leq \frac{2\rho_0^2 + \kappa^2 g^2(k_1^2+k_2^2)k_3^2}{2\rho_0^2 k_3^8\kappa^4} \leq \frac{2\rho_0^2 + \kappa^2 g^2(N_1^2/2)(N_1/2)^2}{2\rho_0^2\kappa^4}. \quad \blacksquare$$

Remark 6 Since k_1 and k_2 have a symmetric part, but not k_3 in the previous computations, we can envisage a scale separation only in the plane (k_1, k_2) or only in the k_3 direction. ■

5. Conclusion and future work

In this paper, we have presented some theoretical estimates based on scale separation. We have established that, for some geophysical flows and for a suitable choice of the cut-off level (chosen to separate the small and large scales), depending on the initial condition, the quantities associated with the small scales, and their time derivatives, are much smaller, in energy norm, than the quantities associated with the large scales. The separation and the comparison between the small and large scales (and their time derivative) have been established in the canonical basis.

For normal mode initialization and approximate initial theory, a separation is used based on the eigenvectors basis (normal modes), see for example [1], [25], [2], [37], [23], [6], [7], [8]).

Following this theoretical work, the aim is to develop new schemes with better stability properties for geophysical problems. Allowing larger time steps, this allows a reduction of the CPU time. In meteorology, several works, using other ideas, have been done in order to obtain more efficient time integration schemes (see, for example, [16], [14], [26], [33], [6], [29], [34], [30], [31], [32]).

As it has been established previously, since, for a certain norm, the quantity $\mathbf{Z}$ associated with the small scales is smaller than the quantity $\mathbf{Y}$ associated with the large scales, we can plan on computing less accurately $\mathbf{Z}$ than $\mathbf{Y}$, without too much damage on the solution $\mathbf{U} = \mathbf{Y} + \mathbf{Z}$. For example, we can use an explicit scheme for $\mathbf{Y}$ and an implicit scheme for $\mathbf{Z}$. Moreover, we have $\dot{\mathbf{Z}}$ less than $\dot{\mathbf{Y}}$ for a certain norm. So we can consider computing $\mathbf{Z}$ with a time step larger than that of $\mathbf{Y}$. This numerical work is in progress and will be presented in [10].

The dynamical behavior of the large and small scales being different (since $||\dot{\mathbf{Z}}||_{L^2(\Omega)^3} < ||\dot{\mathbf{Y}}||_{L^2(\Omega)^3}$), another possibility is to update dynamical equations in function of the size of the scales. For example, shallow water problems correspond to $gH > f^2L^2$, with L a characteristic length. The quasi-geostrophic equations are established from shallow water problem when $gH \simeq f^2L^2$ (see [5], [15]). So the dynamical equations are modified according to the ratio $\frac{H}{L^2}$ *i.e.* the ratio vertical/horizontal characteristic lengths. Now, if we consider the numerical stability constraint (1) for the shallow water problem, rewritten for a computational domain of characteristic length L, we obtain:

$$\left| U \pm \sqrt{gH + f^2 \frac{\Delta x^2}{(\sin(k'\Delta x))^2}} \right| \frac{\Delta t}{\Delta x} \sin(k'\Delta x) \leq 1 \,, \tag{108}$$

with Δx the mesh size, $\Delta x = \frac{L}{N}$, N being the total number of modes retained, and $k' = \frac{2k\pi}{L}$ (periodicity equal to L), $k \in I_N = \left[1 - \frac{N}{2}, \frac{N}{2}\right]$. Under the square root, there are two terms: gH associated with the vertical characteristic length H, and $f^2L_k^2$ associated with the discrete horizontal characteristic length L_k. The characteristic length L_k depends on the wavenumber k:

$$L_k = \frac{\Delta x}{\sin(\frac{2k\pi}{L}\Delta x)} \,. \tag{109}$$

The stability constraint (108) being function of gH and $f^2L_k^2$, we can, as previously, compare gH with $f^2L_k^2$, according to the values of k. For example, we can look at the values of $k \in I_N$ for which we have $gH \simeq f^2L_k^2$ (quasi-geostrophic equations for the scales of size L_k) *i.e.*:

$$\frac{f^2(L/N)^2}{(\sin(\frac{2k\pi}{L}\frac{L}{N}))^2} \simeq gH \Longleftrightarrow \left(\sin\left(\frac{2k\pi}{N}\right)\right)^2 \simeq \frac{f^2L^2}{N^2gH} \,,$$

$$\Longleftrightarrow \sin\left(\frac{2k\pi}{N}\right) \simeq \pm\frac{fL}{N\sqrt{gH}} \Longleftrightarrow \begin{cases} \frac{2k\pi}{N} \simeq \pm \arcsin\left(\frac{fL}{N\sqrt{gH}}\right) + 2\pi l, \\ \\ \frac{2k\pi}{N} \simeq \pi \pm \arcsin\left(\frac{fL}{N\sqrt{gH}}\right) + 2\pi l, \end{cases} \quad \text{for } l \in \mathbb{Z} \text{ and } k \in I_N \,,$$

$$\Longleftrightarrow \begin{cases} k \simeq \pm\frac{N}{2\pi} \arcsin\left(\frac{fL}{N\sqrt{gH}}\right) + Nl, \\ \\ k \simeq \frac{N}{2} \pm \frac{N}{2\pi} \arcsin\left(\frac{fL}{N\sqrt{gH}}\right) + Nl, \end{cases} \quad \text{for } l \in \mathbb{Z} \text{ and } k \in I_N \,.$$

This requires that $\frac{f^2L^2}{N^2gH} \in]0,1]$, *i.e.* $f^2L^2 \leq N^2gH$, implying that N is updated in terms of L and H to satisfy this inequality.

We can consider modifying the dynamical equations according to the values of the ratio $\frac{gH}{f^2L_k^2}$, thus according to the values of the wavenumber k. For theoretical work on the possibility of updating dynamical equations in function of the size of the scales, see [11].

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