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Some results about multiplicity and bifurcation of stationary solutions of a reaction diffusion climatological model

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Abstract. In this survey we collect several results concerning S-type bifurcation curves for the number of solutions of reaction-diffusion stationary equations. In particular, we recall several results in the literature for the case of stationary energy balance models.

Algunos resultados de multiplicidad y bifurcación de soluciones estacionarias de un modelo climático de difusión-reacción.

Resumen. Presentamos algunos resultados relativos a curvas de bifurcación de tipo S en el número de soluciones de ecuaciones estacionarias de tipo reacción-difusión. Un mayor énfasis es hecho sobre los resultados en la literatura para el caso de modelos climáticos estacionarios de balance de energía.

1. Introduction

Semilinear parabolic equations have been widely used during the last thirty years as mathematical models for many problems arising in different fields like mathematical biology, chemical reactions, combustion, nerve impulses, superconductivity, and so on. In the equation

$$\frac{\partial u}{\partial t} - \Delta u = f(x,u) \qquad x \in \Omega, \quad t \geq 0, \tag{1}$$

where Ω is a domain in the space $\mathbb{R}^n$, the Laplacian Δ is used to model (linear) diffusion whereas $f(x,u)$ is the so-called reaction term. The above equation should be supplemented by boundary and initial conditions. Under rather general assumptions, e.g., if the nonlinear term $f(x,u)$ is Lipschitz continuous, the corresponding parabolic equation has a unique local solution and in many cases it is also a global (i.e. defined for all $t > 0$) solution, (see the books [23], [20], where many examples, general results and references can be found).

The asymptotic (for $t \to +\infty$) behavior of solutions to (1) is then of interest from both the theoretical and applied points of view. Convergence to some steady-state solution is one of the possibilities and hence it is important to know about solutions of the stationary problem

$$-\Delta u = f(x,u) \qquad x \in \Omega, \tag{2}$$

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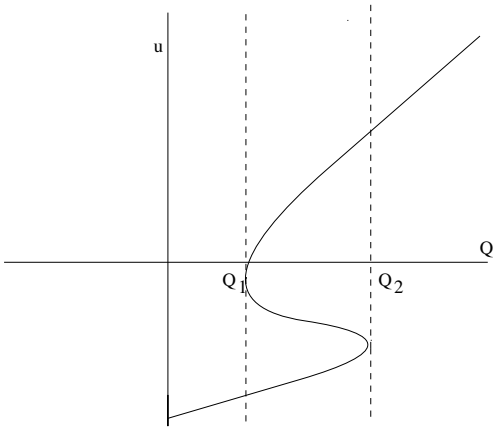
together with some linear (or nonlinear) boundary condition and their stability properties. Very often (but not always, as we will see below) positive (or non-negative) solutions are the only meaningful ones; this is the case, for instance, in population dynamics or chemical reactions.

In general it is very difficult to have a good description of the solution set of (2). Existence of solutions can be proved by using a variety of mathematical tools (sub and supersolutions, degree theory, variational methods, see [1], [20], [23]) but except in the one-dimensional (or the radial) case, where plane phase techniques can be used, in general one cannot list all solutions and associated stability properties.

However, some patterns emerging in different problems can be treated in a unified way by using the same mathematical arguments. One of these situations is the so-called S-type bifurcation curves in the sense that for some problems where the nonlinear term $\lambda f(x,u)$ depends also on a real parameter λ, the corresponding bifurcation diagram



S-shaped diagram

is such that for some *bounded* interval of λ's there are *at least three* solutions. It also happens sometimes that there is a unique solution for λ small and /or large and that the uniqueness proof is easy for λ small and can be rather difficult for λ large. The expression S-type bifurcation curve may be misleading in the sense that in the case of a general domain Ω the solution set is not actually an smooth curve (or even if it is, one cannot know how to provide a proof).

Roughly speaking, there are two main methods in order to show existence of S-bifurcation curves. The first one is to use perturbation methods based on the Implicit Function Theorem as in a paper by Crandall and Rabinowitz [10]. A more simple, and elegant, alternative approach was given in a paper by Brown, Ibrahim and Shivaji [8]. They first prove by using sub and supersolutions that there are two different solutions and then, by using a counting index argument due to Amann ([1], [2]), that if these two solutions are non-degenerate (in the sense that the corresponding linearized operators are isomorphisms) then there exists at least a third solution.

An interesting example to which both methods were applied is the semilinear boundary problem

$$\begin{aligned} -\Delta u &= \lambda(1+u+u^2-\epsilon u^3) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where for $0<\epsilon<\epsilon_0$ for some ϵ_0 "small", there are at least three positive solutions. The general result in [8] applies to functions (like $f(u)=1+u+u^2$), which are C^2 and satisfy $f(0)>0$ and $f''(u)>0$ for $u>0$. It can be shown that similar results can be proved for function f such that $f(0)=0$, f is not even C^1 and is the sum of a concave and a convex nonlinearities, and also to get rid of the above non degeneracy

condition (see [16]). The results in [16] apply for example to the problem

$$\begin{aligned} -\Delta u &= \lambda(u^q + u^p - \epsilon u^m) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $0 < q < 1 < p < m$ for $0 < \epsilon < \epsilon_0$. This is a perturbation of a problem considered by Ambrosetti, Brezis and Cerami [4] (it is even possible to extend the result to $-1 < q$, see [16]).

In this article we collect several recent results concerning existence of S-type bifurcation curves for a model problem arising in climatology. However, this model exhibits some rather non-standard features with respect to the above general picture. First, it is important to consider nonlinear diffusion as well, generalizing the Laplacian by means of the so-called p-Laplacian $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ for $p \geq 2$. Moreover, the reaction term $f(x,u)$ is in some sense discontinuous in u and it is then convenient to provide a mathematical formulation in terms of maximal monotone graphs, which is a way of "filling the gap" in the discontinuity. As a consequence, the model exhibits a loss of uniqueness even for the parabolic problem (see [12], [14], [25]); however, uniqueness of solutions satisfying some additional interesting property is obtained ([14]). On the other hand, the unknown function u is a temperature and hence non-positive solution are specially relevant for the model.

The paper is organized as follows. In section 2 the mathematical model is sketched, together with some general results concerning existence and properties of the corresponding solutions. Much more information on these topics can be found in [13], [25] and the references therein. Section 3 contains a sketch of the results on the existence of S-bifurcation curves obtained in [11] by the authors and section 4 gives an idea of the paper by Arcoya, Díaz and Tello [6] where results in Section, are improved in the sense of showing that there exists a continuum of solutions having at least two bending points, something which provides a more precise setting concerning the S-type curve.

2. The mathematical model.

The so-called energy balance models were introduced independently in 1969 by M. Budyko [9] and W. Sellers [22] for describing the evolution of the climate. They can be considered as diagnostic models and provide a qualitative understanding of their evolution. One of its main characteristics is its high sensitivity with respect to variation of parameters, which can be related with stability properties of solutions and hysteresis cycles.

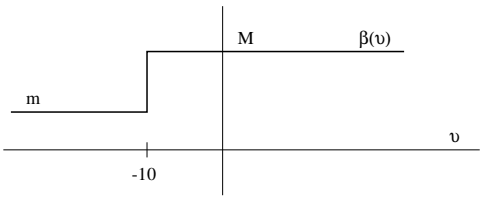
The energy balance where the heat variation on the Earth is given by the absorbed energy minus the emitted energy plus the contribution due to the heat diffusion leads to the parabolic equation

$$c(x)u_t = QS(x)\beta(u) - R_e(u) + D \tag{3}$$

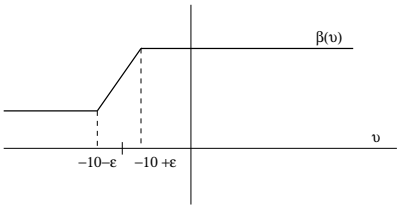
where $u(x,t)$ is the mean surface temperature in $(x,t) \in \mathcal{M} \times (0,T)$, $T > 0$, $\mathcal{M}$ is a smooth manifold without boundary modeling the Earth, $c(x) \geq c_0 > 0$ is the heat capacity, $Q > 0$ (which will play the role of a parameter) is the solar constant, $S(x) \geq S_0 > 0$ is the insolation function, $\beta(u)$ represents the coalbedo, and the diffusion term D is of the form

$$D = div(|\nabla u|^{p-2}\nabla u),$$

the so-called p-Laplacian ($p \geq 2$), which includes also (for $p = 2$) the usual linear diffusion. The particular instance $p = 3$ was suggested by Stone in 1972. Different nonlinear terms were proposed for the coalbedo $\beta(u)$ in the Budyko and the Sellers models



Budyko model



Sellers model

(see below for the precise definition of β). The emitted energy also adopts different expressions in both models: in the Budyko model, $R_e(u) = Bu + C$ with $B > 0$ and $C > 0$ following Newton's law, whereas $R_e(u) = \tau|u|^3 u$, $\tau > 0$ from Stefan-Boltzman's law in Sellers' model.

3. Multiplicity results: I. Existence of an S-bifurcation curve.

We consider here the stationary problem associated to the parabolic equation (3), namely

$$-\Delta_p u + \mathcal{G}(u) \in QS(x)\beta(u) + f(x) \text{ in } \mathcal{M} \tag{4}$$

where

i) $\mathcal{M}$ is a connected, compact C^∞ Riemannian manifold without boundary of dimension 2;

ii) $S : \mathcal{M} \to \mathbb{R}$ and $0 < S_0 \le S(x) \le S_1$;

iii) $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function, $\mathcal{G}(0) = 0$ and $\lim_{s\to\infty} |\mathcal{G}(s)| = +\infty$;

iv) $f \in L^\infty(\mathcal{M})$, and there exists $C_f > 0$ such that

$$-\|f\|_\infty \le f(x) \le -C_f \text{ on } \mathcal{M},$$

v) $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph, which is bounded. More precisely, there exist $0 < m < M$ and $\epsilon > 0$ such that

$$\begin{aligned} \beta(r) &= \{m\} \quad \text{if} \quad r \in (-\infty, -10 - \epsilon) \\ \beta(r) &= \{M\} \quad \text{if} \quad r \in (-10 + \epsilon, +\infty); \end{aligned}$$

vi) $\mathcal{G}(-10 - \epsilon) + C_f > 0$

$$\frac{\mathcal{G}(-10 + \epsilon) + \|f\|_\infty}{\mathcal{G}(-10 - \epsilon) + C_f} \le \frac{S_0 M}{S_1 m}.$$

It is clear that condition v) above allows both increasing Lipschitz functions and the corresponding maximal monotone graph with $\beta(-10) = [m, M]$. We look for weak solutions to (4) in the associated "energy space"

$$V = \{u \in L^2(\mathcal{M}) : \nabla_{\mathcal{M}} u \in L^p(T\mathcal{M})\}$$

where $T\mathcal{M}$ denotes the tangent space to the manifold $\mathcal{M}$. We state the main multiplicity result in [11].

Theorem 1 *Under the above assumptions i)–vi) we have:*
a) For any $Q > 0$, there exists $\underline{u}$ (resp. $\overline{u}$) minimal (resp. maximal) solution to (4) and $\underline{u} \le \overline{u}$.
There exist some constants Q_i $i = 1, .., 4$ explicitly given, such that
b) If $0 < Q < Q_1$, then there exists a unique solution $u = u_m < -10$ which is the minimum of the associated functional on the set

$$K = \{w \in V : \quad \mathcal{G}(w) \in L^1(\mathcal{M})\}.$$

c) If $Q > Q_4$, there exists a unique solution $u = u_M > -10$ which is the minimum of the associated functional on K;
d) If $Q_2 < Q < Q_3$, there exist at least three solutions u_i, $i = 1,..,3$ such that $u_1 = u_M > -10$, $u_2 = u_m < -10$ such that

$$u_2 \leq u_3 \leq u_1 \text{ on } \mathcal{M}.$$

Moreover, u_1 and u_2 are local minima of the associated functional on $V \cap L^\infty(\mathcal{M})$ and, for $p > 2$, on K.

SKETCH OF THE PROOF.

a) Let u_m (resp. u_M) the unique solution of problem

$$-\Delta_p u + \mathcal{G}(u) = QS(x)m + f(x) \text{ in } \mathcal{M}$$

(resp. of)

$$-\Delta_p u + \mathcal{G}(u) = QS(x)M + f(x) \text{ in } \mathcal{M}$$

It is easy to see that u_m (resp. u_M) is a subsolution (resp. supersolution) and this gives existence. Moreover, a comparison argument shows that $u_m \leq u \leq u_M$ for any bounded weak solution.

d) For $Q_2 < Q < Q_3$, we first prove the existence of two solutions u_1 and u_2 which do not cross the level -10 by finding constant ordered sub and supersolutions

In order to deal with the multivalued β we approximate it by the usual Yosida approximation $\beta_\lambda = (I - (I - \lambda\beta)^{-1})$, $\lambda > 0$. We know that, in particular, β_λ is a bounded and nondecreasing function for any $\lambda > 0$. If β is a Lipschitz function, then $\beta_\lambda = \beta$ any λ. We have thus the approximate problem

$$(P_Q^\lambda) \quad -\Delta_p u + \mathcal{G}(u) = QS(x)\beta_\lambda(u) + f(x) \text{ in } \mathcal{M}.$$

By using again sub and supersolutions we show the existence of two solutions u_1^λ and u_2^λ to (P_Q^λ). Moreover $u_1^\lambda = u_1$ and $u_2^\lambda = u_2$ for λ small enough.

Next we show that there is a third solution u_3^λ (different of u_1^λ, u_2^λ) to (P_Q^λ). For this we apply a result due to Amann [2], which is the following (we use the same terminology and notation in [2]).

Lemma 1 *Let X be a retract of some Banach space E and let $F : X \to X$ be a compact map. Suppose that X_1 and X_2 are disjoint retracts of X, and let Y_k, $k = 1, 2$ be open subset of X such that $Y_k \subset X_k$. Moreover, suppose that $F(X_k) \subset X_k$ and that F has no fixed points on $X_k - Y_k$, $k = 1, 2$. Then F has at least three distinct fixed points x, x_1, x_2 with $x_k \in X_k$ and $x \in X - (X_1 \cup X_2)$.* □

By applying the Lemma in the space $L^\infty(\mathcal{M})$ and showing that the nonlinear operator defined by $F(v) = (-\Delta_p + \mathcal{G})^{-1}(QS(\cdot)\beta_\lambda(u) + f(\cdot))$ is compact and finding retracts X, X_1 and X_2 adequately, we find a third solution u_3^λ of (P_Q^λ).

To deal with β we need first to obtain a priori estimates for u_3^λ in order to pass to the limit on a subsequence. This can be done by taking u_3^λ as a test function in a weak formulation of (P_Q^λ) and there is a subsequence u_ϵ such that $u_\epsilon \to u$ weakly in V and strongly in $L^2(\mathcal{M})$. Hence the problem is solved for β Lipschitz.

For the general case we consider the family u_3^λ obtained for β_λ and prove that there is a u_3 such that $u_3^\lambda \to u_3$ in $L^\infty(\mathcal{M})$ and that u_3 is a solution to (P_Q) such that $u_3 \neq u_1$, $u_3 \neq u_2$ and u_3 actually crosses the level -10. ■

4. Multiplicity results: II. Existence of a continuum of solutions with at least two bending points.

We consider again problem (4) but this time from a slightly different viewpoint. By using global bifurcation results due to Rabinowitz [21], Arcoya, Díaz and Tello [6] were able to obtain more information on the structure of the solution set to (4) (see also [3], [5] for previous use of similar ideas).

The main result in [6] can be described by saying that the solution set is a continuum (in the sense of topology, *i.e.* a connected closed subset) of the product space, which is unbounded and having some additional properties, in particular having more than two bending points. We denote by Σ the set

$$\Sigma = \{(Q,u) \in \mathbb{R} \times L^\infty(\mathcal{M}) : Q \geq 0, u \text{ solution of (4) } \}.$$

More precisely we have,

Theorem 2 *Under assumptions i)–vi) with $f = -C > 0$, there exists an unbounded continuum of solutions to (4) having at least two bending points (in the sense that solutions are locally only on one side) containing the point $(0, \mathcal{G}^{-1}(-C)$.* □

SKETCH OF THE PROOF.

As before, the result is proved by using a similar approximation procedure. The graph β is approximated by its Yosida approximation β_n for $\lambda = \frac{1}{n}$. From β_n is bounded lipschitz continuous function and $(\Delta_p + \mathcal{G})^{-1}(QS\beta_n(v) - C)$ is compact in L^∞, we have that Σ is has an unbounded connected component, $\mathcal{C}_n$, containing $(0, \mathcal{G}^{-1}(-C))$ thanks to a theorem in [21].

The comparison arguments with auxiliary zero order problem where Δ_p has disappeared are used to see that the continuum branch is S-shaped.

The second part of this proof consist on the study of the S-shaped branch of Σ_n when $n \to \infty$. We know that if $Q > Q_4$ then the solution is unique. Moreover, the solutions are uniformly bounded for $Q \leq N$, for every N. So, the S structure has to be in a bounded rectangle $\mathcal{R}$ of $\mathbb{R}^+ \times L^\infty(\mathcal{M})$. Then $\mathcal{C}_n \cap \mathcal{R}$ converges in the sense of Whyburn [26] to a continuum $\mathcal{C}$. It is possible to show that the bending points are preserved when passing to the limit.

Again, the comparison arguments with the auxiliary zero order problems used in a) for $Q \in (0, Q_3)$ and $Q \in (Q_2, Q_4)$ where $Q_2 < Q_3$ allow us to prove that the branch C is S-shaped and has at least two bending points. ■

Remark 1

Results of this type were proved by G. Hetzer [17] in case $p = 2$ and β a C^1 mapping by using the implicit function theorem and corollaries due to Amann [2] and Crandall and Rabinowitz [10]. Similar ideas work for time-periodic problems (see [18]) and also for problems with time-delay and systems ([19]). There linearized stability and existence of an even number of bending points is also obtained. ■

5. Infinitely many solutions for a one-dimensional model.

We consider the one-dimensional boundary problem,

$$(P) \begin{cases} -(|u'|^{p-2}u')' + \mathcal{G}(u) + C \ \in \ Q\beta(u) & x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases}$$

where $Q > 0$ and

(H_1) β is a bounded maximal monotone graph of the Heaviside type defined by

$$\beta(u) \ = \ \begin{cases} m & \text{if } u < -10, \\ [m, M] & \text{if } u = -10, \\ M & \text{if } u > -10, \end{cases}$$

with $0 < m < M$,

(H_2) $\mathcal{G}$ is continuous increasing function with $\mathcal{G}(0) = 0$ and $\lim_{s\to\infty} |\mathcal{G}(s)| = +\infty$.

(H_3) $\mathcal{G}(-10) + C > 0$.

We say that u is a solution of (P) if $u \in C^1([0,1])$ and there exists $z \in L^\infty(0,1)$, $z(x) \in \beta(u(x))$ a.e. $x \in (0,1)$ such that u verifies the equation $-(|u'|^{p-2}u')' + \mathcal{G}(u) + C = Qz$ in the weak sense.

As a consequence of Theorem 1, we know that

(i) If $Q < Q_1$ or $Q > Q_2$ then the stationary problem (P) has a unique solution;

(ii) If $Q_1 < Q < Q_2$ then the stationary problem (P) has at least three solutions;

where

$$Q_1 := \frac{\mathcal{G}(-10) + C}{M} \qquad \text{and} \qquad Q_2 := \frac{\mathcal{G}(-10) + C}{m}.$$

Theorem 3 *(i) If $Q_1 < Q < Q_2$ then (P) has infinitely many solutions.*

(ii) Moreover, there exists $K_0 = K_0(p)$ such that for every $K \geq K_0$, $K \in \mathbb{N}$ there exists at least a solution u_K which cross the level $u_K = -10$ exactly K times.

PROOF. We start by computing the intersections between the graphs $\mathcal{G}(u) + C$ and $Q\beta(u)$.
If $Q_1 < Q < Q_2$ then (P) has three constant solutions

$$\begin{aligned} u_1 &= \mathcal{G}^{-1}(Qm - C) < -10, \\ u_2 &= -10, \\ u_3 &= \mathcal{G}^{-1}(QM - C) > -10. \end{aligned}$$

Step 1. We study the phase portrait (u, u') for an auxiliar Cauchy problem. From the equation (P) is conservative, we get the total energy conservation law

$$\frac{|u'|^p}{p} + V(u) = E, \qquad \forall x \in \mathbb{R},$$

for some constant E and for the following potential function

$$V(u) = \begin{cases} (QM - C)u - G(u), & u \geq -10, \\ (Qm - C)u - G(u) - 10Q(M - m), & u < -10, \end{cases}$$

where $G(u) = \displaystyle\int_0^u \mathcal{G}(s)ds$. The shape of graph$(V)$ determines qualitatively the phase portrait (u, u'), and graph(V) depends on Q.

(a) If $Q_1 < Q < Q_3$ then $V(u_3) < V(u_1)$. There exists a homoclinic orbit with ω-limit set equal to u_3, which separates a region of the periodic orbits of the others.

(b) If $Q = Q_3$ then $V(u_1) = V(u_3)$. There exists two heteroclinic orbits with ω-limit equal to u_1 and u_3, respectively.

(c) If $Q_3 < Q < Q_2$ then $V(u_1) < V(u_3)$. There exists a homoclinic orbit with ω-limit equal to u_1.

Step 2. (*Shooting method*). We consider the Cauchy problem depending of the parameter μ,

$$(P_\mu) \begin{cases} -(|u'|^{p-2}u')' + \mathcal{G}(u) + C \in Q\beta(u), & x \in \mathbb{R}^+, \\ u'(0) = 0, \\ u(0) = \mu. \end{cases}$$

Our purpose is to determine the values μ such that the solution of (P_μ) verifies $u'(1) = 0$.

From the phase portrait, we deduce that the solutions which attain at least two times the value $u' = 0$ are the solutions given by the periodic trajectories, that is, the associated ones to energy level $V(u_2) \leq E \leq$

$\min\{V(u_1), V(u_3)\}$. The idea is to choose the periodic trajectories which start in $(\mu, 0)$ and arrive to $(\lambda, 0)$ in the time $x = 1$. That is, integrating the conservation law equation, we obtain

$$\int_{u(0)}^{u(1)} \frac{ds}{\pm \sqrt[p]{p(E - V(s))}} \;=\; \int_0^1 d\sigma,$$

where the sign of $\sqrt[p]{p(E - V(s))}$ is the same of u'. The period of the periodic orbit of the phases portrait is given by the expression:

$$\tau \;=\; 2\int_a^{-10} \frac{ds}{\sqrt[p]{p(E - V(s))}} \;+\; 2\int_{-10}^{b} \frac{ds}{\sqrt[p]{p(E - V(s))}}.$$

Notice that $V(b) = V(a) < \min\{V(u_1), V(u_3)\}$. There exists b^* verifying that $-10 < b^* < u_3$, $V(b^*) = \min\{V(u_1), V(u_3)\}$. If $p = 2$ and $\mathcal{G}(u) = Bu$ where B is a positive constant, it is possible to obtain the explicit expression for τ, in function of μ, $\tau = \tau(\mu)$. If $p \geq 2$ we have obtained the following estimates for the period τ of a periodic trajectory which contains the points $(a, 0)$ and $(b, 0)$ with $a < -10 < b$, $\tau_1(\mu) \leq \tau(\mu) \leq \tau_2(\mu)$. (See [15] for details). Depending on the point where we do the shot, a and b are one of these four cases:

$$\begin{array}{lllllll}
(I) & u(0) & = & \mu & = & b, & u(1) \;=\; a, \\
(II) & u(0) & = & \mu & = & a, & u(1) \;=\; b, \\
(III) & u(0) & = & \mu & = & b \;=\; u(1), & \\
(IV) & u(0) & = & \mu & = & a \;=\; u(1). &
\end{array}$$

If $p = 2$ and $\mathcal{G}(u) = Bu$, the equation

$$\text{(cases I,II)} \qquad N\tau(\mu) = 1 \tag{5}$$

$$\text{(cases III,IV)} \qquad (N + \frac{1}{2})\tau(\mu) = 1, \tag{6}$$

has a solution μ_N for every N. If $p \geq 2$, we analyze, for example, case I. We got $\tau_1(\mu) \leq \tau \leq \tau_2(\mu) \;\; \forall \mu \in (-10, b^*)$, where τ_1 and τ_2 are continuous and increasing functions on $(-10, b^*)$, $\tau_1(-10) = \tau_2(-10) = 0$ and τ_2 has a vertical asymptote $\mu = b^*$.
Thus, we get that there exists N_0 such that for all $N \geq N_0$ there exist μ_1 and μ_2 such that

$$\tau_1(\mu_1) \;=\; \frac{2}{2N + 1} \;=\; \tau_2(\mu_2).$$

Thus, $\exists \mu \in (\mu_1, \mu_2)$ such that $(N + \frac{1}{2})\tau(\mu) = 1$. We conclude that if $Q \in (Q_1, Q_2)$, then $\forall N \geq N_0$ there exists a solution of (P) which crosses $2N + 1$ times the level -10. Moreover, the obtained family of solutions is uniformly bounded: $u_1 \leq u(x) \leq u_3$.

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