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Stationary vector subdivision: quotient ideals, differences and approximation power

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Abstract. The paper considers stationary vector subdivision schemes, that is, subdivision schemes acting on vector valued sequences by using a matrix valued mask, and derives the analog of the well–known "zero condition" for an arbitrary number of variables as well as arbitrary expanding dilation matrices.

Subdivisión vectorial estacionaria: ideales cociente, diferencias y potencia de aproximación

Resumen. Se consideran esquemas de subdivisión vectorial estacionaria, esto es, esquemas que actúan sobre sucesiones vectoriales usando una máscara matricial y se generaliza la conocida "condición del cero" a un número arbitrario de variables así como a matrices de dilatación expansivas arbitrarias.

1. Introduction

A *stationary subdivision scheme* is an iterative way to construct a surface by computing a sequence of discrete functions defined on a nested sequence of finer and finer grids. The "standard" case in this respect, as considered in [1] begins with a sequence

$$c = c^0 = (c_\alpha \ : \ \alpha \in \mathbb{Z})$$

and uses a finitely supported *mask* $a \in \ell_{00}(\mathbb{Z})$ to perform the iteration

$$c^{r+1} = S_a\, c^r := \left(\sum_{\beta \in \mathbb{Z}} a_{\alpha - 2\beta}\, c_\beta \ : \ \alpha \in \mathbb{Z} \right), \qquad r \in \mathbb{N}_0. \tag{1}$$

Associating the coefficients c_α^r to the abscissae $2^{-r}\alpha$, $\alpha \in \mathbb{Z}$, $r \in \mathbb{N}_0$, one can investigate the convergence of this process to a limit function, either in $L_p(\mathbb{R})$, $1 \le p < \infty$, or in $C_u(\mathbb{R})$, the space of *uniformly* continuous functions defined on $\mathbb{R}$. Much interest in stationary subdivision operators arose from their connection to wavelet analysis, in particular, their connection to *refinable functions*. In fact, whenever the subdivision scheme converges, then there exists a function

$$\varphi = S_a^\infty \delta_0, \qquad \delta_0 = (\delta_{0,\alpha} \ : \ \alpha \in \mathbb{Z}),$$

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which solves the *refinement equation*

$$\varphi = \sum_{\alpha \in \mathbb{Z}} a_\alpha \, \varphi(2 \cdot - \alpha) . \tag{2}$$

There have been various generalizations of this concept: stationary subdivision schemes can appear with any number, say s, of variables, als already done in [1], one can use *expanding matrices* as dilation factors, cf. [10], and consider *vector valued* data in addition, that is, *matrix valued* masks, see [4]. In this paper, we will consider this most general situation of stationary vector subdivision with respect to an arbitrary expanding dilation matrix, and characterize those subdivision schemes which possess polynomial eigensequences in terms of a difference operator.

We fix some notation. For $n \in \mathbb{N}$ we denote by $\mathbb{Z}_n$ the set $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$. Matrices $\boldsymbol{A} \in \mathbb{R}^{M \times N}$ will be indexed as

$$\boldsymbol{A} = [A_{jk} \; : \; j \in \mathbb{Z}_M, \, k \in \mathbb{Z}_N]$$

and, analogously, vectors $\boldsymbol{c} \in \mathbb{R}^N$ as

$$\boldsymbol{c} = [c_j \; : \; j \in \mathbb{Z}_N] .$$

Moreover, the Kronecker product $\boldsymbol{A} \otimes \boldsymbol{B}$ of two matrices $\boldsymbol{A} \in \mathbb{R}^{M \times N}$, $\boldsymbol{B} \in \mathbb{R}^{P \times Q}$, defined as

$$\boldsymbol{A} \otimes \boldsymbol{B} = [A_{jk} \boldsymbol{B} \; : \; j \in \mathbb{Z}_M, \, k \in \mathbb{Z}_N] \in \mathbb{R}^{MP \times NQ}$$

will turn out to be a useful notation. For facts about the Kronecker product and the calculus associated to it, see for example [18].

2. Stationary subdivision with expanding matrices

Recall than an *expanding matrix* $M \in \mathbb{Z}^{s \times s}$ is an $s \times s$ integer matrix all whose eigenvalues have modulus strictly larger than one, or, alternatively, a matrix which has the property that

$$\lim_{n \to \infty} \left\| M^{-n} \right\| = 0. \tag{3}$$

In particular, $m := \det M > 1$. The name "expanding" stems from the fact that for any such matrix and any bounded set $\Omega \subset \mathbb{R}^s$ there exists an index $n' \in \mathbb{N}$ such that $\Omega \subset M^n \, [0,1]^s$ for any $n \geq n'$.

By $\ell^{M \times N}(\mathbb{Z}^s)$ we will denote the set of all $M \times N$–matrix–valued sequences, conveniently written as "discrete" functions $\boldsymbol{C} : \mathbb{Z}^s \to \mathbb{R}^{M \times N}$. By $\ell_p^{M \times N}(\mathbb{Z}^s)$ we denote the Banach spaces of those sequences whose p–norm

$$\|\boldsymbol{C}\|_p := \left(\sum_{\alpha \in \mathbb{Z}^s} \sum_{j \in \mathbb{Z}_N} \sum_{k \in \mathbb{Z}_M} |C_{jk}(\alpha)|^p \right)^{1/p}, \qquad 1 \leq p < \infty,$$

or

$$\|\boldsymbol{C}\|_\infty := \sup_{\alpha \in \mathbb{Z}^s} \sum_{j \in \mathbb{Z}_N} \sum_{k \in \mathbb{Z}_M} |C_{jk}(\alpha)| ,$$

is finite. In the same fashion, we denote the space of *vector valued* sequences by $\ell^N(\mathbb{Z}^s) := \ell^{N \times 1}(\mathbb{Z}^s)$ and $\ell_p^N(\mathbb{Z}^s) := \ell_p^{N \times 1}(\mathbb{Z}^s)$, respectively, and also use $\ell_{00}^{M \times N}(\mathbb{Z}^s) \subset \ell_p^{M \times N}(\mathbb{Z}^s)$, $1 \leq p \leq \infty$, for the subspace of sequences of *finite support*, i.e., those sequences $\boldsymbol{C} \in \ell^{M \times N}(\mathbb{Z}^s)$ such that

$$s(\boldsymbol{C}) := \#\mathrm{supp}\, \boldsymbol{C} < \infty, \qquad \mathrm{supp}\, \boldsymbol{C} := \{\alpha \in \mathbb{Z}^s \; : \; \boldsymbol{C}(\alpha) \neq 0\} .$$

A similar notation will be used for functions. Here we use

$$H_p(\mathbb{R}^s) := \begin{cases} L_p(\mathbb{R}^s), & 1 \leq p < \infty, \\ C_u(\mathbb{R}^s), & p = \infty, \end{cases}$$

where $C_u(\mathbb{R}^s)$ denotes the uniformly continuous and uniformly bounded functions on $\mathbb{R}^s$. In accordance with this, we write $H_p^{M\times N}(\mathbb{R}^s)$ for the $M \times N$–matrix valued functions $\boldsymbol{F}$ with components in $H_p(\mathbb{R}^s)$ and with norms

$$\|\boldsymbol{F}\|_p := \left(\int_{\mathbb{R}^s} \sum_{j\in\mathbb{Z}_M} \sum_{k\in\mathbb{Z}_N} |F_{jk}(x)|^p \, dx \right)^{1/p}, \qquad 1 \le p < \infty,$$

and

$$\|\boldsymbol{F}\|_\infty := \sup_{x\in\mathbb{R}^s} \sum_{j\in\mathbb{Z}_M} \sum_{k\in\mathbb{Z}_N} |F_{jk}(x)|,$$

respectively.

The *stationary vector subdivision operator* $S_{\boldsymbol{A}} = S_{M,\boldsymbol{A}}$ based on an expanding matrix M and a finitely supported *matrix valued* mask $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$ is defined, for any $\boldsymbol{c} \in \ell_p^N(\mathbb{Z})$, as

$$S_{M,\boldsymbol{A}}\, \boldsymbol{c} := \boldsymbol{A} *_M \boldsymbol{c} := \sum_{\beta\in\mathbb{Z}^s} \boldsymbol{A}(\cdot - M\beta)\, \boldsymbol{c}(\beta). \tag{4}$$

The matrix M is considered to be fixed throughout this paper, because of which I will drop the subscripts referring to M in order to keep the notation uncluttered.

Since

$$\|\boldsymbol{A} *_M \boldsymbol{c}\| \le s(\boldsymbol{A})\, \|\boldsymbol{A}\|_\infty\, \|\boldsymbol{c}\|_p, \tag{5}$$

which follows immediately from (4), the p–operator norm satisfies

$$\|S_{\boldsymbol{A}}\|_p \le s(\boldsymbol{A})\, \|\boldsymbol{A}\|_\infty < \infty \tag{6}$$

as long as $\boldsymbol{A} \in \ell_{00}^{N\times N}$ and so $S_{\boldsymbol{A}}$ is a continuous linear operator from $\ell_p^N(\mathbb{Z}^s)$ to itself for any $1 \le p \le \infty$. In order to approach a *limit function*, the subdivision operator is iterated, starting with $\boldsymbol{c}^0 = \boldsymbol{c} \in \ell_p^N(\mathbb{Z}^s)$ and generating a sequence

$$\boldsymbol{c}^{r+1} := S_{\boldsymbol{A}}^{r+1}\boldsymbol{c} := S_{\boldsymbol{A}}\, \boldsymbol{c}^r = \boldsymbol{A} *_M \boldsymbol{c}^r, \qquad r \in \mathbb{N}_0. \tag{7}$$

In fact, we can iterate the vector subdivision operator not only on N–vectors but even on $N \times M$–matrices, $M \in \mathbb{N}$, producing the sequence

$$\boldsymbol{C}^r := S_{\boldsymbol{A}}^r \boldsymbol{C}, \qquad r \in \mathbb{N}, \qquad \boldsymbol{C}^0 = \boldsymbol{C} \in \ell_p^{N\times M}(\mathbb{Z}^s). \tag{8}$$

Note that (8) could also be understood as applying the iteration (7) to the columns of $\boldsymbol{C}^r$ separately and then to rearrange the results as column vectors into the matrix $\boldsymbol{C}^{r+1}$.

The perceptive, geometric idea behind the stationary subdivision process is that for $\alpha \in \mathbb{Z}^s$ the value $\boldsymbol{c}^r(\alpha)$ or $\boldsymbol{C}^r(\alpha)$, respectively, corresponds to the abscissa $M^{-r}\alpha$ and since M is expanding, (3) ensures that these points form a denser and denser grid in $\mathbb{R}^s$ which makes it reasonable to speak of convergence of the subdivision process towards a limit function. For the purpose of a rigorous definition, we recall the notion of a *test function* from [4], see also [12, 13].

Definition 1 *A scalar valued function $\phi \in L_p(\mathbb{Z}^s)$ is called a* test function *if*

1. *ϕ has compact support,*
2. *ϕ is stable, i.e., there exist constants $A, B > 0$ such that*

$$A\,\|c\|_p \le \|\phi * c\|_p \le B\,\|c\|_p, \qquad c \in \ell_p(\mathbb{Z}^s),$$

where, for $\phi \in H(\mathbb{Z}^s)$ and $c \in \ell(\mathbb{Z}^s)$ we define

$$\phi * c := \sum_{\alpha\in\mathbb{Z}^s} \phi(\cdot - \alpha)\, c(\alpha),$$

whenever this sum converges,

3. $\phi * 1 = 1$, *where* $1 \in \ell(\mathbb{Z}^s)$ *denotes the constant sequence:* $1(\alpha) = 1$, $\alpha \in \mathbb{Z}^s$.

Finally, we introduce the *mean value operator* of level $r \in \mathbb{N}$, written as $\mu : H_p(\mathbb{R}^s) \to \ell_p(\mathbb{Z}^s)$ by

$$\mu^r(f) := m^r \int_{\cdot + M^{-r}[0,1]^s} f(t)\, dt, \tag{9}$$

which is normalized such that $\mu^r(1) = 1$. As was pointed out in [4], these operators are bounded, more precisely,

$$\|\mu^r\|_p = \sup_{f \neq 0} \frac{\|\mu^r(f)\|_{\ell_p(\mathbb{Z}^s)}}{\|f\|_{H_p(\mathbb{R}^s)}} = m^{r/p},$$

and that for any test function ϕ and any $f \in H_p(\mathbb{R}^s)$ we have that

$$\lim_{r \to \infty} \|f - \phi * \mu^r(f)(M^r \cdot)\| = 0. \tag{10}$$

Obviously, one can easily extend (9) and (10) to matrix valued functions by letting μ^r act on the components of the matrix separately. Now we are in position to define the convergence of the subdivision operator.

Definition 2 *We say that* $\boldsymbol{A} \in \ell_{00}^{N \times N}(\mathbb{Z}^s)$ *admits a* p*–convergent stationary subdivision scheme if for any initial vector* $\boldsymbol{c} \in \ell_p^N(\mathbb{Z}^s)$ *there exists a vector field* $\boldsymbol{f}_{\boldsymbol{c}} \in H_p^N(\mathbb{R}^s)$ *such that*

1. *for any test function* $\phi \in H_p(\mathbb{R}^s)$ *one has that*

$$\lim_{r \to \infty} \|\boldsymbol{f}_c - \phi * (S_{\boldsymbol{A}}^r \boldsymbol{c})(M^r \cdot)\|_p = 0, \tag{11}$$

2. *one has that*

$$\lim_{r \to \infty} m^{-r/p} \|\mu^r(\boldsymbol{f}_{\boldsymbol{c}}) - S_{\boldsymbol{A}}^r \boldsymbol{c}\|_p = 0, \tag{12}$$

and that $\boldsymbol{f}_{\boldsymbol{c}} \neq 0$ *for at least one* $\boldsymbol{c} \in \ell_p^N(\mathbb{Z}^s)$.

It is worthwhile to mention that the two limit processes (11) and (12) defining convergence serve as an *alternative*, the first one being a definition in terms of convergence of functions, that is, taking norms in $H_p^N(\mathbb{R}^s)$, while the second one works in the sequence space $\ell_p^N(\mathbb{Z}^s)$. Their equivalence has been proved in [4] together with the fact that the actual choice of the test function appearing in (11) is irrelevant: if the limit is zero for *one* test function, it is zero for *all* test functions.

If $\boldsymbol{A}$ admits a p–convergent subdivision scheme, then it converges in particular for the sequences $\delta \boldsymbol{e}_j$, $j \in \mathbb{Z}_n$, where $\delta \in \ell_{00}(\mathbb{Z}^s)$ is the scalar valued sequence defined by $\delta(\alpha) = \delta_{\alpha,0}$, and $\boldsymbol{e}_j \in \mathbb{R}^N$ denotes the j–th coordinate vector, $j \in \mathbb{Z}_N$. Hence the *matrix sequence* $\delta \boldsymbol{I} \in \ell_{00}^{N \times N}(\mathbb{Z}^s)$ converges to a function $\boldsymbol{F} \in H_p^{N \times N}(\mathbb{R}^s)$, i.e.,

$$\lim_{r \to \infty} \|\boldsymbol{F} - \phi * (S_{\boldsymbol{A}}\, \delta\, \boldsymbol{I})(M^r \cdot)\| = 0.$$

We call this function $\boldsymbol{F}$ the *canonical limit function* associated to the stationary subdivision scheme induced by $\boldsymbol{A}$. Some immediate consequences of convergence are recorded in the following result, cf. [4] which we prove for the reader's convenience.

Proposition 1 *Suppose that* $\boldsymbol{A} \in \ell_{00}^{N \times N}(\mathbb{Z}^s)$ *admits a* p*–convergent subdivision with associate canonical limit function* $\boldsymbol{F} \in H_p^{N \times N}(\mathbb{R}^s)$.

1. *For any* $\boldsymbol{c} \in \ell_p^N(\mathbb{Z}^s)$*, the limit function* $\boldsymbol{f}_{\boldsymbol{c}}$ *takes the form*

$$\boldsymbol{f}_{\boldsymbol{c}} = \boldsymbol{F} * \boldsymbol{c} = \sum_{\alpha \in \mathbb{Z}} \boldsymbol{F}(\cdot - \alpha)\, \boldsymbol{c}(\alpha).$$

2. *The function* $\boldsymbol{F}$ *is* M–refinable *with respect to* $\boldsymbol{A}$*, that is,*

$$\boldsymbol{F} = \boldsymbol{F} * \boldsymbol{A}(M\cdot) = \sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{F}(M\cdot-\alpha)\ \boldsymbol{A}(\alpha).$$

PROOF. The first statement follows immediately from writing

$$\boldsymbol{c} = \delta\boldsymbol{I} * \boldsymbol{c} = \sum_{\alpha\in\mathbb{Z}^s} \delta(\cdot-\alpha)\ \boldsymbol{I}\,\boldsymbol{c}(\alpha)$$

and using the linearity of the subdivision operator, while the second statement follows from observing that for $r \in \mathbb{N}$

$$\begin{aligned}\|\boldsymbol{F} - \boldsymbol{F} * \boldsymbol{A}(M\cdot)\|_p &\le \left\|\phi * S_{\boldsymbol{A}}^r\delta\boldsymbol{I}(M^r\cdot) - \phi * S_{\boldsymbol{A}}^{r+1}\delta\boldsymbol{I}\left(M^{r+1}\cdot\right)\right\|_p \\ &+ \|\boldsymbol{F} - \phi * S_{\boldsymbol{A}}^r\delta\boldsymbol{I}(M^r\cdot)\|_p + \left\|\boldsymbol{F} * \boldsymbol{A}(M\cdot) - \phi * S_{\boldsymbol{A}}^{r+1}\delta\boldsymbol{I}\left(M^{r+1}\cdot\right)\right\|_p,\end{aligned}$$

where the first and the second term on the right hand side converge to zero as $r \to \infty$ since the subdivision scheme converges and because of

$$\boldsymbol{A} *_M \delta\boldsymbol{I} = \sum_{\beta\in\mathbb{Z}^s} \boldsymbol{A}(\cdot - M\beta)\ \delta(\beta) = \boldsymbol{A} = \sum_{\beta\in\mathbb{Z}^s} \delta(\cdot-\beta)\ \boldsymbol{A}(\beta) = \delta * \boldsymbol{A} = \delta\boldsymbol{I} * \boldsymbol{A},$$

the third term can be rewritten as

$$\begin{aligned}&\left\|\boldsymbol{F} * \boldsymbol{A}(M\cdot) - \phi * \boldsymbol{A} *_M \cdots *_M \boldsymbol{A} *_M \delta\boldsymbol{I}\left(M^{r+1}\cdot\right)\right\|_p \\ &= m^{-1/p}\,\|(\boldsymbol{F} - \phi * S_{\boldsymbol{A}}^r\delta\boldsymbol{I}(M^r\cdot)) * \boldsymbol{A}\|_p \le m^{-1/p}\,s(A)\,\|\boldsymbol{A}\|_\infty\ \|\boldsymbol{F} - \phi * S_{\boldsymbol{A}}^r\delta\boldsymbol{I}(M^r\cdot)\|_p,\end{aligned}$$

which also converges to zero for $r \to \infty$. ■

In order to give a classification of vector subdivision schemes, we have to introduce some more notation. For a square matrix $\boldsymbol{A} \in \mathbb{R}^{N\times N}$ we denote the *right eigenspace* of $\boldsymbol{A}$ with respect to an eigenvalue λ by

$$\mathcal{E}(\boldsymbol{A},\lambda) := \left\{0 \ne \boldsymbol{y} \in \mathbb{R}^N\ :\ \boldsymbol{A}\boldsymbol{y} = \lambda\boldsymbol{y}\right\}.$$

Using, moreover, $E = M\,[0,1)^s \cap \mathbb{Z}^s$ as the usual standard representer set of the finite group $\mathbb{Z}^s/M\mathbb{Z}^s$, we can define the matrices

$$\boldsymbol{A}_\epsilon := \sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{A}(\epsilon + M\alpha), \qquad \epsilon \in E,$$

and the vector space

$$\mathcal{Q}_{\boldsymbol{A}} := \bigcap_{\epsilon\in E} \mathcal{E}(\boldsymbol{A}_\epsilon, 1) \subset \mathbb{R}^N,$$

whenever $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$. We will classify stationary vector subdivision schemes with respect to the number $n = \dim \mathcal{Q}_{\boldsymbol{A}}$ and we will see that this quantity is a fundamental property of the stationary vector subdivision scheme. In particular, we will speak of *rank 1* subdivision schemes if $\dim \mathcal{Q}_{\boldsymbol{A}} = 1$ and of a *full rank* subdivision scheme if $\dim \mathcal{Q}_{\boldsymbol{A}} = N$. In fact, there is no convergent “rank zero” subdivision scheme, as the following result shows.

Proposition 2 *Suppose that* $\boldsymbol{A} \in \ell_{00}^{N\times N}$ *admits a p–convergent subdivision scheme with canonical limit function* $\boldsymbol{F}$*. Then*

1. *we have that*

$$(\boldsymbol{I} - \boldsymbol{A}_\epsilon)\,\boldsymbol{F} = 0, \qquad \epsilon \in E, \tag{13}$$

which holds true in the sense of $H_p^{N\times N}(\mathbb{R}^s)$.

2. $\mathcal{Q}_{\boldsymbol{A}} \neq \emptyset$, *i.e.,*

$$1 \leq \dim \mathcal{Q}_{\boldsymbol{A}} \leq N. \tag{14}$$

3. *there exist exactly* $n = \dim \mathcal{Q}_{\boldsymbol{A}}$ constant *linearly independent eigenvectors* $\boldsymbol{c}_j \in \ell^N(\mathbb{Z})$, $j \in \mathbb{Z}_n$, *of the subdivision operator* $S_{\boldsymbol{A}}$ *with respect to the eigenvalue* 1, *i.e.,*

$$S_{\boldsymbol{A}}\, \boldsymbol{c}_j = \boldsymbol{c}_j, \qquad \textit{and} \qquad \boldsymbol{c}_j(\alpha) = \boldsymbol{c}_j(\beta), \quad \alpha, \beta \in \mathbb{Z}^s, \qquad j \in \mathbb{Z}_n.$$

PROOF. The crucial point of this Proposition is equation (13) which is due to [4]. Since (13) means that every column vector of $\boldsymbol{F}$ is an eigenvector of every $\boldsymbol{A}_\epsilon$, $\epsilon \in E$, for the eigenvalue 1 and since the nontriviality condition for a convergent subdivision scheme requests that $\boldsymbol{F} \neq 0$, it follows that there is a nontrivial joint eigenspace of the matrices $\boldsymbol{A}_\epsilon$, $\epsilon \in E$, and thus we obtain (14). Moreover, note that

$$\dim \mathcal{Q}_{\boldsymbol{A}} \geq \dim \mathcal{R}(\boldsymbol{F}), \tag{15}$$

where $\mathcal{R}(\boldsymbol{F})$, the *range of* $\boldsymbol{F}$, is the smallest linear subspace of $\mathbb{R}^N$ which contains the columns of $\boldsymbol{F}$ for (almost) all $x \in \mathbb{R}^s$, the "almost" depending on whether $1 \leq p < \infty$ or $p = \infty$.

Next, let $\boldsymbol{y}_j$, $j \in \mathbb{Z}_n$, be a basis of $\mathcal{Q}_{\boldsymbol{A}}$ and set $\boldsymbol{c}_j(\cdot) = \boldsymbol{y}_j$, $j \in \mathbb{Z}_n$. Then

$$S_{\boldsymbol{A}}\, \boldsymbol{c}_j = \boldsymbol{A} *_M \boldsymbol{c}_j = \sum_{\beta \in \mathbb{Z}^s} \boldsymbol{A}(\cdot - M\beta)\, \boldsymbol{c}_j(\beta) = \left(\sum_{\beta \in \mathbb{Z}^s} \boldsymbol{A}(\cdot - M\beta) \right) \boldsymbol{y}_j = \boldsymbol{y}_j = \boldsymbol{c}_j(\cdot)$$

since the rightmost sum is of the form $\boldsymbol{A}_\epsilon$ for some $\epsilon \in E$. Consequently, these constant vectors $\boldsymbol{c}_j$, $j \in \mathbb{Z}_n$, are eigenvectors of $S_{\boldsymbol{A}}$ with respect to the eigenvalue 1 which implies that there are *at least* n such eigenvectors. On the other hand, let $\boldsymbol{c} \in \ell(\mathbb{Z}^s)$ be any constant eigenvector, of $S_{\boldsymbol{A}}$, say $\boldsymbol{c}(\cdot) = \boldsymbol{y}$, and set, for $R > ms(\boldsymbol{A})$,

$$\boldsymbol{c}_R := \chi_{[-mR, mR]^s}\, \boldsymbol{c}.$$

Then it follows by a standard argument for any test function ϕ and any sufficiently large value of R we have that

$$\boldsymbol{F} * \boldsymbol{c}_R(x) = \lim_{n \to \infty} \phi * S_{\boldsymbol{A}}^r \boldsymbol{c}(M^r x) = \boldsymbol{y}, \qquad x \in [-R, R]^s,$$

hence, letting $R \to \infty$, we can conclude that $\boldsymbol{F} * \boldsymbol{c} = \boldsymbol{c}(0)$. Consequently, it follows that $\boldsymbol{c} \in \mathcal{R}(\boldsymbol{F})$ for any $\boldsymbol{c} \in \ell^N(\mathbb{Z}^s)$ such that $\boldsymbol{c} = S_{\boldsymbol{A}}\boldsymbol{c}$ and therefore the dimension of this eigenspace of *constant* vectors is $\leq \dim \mathcal{R}(\boldsymbol{F})$. By (15) this implies, however, that the dimension of this eigenspace must be exactly n. ■

3. Factorization

In this section, we will connect convergent stationary subdivision schemes to a *factorization* property which will be expressed in terms of appropriate difference operators, but also in terms of quotient ideals. These results generalize the well–known fact that preservation of constant data by a scalar, univariate stationary subdivision operator is equivalent to the associated mask having a zero at $z = -1$, that is, the mask contains a factor $(z + 1)$. The counterpart of this result in univariate *vector* subdivision are the matrix factorizations, given in [16] for the case that $\dim \mathcal{Q}_{\boldsymbol{A}} = 1$ and in [12, 13]; except the *full rank* case where $\dim \mathcal{Q}_{\boldsymbol{A}} = N$, those factorizations consist of multiplying matrices from the left and the right. It was, however, shown in [13] that the common background behind all these factorizations was the existence of a subdivision operator $S_{\boldsymbol{B}}$ such that

$$D S_{\boldsymbol{A}} = S_{\boldsymbol{B}} D,$$

where D denotes an appropriate difference operator. This interpretation of factorizations, both in the scalar and vector case, turned out to be useful for the determination of convergence and the regularity of (stable) refinable vector fields.

Also in the multivariate case, even with arbitrary expanding scaling matrices, part of this is already known, especially in the *scalar* situation $N = 1$, see for example [1, 3, 5, 10]. The vector case has been addressed recently in [7], but with techniques significantly different from those used here. Also, we will not make use of the method of *invariant subspaces* and *joint spectral radius* which has been initiated by Jia and collaborators, cf. [6, 8, 9], see also [11, 14] for the connection between matrix subdivision schemes, joint spectral radii and stationary subdivision schemes.

The approach presented here will combine the classification of vector subdivision schemes with respect to the number $n := \dim \mathcal{Q}_{\boldsymbol{A}}$, as developed in [13] with the *quotient ideal* methods which have been introduced in [17] for the case $M = 2I$, and were elaborated further in [15]. This approach will turn out to be appropriate for the treatment of multivariate stationary vector subdivision schemes.

Let a sequence $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$ be given such that

$$1 \le n = \dim \mathcal{Q}_{\boldsymbol{A}} = \dim \bigcap_{\epsilon\in E} \mathcal{E}\left(\boldsymbol{A}_\epsilon, 1\right), \qquad \boldsymbol{A}_\epsilon = \sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{A}\left(\epsilon + M\alpha\right), \quad \epsilon \in E.$$

According to Proposition 2, all "reasonable" stationary subdivision schemes are of this form. Let $\boldsymbol{V} \in \mathbb{R}^{N\times N}$ be any *orthogonal* matrix, that is, $\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{V}\boldsymbol{V}^T = \boldsymbol{I}$, such that the first n columns of $\boldsymbol{V}$ span $\mathcal{Q}_{\boldsymbol{A}}$, that is,

$$\mathcal{Q}_{\boldsymbol{A}} = \boldsymbol{V} \begin{bmatrix} \mathbb{R}^n \\ \boldsymbol{0}_{N-n} \end{bmatrix}, \tag{16}$$

where, for $k, \ell \in \mathbb{N}$, we use the notation

$$\boldsymbol{0}_{k,\ell} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{k\times\ell}, \qquad \boldsymbol{1}_{k,\ell} = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{k\times\ell}, \qquad \boldsymbol{0}_k = \boldsymbol{0}_{k,1}, \quad \boldsymbol{1}_k = \boldsymbol{1}_{k,1}.$$

With the help of this matrix $\boldsymbol{V}$, the matrices $\boldsymbol{A}_\epsilon$, $\epsilon \in E$, are *partially* jointly diagonalizable. In fact, we have

$$\boldsymbol{V}^T \boldsymbol{A}_\epsilon \boldsymbol{V} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0}_{n,N-n} \\ \boldsymbol{0}_{N-n,n} & \widetilde{\boldsymbol{A}}_\epsilon \end{bmatrix}, \qquad \widetilde{\boldsymbol{A}}_\epsilon \in \mathbb{R}^{N-n\times N-n}. \tag{17}$$

To keep the notation uncluttered, we will drop the subscripts of the matrix blocks in decompositions like the one above whenever their dimension will be clear from the context.

Recall that an element of the ring $\Lambda = \mathbb{C}\left[z_j, z_j^{-1} \,:\, j \in \mathbb{Z}_s\right]$ of all finite linear combinations of z^α, $\alpha \in \mathbb{Z}^s$, is called a *Laurent polynomial.* To our sequence, the *mask* $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$, we define its *symbol* $\boldsymbol{A}^* \in \Lambda^{N\times N}$ as the the *matrix valued* Laurent polynomial or, equivalently, the matrix of Laurent polynomials,

$$\boldsymbol{A}^*(z) = \sum_{\alpha\in\mathbb{Z}} \boldsymbol{A}(\alpha)\, z^\alpha, \qquad z \in \mathbb{C}_\times^s,$$

where $\mathbb{C}_\times = \mathbb{C} \setminus \{0\}$ denotes the units of $\mathbb{C}$. With the *subsymbols*

$$\boldsymbol{A}_\epsilon^*(z) := \sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{A}\left(\epsilon + M\alpha\right)\, z^\alpha, \qquad \epsilon \in E, \quad z \in \mathbb{C}_\times^s,$$

we the obtain the well–known decomposition

$$\boldsymbol{A}^*(z) = \sum_{\epsilon\in E}\sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{A}\left(\epsilon + M\alpha\right) z^{\epsilon+M\alpha} = \sum_{\epsilon\in E} z^\epsilon \sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{A}\left(\epsilon + M\alpha\right) z^{M\alpha} = \sum_{\epsilon\in E} z^\epsilon\, \boldsymbol{A}_\epsilon^*\left(z^M\right). \tag{18}$$

From this convenient formula we almost immediately get the following result.

Lemma 1 *If* $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$ *has the property* $1 \le n = \dim \mathcal{Q}_{\boldsymbol{A}}$ *and* $\boldsymbol{V}$ *is any orthonormal matrix satisfying (16), then*

$$\boldsymbol{V}^T \boldsymbol{A}^*(z_0)\, \boldsymbol{V} = m \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & * \end{bmatrix}, \qquad z_0 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{19}$$

and

$$\boldsymbol{V}^T \boldsymbol{A}^*(z_{\epsilon'})\, \boldsymbol{V} = \begin{bmatrix} \boldsymbol{0}_{n,n} & \boldsymbol{0} \\ \boldsymbol{0} & * \end{bmatrix}, \qquad z_{\epsilon'} := e^{-2\pi i M^{-T}\epsilon'}, \quad \epsilon' \in E' \setminus \{0\}, \tag{20}$$

where $E' = M^T [0,1)^s \cap \mathbb{Z}^s$ *are the standard representers of* $\mathbb{Z}/M^T\mathbb{Z}$.

PROOF. By (18), we obtain for $z_0 = (1, \ldots, 1)$ that

$$\boldsymbol{A}^*(z_0) = \sum_{\epsilon \in E} \boldsymbol{A}_\epsilon^*\left(z_0^M\right) = \sum_{\epsilon \in E} \boldsymbol{A}_\epsilon,$$

from which (19) follows immediately. For (20) we first note that

$$z_{\epsilon'}^M = \left(e^{-2\pi i M^{-T}\epsilon'}\right)^M = e^{-2\pi i M^T M^{-T}\epsilon'} = e^{-2\pi i \epsilon'} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = z_0,$$

and so, with (18) and (17),

$$\begin{aligned} \boldsymbol{V}^T \boldsymbol{A}^*(z_{\epsilon'})\, \boldsymbol{V} &= \sum_{\epsilon \in E} z_{\epsilon'}^{\epsilon} \boldsymbol{V}^T \boldsymbol{A}_\epsilon^*(z_0)\, \boldsymbol{V}^T = \sum_{\epsilon \in E} z_{\epsilon'}^{\epsilon} \boldsymbol{V}^T \boldsymbol{A}_\epsilon \boldsymbol{V}^T \\ &= \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \left(\sum_{\epsilon \in E} z_{\epsilon'}^{\epsilon}\right) + \sum_{\epsilon \in E} z_{\epsilon'}^{\epsilon} \begin{bmatrix} \boldsymbol{0}_{n,n} & \boldsymbol{0} \\ \boldsymbol{0} & \widetilde{\boldsymbol{A}}_\epsilon \end{bmatrix} \end{aligned}$$

and since

$$\sum_{\epsilon \in E} z_{\epsilon'}^{\epsilon} = m\, \delta_{\epsilon',0}, \qquad \epsilon \in E',$$

cf. [10, Lemma 1], we obtain (20). ∎

Consequently, all the Laurent polynomials in $\boldsymbol{V}^T \boldsymbol{A}^* \boldsymbol{V}$ except those in the "lower right corner" of size $N-n \times N-n$ vanish at the points $z_{\epsilon'}$, $\epsilon' \in E' \setminus \{0\}$, and the off–diagonal entries among them even vanish at z_0 in addition. Since the sets of all Laurent polynomials satisfying this type of zero conditions form an *ideal* in Λ, we can describe these properties in terms of ideals. For that purpose, let us recall some concepts from ideal theory.

A subset $I \subseteq \Lambda$ of the ring Λ is called an *ideal* if it is closed under addition, that is $I + I = I$, and if $I \cdot \Lambda \subseteq I$. Particular ideals are those which are generated by a finite set $\mathcal{F} \subset \Lambda$ as

$$\langle \mathcal{F} \rangle := \left\{ \sum_{f \in \mathcal{F}} q_f\, f \; : \; q_f \in \Lambda,\; f \in \mathcal{F} \right\}.$$

In fact, for any ideal $I \subset \Lambda$ there exists a finite set $\mathcal{F} \subset \Lambda$ such that $I = \langle \mathcal{F} \rangle$; this is Hilbert's famous *Basissatz.* Since the ring Λ is structurally different from the ring $\mathbb{C}[z_j \; : \; j \in \mathbb{Z}_s]$ of all polynomials with complex coefficients (in contrast to the ring of polynomials, Λ is spanned by finite linear combinations of units), there are some more elaborate arguments necessary, for example in the construction of "good" ideal bases. Nevertheless, these aspects are discussed and resolved in [15]. The crucial concept in what follows is that of a *quotient ideal*, cf. [2].

Definition 3 *Given two ideals $I, J \subset \Lambda$, their* quotient ideal *or* colon ideal $I : J$ *is defined as*

$$I : J := \{ f \in \Lambda \ : \ f \cdot J \subseteq I \},$$

where obviously $I \subseteq I : J$.

Let $M = [m_j \ : \ j \in \mathbb{Z}_s]$, where $m_j \in \mathbb{Z}^s$ denotes the j–th column column vector of M. In what follows, we will consider two particular ideals, namely

$$\left\langle z^M - 1 \right\rangle := \langle z^{m_j} - 1 \ : \ j \in \mathbb{Z}_s \rangle,$$

and

$$\langle z - 1 \rangle := \langle z_j - 1 \ : \ j \in \mathbb{Z}_s \rangle.$$

These two ideals describe the polynomials which satisfy the zero conditions at the points $z_{\epsilon'}$, $\epsilon' \in E'$, or $\epsilon' \in E' \setminus \{0\}$.

Theorem 1 [15, Theorem 2, Proposition 3] *For $f \in \Lambda$ we have that*

$$f(z_{\epsilon'}) = 0, \quad \epsilon' \in E' \qquad \Longleftrightarrow \qquad f \in \left\langle z^M - 1 \right\rangle, \tag{21}$$

and

$$f(z_{\epsilon'}) = 0, \quad \epsilon' \in E' \setminus \{0\}, \qquad \Longleftrightarrow \qquad f \in \left\langle z^M - 1 \right\rangle : \langle z - 1 \rangle. \quad \square \tag{22}$$

Remark 1 To my knowledge, the equivalence (21) has first been proved in [10]; a different, algorithmical proof based on the Smith factorization for matrices and Gröbner bases was later provided in [15, Lemma 1]. The second statement, (22), reflects the geometric fact that, loosely spoken, the quotient of two ideals corresponds to the *difference* of the associated varieties, cf. [2], and since the variety associated to the radical ideal $\langle z - 1 \rangle$ consists just of the point z_0, this geometric operation precisely consists of releasing the zero condition at $z = z_0$. ■

Combining Lemma 1 with Theorem 1, we thus obtain the following result.

Corollary 1 *If $\boldsymbol{A} \in \ell_{00}^{N \times N}(\mathbb{Z}^s)$ has the property $1 \le n = \dim \mathcal{Q}_{\boldsymbol{A}}$ and $\boldsymbol{V}$ is any orthonormal matrix satisfying (16), then*

$$\boldsymbol{V}^T \boldsymbol{A}^* \boldsymbol{V} \in \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \left(\left\langle z^M - 1 \right\rangle : \langle z - 1 \rangle \right) + \begin{bmatrix} \boldsymbol{1}_{n,n} - \boldsymbol{I}_n & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{0} \end{bmatrix} \left\langle z^M - 1 \right\rangle + \begin{bmatrix} \boldsymbol{0}_{n,n} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix} \Lambda. \quad \square$$

Our goal will be to rewrite this relationship in terms of a difference operator. For that purpose we let δ_j, $j \in \mathbb{Z}_s$, denote the j–th backwards partial difference operator, that is

$$\delta_j \, c := c(\cdot - e_j) - c(\cdot), \qquad c \in \ell(\mathbb{Z}^s). \tag{23}$$

We combine these operators into a matrix valued difference operator

$$\Delta_n \ : \ \ell^{N \times N}(\mathbb{Z}^s) \ \to \ \ell^{Ns \times N}(\mathbb{Z}^s).$$

which we define with the help of the matrix $\boldsymbol{V}$ as

$$\Delta_n := \boldsymbol{D}_n \boldsymbol{V}^T := \begin{bmatrix} \boldsymbol{D}_0 \\ \vdots \\ \boldsymbol{D}_{s-1} \end{bmatrix} \boldsymbol{V}^T, \qquad \boldsymbol{D}_j = \begin{bmatrix} \delta_j \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{N-n} \end{bmatrix}, \quad j \in \mathbb{Z}_s. \tag{24}$$

This difference operator plays the key role in our considerations.

Theorem 2 *If $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$ has the property $1 \le n = \dim \mathcal{Q}_{\boldsymbol{A}}$ and $\boldsymbol{V}$ is any orthonormal matrix satisfying (16), then there exists $\boldsymbol{B} \in \ell_{00}^{Ns\times Ns}(\mathbb{Z}^s)$ such that*

$$\Delta_n S_{\boldsymbol{A}} = S_{\boldsymbol{B}} \Delta_n. \tag{25}$$

For the proof of this theorem, we need an observation about the symbols of operators which are obtained by convolving matrix sequences, taking into account the *slantedness* of subdivision operators. For the readers convenience we give this result, which can also be found in [15] in a more detailed form, together with its simple proof.

Lemma 2 *Let $\boldsymbol{A} \in \ell_{00}^{L\times M}(\mathbb{Z}^s)$ and $\boldsymbol{B} \in \ell_{00}^{M\times N}(\mathbb{Z}^s)$. Then*

$$(\boldsymbol{A} * \boldsymbol{B})^*(z) = \boldsymbol{A}^*(z)\, \boldsymbol{B}^*(z), \qquad (\boldsymbol{A} *_M \boldsymbol{B})^*(z) = \boldsymbol{A}^*(z)\, \boldsymbol{B}^*\left(z^M\right), \qquad z \in \mathbb{C}_\times^s. \tag{26}$$

PROOF. Straightforward computations yield

$$\sum_{\alpha\in\mathbb{Z}^s} \left(\sum_{\beta\in\mathbb{Z}^s} \boldsymbol{A}(\alpha-\beta)\, \boldsymbol{B}(\beta) \right) z^\alpha = \sum_{\alpha,\beta\in\mathbb{Z}^s} \boldsymbol{A}(\alpha-\beta)\, z^{\alpha-\beta} \boldsymbol{B}(\beta) z^\beta = \boldsymbol{A}^*(z)\, \boldsymbol{B}^*(z)$$

and

$$\sum_{\alpha\in\mathbb{Z}^s} \left(\sum_{\beta\in\mathbb{Z}^s} \boldsymbol{A}(\alpha-M\beta)\, \boldsymbol{B}(\beta) \right) z^\alpha = \sum_{\alpha,\beta\in\mathbb{Z}^s} \boldsymbol{A}(\alpha-M\beta)\, z^{\alpha-M\beta} \boldsymbol{B}(\beta) z^{M\beta} = \boldsymbol{A}^*(z)\, \boldsymbol{B}^*\left(z^M\right),$$

which is (26). ■

PROOF OF THEOREM 2 In view of Corollary 1, we write

$$\boldsymbol{V}^T \boldsymbol{A}^* \boldsymbol{V} = \begin{bmatrix} \boldsymbol{F}^* & \boldsymbol{G}_1^* \\ \boldsymbol{G}_2^* & \boldsymbol{H}^* \end{bmatrix}, \tag{27}$$

where

$$\boldsymbol{F}^* \in \boldsymbol{I}_n \left(\left\langle z^M - 1 \right\rangle : \langle z-1 \rangle \right) + \left(\mathbf{1}_{n,n} - \boldsymbol{I}_n \right) \left\langle z^M - 1 \right\rangle, \qquad \boldsymbol{H}^* \in \Lambda,$$

as well as

$$\boldsymbol{G}_1^* \in \mathbf{1}_{n,N-n} \left\langle z^M - 1 \right\rangle, \qquad \boldsymbol{G}_2^* \in \mathbf{1}_{N-n,n} \left\langle z^M - 1 \right\rangle.$$

For $j \in \mathbb{Z}_s$ we have that

$$\boldsymbol{D}_j^*(z) = \begin{bmatrix} (z_j - 1)\, \boldsymbol{I}_n & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{N-n} \end{bmatrix}, \qquad z \in \mathbb{C}_\times^s.$$

Therefore, by Lemma 2, we have for $z \in \mathbb{C}_\times^s$ that

$$\begin{aligned} \boldsymbol{C}^*(z) &:= \left(\boldsymbol{D} * \left(\boldsymbol{V}^T S_{\boldsymbol{A}} \boldsymbol{V} \right) \right)^*(z) = \boldsymbol{D}(z)\, \boldsymbol{V}^T \boldsymbol{A}^*(z) \boldsymbol{V} = \begin{bmatrix} \boldsymbol{D}_0^*(z) \\ \vdots \\ \boldsymbol{D}_{s-1}^*(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{F}^*(z) & \boldsymbol{G}_1^*(z) \\ \boldsymbol{G}_2^*(z) & \boldsymbol{H}^*(z) \end{bmatrix} \\ &= \begin{bmatrix} (z_0-1)\, \boldsymbol{F}^*(z) & (z_0-1)\, \boldsymbol{G}_1^*(z) \\ \boldsymbol{G}_2^*(z) & \boldsymbol{H}^*(z) \\ \vdots & \vdots \\ (z_{s-1}-1)\, \boldsymbol{F}^*(z) & (z_{s-1}-1)\, \boldsymbol{G}_1^*(z) \\ \boldsymbol{G}_2^*(z) & \boldsymbol{H}^*(z) \end{bmatrix}, \end{aligned}$$

hence,

$$C^* \in \mathbf{1}_s \otimes \left(\begin{bmatrix} \mathbf{1}_{n,n} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \langle z^M - 1 \rangle + \begin{bmatrix} \mathbf{0}_{n,n} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \Lambda \right).$$

Thus, there exist Laurent polynomials

$$\boldsymbol{B}^*_{j,k} \in \Lambda^{n\times n}, \quad \widetilde{\boldsymbol{B}}^*_j \in \Lambda^{N-n\times n}, \qquad j,k \in \mathbb{Z}_s,$$

such that

$$(z_j - 1)\ \boldsymbol{F}^*(z) = \sum_{k=1}^{s} (z^{m_k} - 1)\ \boldsymbol{B}^*_{j,k}(z), \qquad j \in \mathbb{Z}_s, \tag{28}$$

and

$$\boldsymbol{G}^*_2(z) = \sum_{k=1}^{s} (z^{m_k} - 1)\ \widetilde{\boldsymbol{B}}^*_k(z). \tag{29}$$

Taking (28) and (29) into account, we therefore obtain for $z \in \mathbb{C}^s_\times$ that

$$\begin{aligned} \boldsymbol{C}^*(z) \quad &= \quad \begin{bmatrix} \boldsymbol{B}^*_{0,0}(z) & \frac{z_0-1}{s}\boldsymbol{G}^*_1(z) & \dots & \boldsymbol{B}^*_{0,s-1}(z) & \frac{z_0-1}{s}\boldsymbol{G}^*_1(z) \\ \widetilde{\boldsymbol{B}}^*_0 & \frac{1}{s}\boldsymbol{H}^*(z) & \dots & \widetilde{\boldsymbol{B}}^*_{s-1} & \frac{1}{s}\boldsymbol{H}^*(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{B}^*_{s-1,0}(z) & \frac{z_{s-1}-1}{s}\boldsymbol{G}^*_1(z) & \dots & \boldsymbol{B}^*_{s-1,s-1}(z) & \frac{z_{s-1}-1}{s}\boldsymbol{G}^*_1(z) \\ \widetilde{\boldsymbol{B}}^*_0 & \frac{1}{s}\boldsymbol{H}^*(z) & \dots & \widetilde{\boldsymbol{B}}^*_{s-1} & \frac{1}{s}\boldsymbol{H}^*(z) \end{bmatrix} \times \\ & \times \begin{bmatrix} (z^{m_0} - 1)\ \boldsymbol{I}_n & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{N-n} \\ \vdots & \vdots \\ (z^{m_{s-1}} - 1)\ \boldsymbol{I}_n & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{N-n} \end{bmatrix} \\ &=: \quad \boldsymbol{B}^*(z)\, \boldsymbol{D}^*\left(z^M\right) = \left(\boldsymbol{B} *_M \boldsymbol{D}\right)^*(z) = \left(S_{\boldsymbol{B}}\, \boldsymbol{D}\right)^*(z), \end{aligned} \tag{30}$$

where $\boldsymbol{B} \in \Lambda^{Ns\times Ns}$, by another application of Lemma 2. Thus,

$$\boldsymbol{D}\boldsymbol{V}^T S_{\boldsymbol{A}} \boldsymbol{V} = S_{\boldsymbol{B}} \boldsymbol{D}$$

and multiplying this from the right with $\boldsymbol{V}^T$ yields (25). ■

Next, let us make some comments on Theorem 2 and its proof.

Remark 2

1. Note that Theorem 2 is constructive in the sense that the finitely supported sequence $\boldsymbol{B} \in \ell_{00}^{Ns\times Ns}(\mathbb{Z}^s)$ can be determined by computing the symbol $\boldsymbol{B}^*(z)$ via (30), where the Laurent polynomials in (28) and (29) can be obtained, for example by using Gröbner bases or H–bases for the ideal $I = \langle z^M - 1 \rangle$. This is straightforward if $M \in \mathbb{N}_0^{s\times s}$ has only nonnegative entries and thus I is a *polynomial* ideal. Otherwise one has to make use of a basis of the *polynomial part* $P(I)$ as described in [15].
2. The sequence $\boldsymbol{B}$ above is in no way unique. Besides the fact that there usually will be ambiguities in (28) and (29) due to the appearance of syzygies, even the choice of $\boldsymbol{B}^*(z)$ in (30) is clearly not the only one possible. For example, one can distribute arbitrary multiples $\lambda_j\,(z_1 - 1)\ \boldsymbol{G}^*_1(z)$, $j \in \mathbb{Z}_s$, over the even indexed entries of the first block row of $\boldsymbol{B}^*(z)$, the only requirement is that $\sum_{j\in\mathbb{Z}_s} \lambda_j = 1$.

3. As pointed out in [17] for the case $N = 1$ and $M = 2I$, one can choose the sequence $\boldsymbol{B}$ in such a way that it has *smaller* support than $\boldsymbol{A}$. This is due to the fact that by choosing an appropriate reduction process in (28), the polynomial coefficients appearing there can be chosen to have *strictly lower* total degree than those of $\boldsymbol{A}^*$. As (30) indicates, this property carries over to the full rank case $n = N$, but for $n < N$ an *increase* of support will be unavoidable in general as the appearance of the terms $(z_j - 1)\ \boldsymbol{G}_1^*(z)$, $j \in \mathbb{Z}_s$, shows which are of strictly greater degree that $\boldsymbol{G}_1^*$.
4. Theorem 2 also has a partial converse: if there exist $n \geq 1$ and an orthogonal matrix $\boldsymbol{V} \in \mathbb{R}^{N\times N}$ such that (25) holds true, then we have, for any vector $\boldsymbol{y} \in \text{span}\,\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ that

$$\mathbf{0}_N = \Delta_n \boldsymbol{y} = S_{\boldsymbol{B}}\, \Delta_n \boldsymbol{y} = \Delta_n\, S_{\boldsymbol{A}} \boldsymbol{y},$$

hence

$$S_{\boldsymbol{A}} \boldsymbol{y} \in \ker \Delta_n = \text{span}\,\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}\,,$$

that is, $S_{\boldsymbol{A}}$ maps any such constant sequence to another constant sequence in the same vector space.
5. Theorem 2 also plays a crucial role in the description of convergence of the subdivision operator. Indeed, convergence is equivalent to the existence of the above $\boldsymbol{B}$ and its contractivity, that is, the (restricted) spectral radius condition

$$m^{1/p} > \rho_p\left(S_{\boldsymbol{B}}, \Delta_n\right) := \limsup_{r\to\infty} \sup \left\{ \frac{\|S_{\boldsymbol{B}}\, \boldsymbol{c}\|_p}{\|\boldsymbol{c}\|_p} \ :\ \boldsymbol{c} \in \Delta_n\, \ell_p^N\left(\mathbb{Z}^s\right) \right\}.$$

We will not give such a proof here since it is lengthy and tedious, but refer to [10] where the case $N = 1$ and $p = \infty$ is elaborated. ■

The explicit formula for $\boldsymbol{B}$ in (30) allows for a first observation on $\mathcal{Q}_{\boldsymbol{B}}$.

Proposition 3 *Assume that the assumptions of Theorem 2 are satisfied, let $\boldsymbol{B}$ be as defined in (30) and let*

$$\boldsymbol{w} \in \mathcal{Q}_{\boldsymbol{B}} \cap \left(\begin{bmatrix} \mathbf{0}_n \\ \mathbb{R}^{N-n} \end{bmatrix} \right)^s.$$

Then there exists $\boldsymbol{w}' \in \mathbb{R}^{N-n}$ such that

$$\boldsymbol{w} = \mathbf{1}_s \otimes \begin{bmatrix} \mathbf{0}_n \\ \boldsymbol{w}' \end{bmatrix}.$$

PROOF. Let

$$\boldsymbol{w} = \left[\begin{bmatrix} \mathbf{0}_n \\ \boldsymbol{w}_j \end{bmatrix} \ :\ j \in \mathbb{Z}_s \right] \in \left(\begin{bmatrix} \mathbf{0}_n \\ \mathbb{R}^{N-n} \end{bmatrix} \right)^s, \qquad \boldsymbol{w}_j \in \mathbb{R}^{N-n}, \quad j \in \mathbb{Z}_s,$$

belong to $\mathcal{Q}_{\boldsymbol{B}}$, that is, $S_{\boldsymbol{B}} \boldsymbol{w} = \boldsymbol{w}$. With $\boldsymbol{w}^*(z) = \boldsymbol{w}$, $z \in \mathbb{C}_\times^s$, we thus get from the definition of $\boldsymbol{B}$ in (30) that for $z \in \mathbb{C}_\times^s$

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{w}^*(z) = \left(S_{\boldsymbol{B}} \boldsymbol{w}\right)^*(z) = \left(\boldsymbol{B} *_M \boldsymbol{w}\right)^*(z) = \boldsymbol{B}^*(z)\, \boldsymbol{w}^*\left(z^M\right) = \boldsymbol{B}^*(z)\, \boldsymbol{w} \\ &= \begin{bmatrix} (z_0 - 1)\ \boldsymbol{G}_0^*(z) \\ \boldsymbol{H}^*(z) \\ \vdots \\ (z_{s-1} - 1)\ \boldsymbol{G}_{s-1}^*(z) \\ \boldsymbol{H}^*(z) \end{bmatrix} \sum_{j\in\mathbb{Z}_s} \boldsymbol{w}_j = \begin{bmatrix} \mathbf{0}_n \\ \boldsymbol{w}_0 \\ \vdots \\ \mathbf{0}_n \\ \boldsymbol{w}_{s-1} \end{bmatrix} \end{aligned}$$

which implies that

$$\boldsymbol{w}_j = \boldsymbol{H}^*(z) \sum_{k\in\mathbb{Z}_s} \boldsymbol{w}_k =: \boldsymbol{w}', \qquad z \in \mathbb{C}_\times^s, \quad j \in \mathbb{Z}_s. \quad ■$$

4. Polynomial reproduction and approximation power

In this section, we will use Theorem 2 to describe when a subdivision operator $S_{\boldsymbol{A}}$ preserves polynomial sequences of a certain total degree. To that end, we let $\mathcal{V}$ be a subspace of $\mathbb{R}^N$ of dimension $n \geq 1$, and let $\boldsymbol{V} \in \mathbb{R}^{N \times N}$ be an orthogonal matrix such that

$$\mathcal{V} = \boldsymbol{V} \begin{bmatrix} \mathbb{R}^n \\ \boldsymbol{0}_{N-n} \end{bmatrix}.$$

Such a matrix $\boldsymbol{V}$ whose first n columns span $\mathcal{V}$ will be called $\mathcal{V}$*–generating*. By $\Pi = \mathbb{R}[z_j \ : \ j \in \mathbb{Z}_s]$ we will denote the ring of all polynomials with real coefficients and by Π_r, $r \in \mathbb{N}_0$, the vector space of all polynomials of *total degree* at most $\leq r$. Moreover, we denote by Π_r^0 the vector space of all *homogeneous* polynomials of degree r. The vector space of all polynomials with coefficients in $\mathcal{V}$ will be written as

$$\Pi[\mathcal{V}] = \left\{ \boldsymbol{f}(z) = \sum_{\alpha \in \mathbb{Z}^s} \boldsymbol{v}_\alpha \, z^\alpha \ : \ \boldsymbol{v}_\alpha \in \mathcal{V} \right\}$$

where, as usually, only finitely many of the coefficients $\boldsymbol{v}_\alpha$ are nonzero. From this, the definition of $\Pi_r[\mathcal{V}]$ and $\Pi_r^0[\mathcal{V}]$ are immediate; also note that $\Pi_0[\mathcal{V}] = \mathcal{V}$.

Definition 4 *Let* $\boldsymbol{A} \in \ell_{00}^{N \times N}(\mathbb{Z}^s)$ *and* $\mathcal{V} \subset \mathbb{R}^N$. *We say that* $S_{\boldsymbol{A}}$ *has* approximation power r *with respect to* $\mathcal{V}$ *if*

$$S_{\boldsymbol{A}} \Pi_k[\mathcal{V}] = \Pi_k[\mathcal{V}], \qquad k \in \mathbb{Z}_r. \tag{31}$$

The vector space $\mathcal{V}$ *will be called* maximal *if any subspace* $\mathcal{W} \subset \mathbb{R}^n$ *that satisfies (31) is also a subspace of* $\mathcal{V}$, $\mathcal{W} \subset \mathcal{V}$, *and if (31) cannot be satisfied for any true superspace* $\mathcal{W}$ *of* $\mathcal{V}$ *such that* $\mathcal{V} \subset \mathcal{W} \subseteq \mathbb{R}^N$.

In order to formulate our main result of this section, we need to introduce some more notation.

Definition 5 *For* $r \in \mathbb{N}$ *we set* $\Gamma_r = \{\alpha \in \mathbb{N}_0^s \ : \ |\alpha| = r\}$ *and define the* r–th order backward difference operator *as*

$$\boldsymbol{D}_n^r = [\boldsymbol{D}_\alpha \ : \ \alpha \in \Gamma_r], \qquad \boldsymbol{D}_\alpha := \begin{bmatrix} \delta^\alpha \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{N-n} \end{bmatrix}, \qquad \delta^\alpha = \prod_{j \in \mathbb{Z}_s} \delta_j^{\alpha_j}, \tag{32}$$

and

$$\Delta_n^r := \boldsymbol{D}_n^r \boldsymbol{V}^T. \tag{33}$$

Here, $\boldsymbol{D}_n^r$ *and* Δ_n^r *are operators that map* $\ell_{00}^N(\mathbb{Z}^s)$ *to* $\ell_{00}^{Nd_r}(\mathbb{Z}^s)$, *where* $d_r := \binom{r+s}{s}$ *is the dimension of* Π_r.

Moreover, we define for a matrix $\boldsymbol{T} \in \mathbb{R}^{d_r n \times d_r n}$, *conveniently written as*

$$\boldsymbol{T} = [\boldsymbol{T}_{\alpha,\beta} \ : \ \alpha, \beta \in \Gamma_r], \qquad \boldsymbol{T}_{\alpha,\beta} \in \mathbb{R}^{n \times n},$$

the matrix $\boldsymbol{J}_{\boldsymbol{T}} \in \mathbb{R}^{d_r N \times d_r N}$ *as*

$$\boldsymbol{J}_{\boldsymbol{T}} = \left[\widehat{\boldsymbol{T}}_{\beta,\alpha}^T \ : \ \alpha, \beta \in \Gamma_r \right], \qquad \widehat{\boldsymbol{T}}_{\alpha,\beta} = \begin{bmatrix} \boldsymbol{T}_{\alpha,\beta} & \boldsymbol{0} \\ \boldsymbol{0} & \delta_{\alpha,\beta} \boldsymbol{I}_{N-n} \end{bmatrix}$$

Remark 3 Note that though $\boldsymbol{D}_n^r$ and Δ_n^r are *not* the r–th powers of $\boldsymbol{D}_n$ and Δ_n which map $\ell_{00}^N(\mathbb{Z}^s)$ to $\ell_{00}^{Ns^r}(\mathbb{Z}^s)$, they are *equivalent* to them. This is due to the fact that, just like derivatives, the difference operators commute. ■

We now are in a position to describe the existence of polynomial eigenfunctions in terms of a factorization property.

Theorem 3 *Suppose that* $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$ *satisfies* $1 \le n = \dim \mathcal{Q}_{\boldsymbol{A}}$ *and let* $\boldsymbol{V} \in \mathbb{R}^N$ *be a* $\mathcal{Q}_{\boldsymbol{A}}$*–generating matrix. Then*

$$S_{\boldsymbol{A}} \Pi_k [\mathcal{Q}_{\boldsymbol{A}}] = \Pi_k [\mathcal{Q}_{\boldsymbol{A}}], \qquad k \in \mathbb{Z}_r, \tag{34}$$

if and only if there exist $\boldsymbol{B}_j \in \ell_{00}^{Nd_j \times Nd_j}(\mathbb{Z}^s)$ *and* $\boldsymbol{T}_j \in \mathbb{R}^{d_j n \times d_j n}$, $j \in \mathbb{Z}_r$ *such that*

$$\Delta_n^j S_{\boldsymbol{A}} = \boldsymbol{J}_{\boldsymbol{T}_j} S_{\boldsymbol{B}_j} \Delta_n^j \tag{35}$$

and

$$\mathcal{Q}_{\boldsymbol{B}_j} = \left(\boldsymbol{V}^T \mathcal{Q}_{\boldsymbol{A}}\right)^{d_j} = \underbrace{\begin{bmatrix} \mathbb{R}^n \\ \boldsymbol{0}_{N-n} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} \mathbb{R}^n \\ \boldsymbol{0}_{N-n} \end{bmatrix}}_{d_j}. \tag{36}$$

Remark 4

1. For the univariate case it was observed in [13] that the dimension of $\mathcal{Q}_{\boldsymbol{A}}$ forms an invariant under factorization. As (36) shows, this essentially holds true also in the multivariate case, however, now the dimension grows by a factor s with each single factorization step, which is exactly the growth of the matrix $\boldsymbol{B}$ itself. In other words: the *ratio* $\dim \mathcal{Q}_{\boldsymbol{B}_j}/d_j$, $j \in \mathbb{Z}_{r+1}$, remains an invariant under factorization.
2. The factorization condition (35) should be interpreted in a twofold way: one part is the *existence* of such a factorization, that is, of a mask $\widetilde{\boldsymbol{B}}$ such that $\Delta_n^r S_{\boldsymbol{A}} = S_{\widetilde{\boldsymbol{B}}} \Delta_n^r$ (which is the "algebraic part" of this result, see also Proposition 6), the other part is a proper normalization of this mask by left multiplying each entry of the sequence $\boldsymbol{B}$ with the matrix $\boldsymbol{J}_{\boldsymbol{T}}$. ■

The proof of Theorem 3, which is the purpose of the remainder of this section, will proceed by induction on r. In order to do so, we will first consider case $r = 0$ for which (34) ist satisfied trivially, then the start the induction by proving case $r = 1$ which will also serve as an illustration of the general concept used in the end.

We begin with the following observation on the *maximal* subspace which is preserved by $S_{\boldsymbol{A}}$, which is even stronger than what would be needed for the case $r = 0$.

Proposition 4 *For* $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$ *the subdivision operator* $S_{\boldsymbol{A}}$ *has approximation power* 0 *with respect to a maximal subspace* $\mathcal{V} \subset \mathbb{R}^N$ *if and only if* $\mathcal{Q}_{\boldsymbol{A}} = \mathcal{V}$.

PROOF. Since $S_{\boldsymbol{A}} \boldsymbol{y} = \boldsymbol{y}$ for $\boldsymbol{y} \in \mathcal{Q}_{\boldsymbol{A}}$, the maximality of $\mathcal{V}$ implies that $\mathcal{Q}_{\boldsymbol{A}} \subseteq \mathcal{V}$. On the other hand, we have for any $\boldsymbol{v} \in \mathcal{V}$ that $\boldsymbol{v} = S_{\boldsymbol{A}} \boldsymbol{v}$, in particular, for $\epsilon \in E$,

$$\boldsymbol{v} = \boldsymbol{v}(\epsilon) = (S_{\boldsymbol{A}} \boldsymbol{v})(\epsilon) = \sum_{\alpha \in \mathbb{Z}^s} \boldsymbol{A}(\epsilon - M\alpha)\, \boldsymbol{v}(\alpha) = \boldsymbol{A}_\epsilon \boldsymbol{v}, \tag{37}$$

hence, $\boldsymbol{v} \in \mathcal{Q}_{\boldsymbol{A}}$, that is, $\mathcal{V} \subseteq \mathcal{Q}_{\boldsymbol{A}}$. Conversely, $\mathcal{V} = \mathcal{Q}_{\boldsymbol{A}}$ is obviously a subspace with respect to which $S_{\boldsymbol{A}}$ has approximation power 0, and maximality again follows from (37). ■

Next, we point out the simple way how the matrix $\boldsymbol{T}_r \in \mathbb{R}^{d_j n \times d_r n}$ is obtained.

Lemma 3 *If the mask* $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$ *satisfies (34), then there exist matrices* $\boldsymbol{T}_{\alpha,\beta} \in \mathbb{R}^{n\times n}$, $\alpha, \beta \in \Gamma_r$, *such that*

$$S_{\boldsymbol{A}} \left(\boldsymbol{V} \begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{0}_{N-n} \end{bmatrix} (\cdot)^\alpha \right) - \sum_{\beta \in \Gamma_r} \boldsymbol{V} \boldsymbol{T}_{\alpha,\beta}^T (\cdot)^\beta \quad \in \Pi_{r-1}[\mathcal{Q}_{\boldsymbol{A}}], \qquad \alpha \in \Gamma_r, \tag{38}$$

and the matrix $\boldsymbol{T} = [\boldsymbol{T}_{\alpha,\beta} \;:\; \alpha, \beta \in \Gamma_r]$ *is nonsingular.*

PROOF. The monomials $\boldsymbol{m}_{j,\alpha}(z) := \boldsymbol{v}_j\, z^\alpha$, $\boldsymbol{v}_j = \boldsymbol{V}\boldsymbol{e}_j$, $\alpha \in \Gamma_r$, $j \in \mathbb{Z}_n$, span the vector space $\Pi_r^0\left[\mathcal{Q}_{\boldsymbol{A}}\right]$. By (34), their images $\boldsymbol{f}_{j,\alpha} := S_{\boldsymbol{A}}\boldsymbol{m}_{j,\alpha}$ belong to $\Pi_r\left[\mathcal{Q}_{\boldsymbol{A}}\right]$ and therefore can be written as

$$\boldsymbol{f}_{j,\alpha}(z) = \boldsymbol{g}(z) + \sum_{\beta\in\Gamma_r}\sum_{k\in\mathbb{Z}_n} \boldsymbol{V}\boldsymbol{e}_k T_{\alpha,\beta,j,k}\, z^\beta = \boldsymbol{g}(z) + \sum_{\beta\in\Gamma_r} z^\beta \boldsymbol{V}\boldsymbol{T}_{\alpha,\beta}^T\, \boldsymbol{e}_j, \qquad \boldsymbol{g} \in \Pi_{r-1}\left[\mathcal{Q}_{\boldsymbol{A}}\right],$$

where $\boldsymbol{T}_{\alpha,\beta} = [T_{\alpha,\beta,j,k} \;:\; j,k \in \mathbb{Z}_n]$. Written in matrix form, this gives (38) by comparing coefficients of highest degree. Since

$$S_{\boldsymbol{A}}\Pi_r^0\left[\mathcal{Q}_{\boldsymbol{A}}\right] + \Pi_{r-1}\left[\mathcal{Q}_{\boldsymbol{A}}\right] = \Pi_r\left[\mathcal{Q}_{\boldsymbol{A}}\right],$$

the matrix $\boldsymbol{T} = [\boldsymbol{T}_{\alpha,\beta} \;:\; \alpha,\beta \in \Gamma_r]$ must be nonsingular because otherwise there would exist a vector $\boldsymbol{y} = [\boldsymbol{y}_\alpha \;:\; \alpha \in \Gamma_r] \in \mathbb{R}^{d_r n}$ such that

$$\boldsymbol{0} = \boldsymbol{T}^T\boldsymbol{y} = \left[\sum_{\alpha\in\Gamma_r} \boldsymbol{T}_{\alpha,\beta}^T\,\boldsymbol{y}_\alpha \;:\; \beta \in \Gamma_r\right].$$

But then

$$\boldsymbol{f}(z) := \sum_{\alpha\in\Gamma_r} \boldsymbol{V}\begin{bmatrix} \boldsymbol{y}_\alpha \\ \boldsymbol{0}_{N-n}\end{bmatrix} z^\alpha = \sum_{\alpha\in\Gamma_r} \boldsymbol{V}\begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{0}_{N-n}\end{bmatrix}\boldsymbol{y}_\alpha \qquad \in \Pi_r^0\left[\mathcal{Q}_{\boldsymbol{A}}\right]$$

has, by (38) the property that

$$S_{\boldsymbol{A}}\boldsymbol{f} - \sum_{\alpha\in\Gamma_r}\sum_{\beta\in\Gamma_r} \boldsymbol{V}\boldsymbol{T}_{\alpha,\beta}^T\,\boldsymbol{y}_\alpha\,(\cdot)^\beta = S_{\boldsymbol{A}}\boldsymbol{f} - \boldsymbol{V}\sum_{\beta\in\Gamma_r}\left(\sum_{\alpha\in\Gamma_r}\boldsymbol{T}_{\alpha,\beta}^T\,\boldsymbol{y}_\alpha\right)(\cdot)^\beta = S_{\boldsymbol{A}}\boldsymbol{f}$$

belongs to $\Pi_{r-1}\left[\mathcal{Q}_{\boldsymbol{A}}\right]$ which contradicting the fact

$$\dim\Big(S_{\boldsymbol{A}}\Pi_r^0\left[\mathcal{Q}_{\boldsymbol{A}}\right] \cap \left(\Pi_r\left[\mathcal{Q}_{\boldsymbol{A}}\right] \setminus \Pi_{r-1}\left[\mathcal{Q}_{\boldsymbol{A}}\right]\right)\Big) = \dim \Pi_r^0\left[\mathcal{Q}_{\boldsymbol{A}}\right] = n\binom{r+s-1}{s-1}$$

which follows directly from (34). ■

Let us now turn to the case $r = 1$ which we record in the following proposition.

Proposition 5 *Suppose that* $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$ *satisfies* $1 \le n = \dim \mathcal{Q}_{\boldsymbol{A}}$ *and let* $\boldsymbol{V} \in \mathbb{R}^N$ *be a* $\mathcal{Q}_{\boldsymbol{A}}$*–generating matrix. Then*

$$S_{\boldsymbol{A}}\Pi_1\left[\mathcal{Q}_{\boldsymbol{A}}\right] = \Pi_1\left[\mathcal{Q}_{\boldsymbol{A}}\right] \tag{39}$$

if and only if there exists $\boldsymbol{B} \in \ell_{00}^{Ns\times Ns}(\mathbb{Z}^s)$ *and* $\boldsymbol{T} \in \mathbb{R}^{ns\times ns}$ *such that*

$$\Delta_n\, S_{\boldsymbol{A}} = \boldsymbol{J}_{\boldsymbol{T}}\, S_{\boldsymbol{B}}\,\Delta_n, \qquad \mathcal{Q}_{\boldsymbol{B}} = \left(\boldsymbol{V}^T\mathcal{Q}_{\boldsymbol{A}}\right)^s. \tag{40}$$

PROOF. By Theorem 2 there exists $\widetilde{\boldsymbol{B}} \in \ell_{00}^{Ns\times Ns}(\mathbb{Z}^s)$, such that

$$\Delta_n\, S_{\boldsymbol{A}} = S_{\widetilde{\boldsymbol{B}}}\Delta_n. \tag{41}$$

Now assume that (39) holds true. For arbitrary vectors $\boldsymbol{v}_j = \boldsymbol{V}\begin{bmatrix} \boldsymbol{y}_j \\ \boldsymbol{0}_{N-n}\end{bmatrix} \in \mathcal{Q}_{\boldsymbol{A}}$, $\boldsymbol{y}_j \in \mathbb{R}^n$, $j \in \mathbb{Z}_s$, let $\boldsymbol{f} \in \Pi_1\left[\mathcal{Q}_{\boldsymbol{A}}\right]$ denote the linear polynomial

$$\boldsymbol{f}(z) = -\sum_{j\in\mathbb{Z}_s} \boldsymbol{v}_j\, z_j, \qquad z \in \mathbb{C}^s. \tag{42}$$

as well as the associated sequence $\boldsymbol{f} \in \ell^N(\mathbb{Z}^s)$ Then it follows from

$$\delta_j(\cdot)_k = (\cdot - e_j)_k - (\cdot)_k = -\delta_{jk}, \qquad j,k \in \mathbb{Z}_s,$$

that

$$\Delta_n \boldsymbol{f} = \boldsymbol{D}_n \boldsymbol{V}^T \boldsymbol{f} = -\boldsymbol{D}_n \left(\sum_{j=1}^s \boldsymbol{V}^T \boldsymbol{v}_j (\cdot)_j \right) = \left[\boldsymbol{V}^T \boldsymbol{v}_j \ : \ j \in \mathbb{Z}_s \right] = \left[\begin{bmatrix} \boldsymbol{y}_j \\ \boldsymbol{0}_{N-n} \end{bmatrix} \ : \ j \in \mathbb{Z}_n \right].$$

Moreover, let $\boldsymbol{T}$ be the matrix from Lemma 3, then the polynomial

$$\boldsymbol{g}(z) = \sum_{j,k \in \mathbb{Z}_s} \boldsymbol{V} \boldsymbol{T}_{jk}^T \boldsymbol{y}_j \, z_k \qquad \in \Pi_1^0 \left[\mathcal{Q}_{\boldsymbol{A}} \right]$$

satisfies $S_{\boldsymbol{A}} \boldsymbol{f} - \boldsymbol{g} \in \mathcal{Q}_{\boldsymbol{A}} = \ker \Delta_n$. Moreover,

$$\Delta_n \boldsymbol{g} = \sum_{j \in \mathbb{Z}_s} \left[\boldsymbol{T}_{jk}^T \boldsymbol{y}_j \ : \ k \in \mathbb{Z}_s \right] = \boldsymbol{T}^T \left[\boldsymbol{y}_j \ : \ j \in \mathbb{Z}_s \right] = \boldsymbol{J}_{\boldsymbol{T}} \left[\begin{bmatrix} \boldsymbol{y}_j \\ \boldsymbol{0}_{N-n} \end{bmatrix} \ : \ j \in \mathbb{Z}_n \right].$$

Therefore we get, substituting the above identities into (41), that

$$\boldsymbol{J}_{\boldsymbol{T}} \begin{bmatrix} \boldsymbol{V}^T \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{V}^T \boldsymbol{v}_{s-1} \end{bmatrix} = \Delta_n \boldsymbol{g} = \Delta_n S_{\boldsymbol{A}} \boldsymbol{f} = S_{\widetilde{\boldsymbol{B}}} \Delta_n \boldsymbol{f} = S_{\widetilde{\boldsymbol{B}}} \begin{bmatrix} \boldsymbol{V}^T \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{V}^T \boldsymbol{v}_{s-1} \end{bmatrix}$$

and defining $\boldsymbol{B} = \boldsymbol{J}_{\boldsymbol{T}}^{-1} \widetilde{\boldsymbol{B}}$, that is, $\Delta_n S_{\boldsymbol{A}} = \boldsymbol{J}_{\boldsymbol{T}} S_{\boldsymbol{B}} \Delta_n$, a left multiplication with the nonsingular matrix $\boldsymbol{J}_{\boldsymbol{T}}^{-1}$ results in

$$\left[\boldsymbol{V}^T \boldsymbol{v}_j \ : \ j \in \mathbb{Z}_s \right] = S_{\boldsymbol{B}} \left[\boldsymbol{V}^T \boldsymbol{v}_j \ : \ j \in \mathbb{Z}_s \right],$$

that is $\left(\boldsymbol{V}^T \mathcal{Q}_{\boldsymbol{A}} \right)^s \subseteq \mathcal{Q}_{\boldsymbol{B}}$. Suppose that $\mathcal{Q}_{\boldsymbol{B}} \supset \left(\boldsymbol{V}^T \mathcal{Q}_{\boldsymbol{A}} \right)^s$. This assumption implies that there exists a nonzero vector $\boldsymbol{w} \in \mathcal{Q}_{\boldsymbol{B}} \cap \left(\begin{bmatrix} \boldsymbol{0}_n \\ \mathbb{R}^{N-n} \end{bmatrix} \right)^s$ which, by Proposition 3, must be of the form $\boldsymbol{w} = \boldsymbol{1}_s \otimes \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix}$ for some $\boldsymbol{w}' \in \mathbb{R}^{N-n}$. Thus,

$$\boldsymbol{w} = \boldsymbol{D}_n \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix} = \boldsymbol{D}_n \boldsymbol{V}^T \boldsymbol{V} \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix} = \Delta_n \boldsymbol{V} \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix},$$

which yields, taking into account (17), that

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{J}_{\boldsymbol{T}} \boldsymbol{w} = \boldsymbol{J}_{\boldsymbol{T}} S_{\boldsymbol{B}} \Delta_n \boldsymbol{V} \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix} = \Delta_n S_{\boldsymbol{A}} \boldsymbol{V} \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix} = \boldsymbol{D}_n \boldsymbol{V}^T S_{\boldsymbol{A}} \boldsymbol{V} \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix} \\ &= \boldsymbol{D}_n \begin{bmatrix} m \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \widetilde{\boldsymbol{A}} \end{bmatrix} \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix} = \boldsymbol{1}_s \otimes \begin{bmatrix} \boldsymbol{0}_n \\ \widetilde{\boldsymbol{A}} \boldsymbol{w}' \end{bmatrix}, \end{aligned}$$

and thus the contradiction that

$$\begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{0}_n \\ \widetilde{\boldsymbol{A}} \boldsymbol{w}' \end{bmatrix}, \qquad \text{i.e.,} \qquad \boldsymbol{V} \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{w}' \end{bmatrix} \in \mathcal{Q}_{\boldsymbol{A}}.$$

The converse is easy: since any polynomial $\boldsymbol{f} \in \Pi_1 \left[\mathcal{Q}_{\boldsymbol{V}} \right]$ can be written in the form (42), (40) implies that

$$\Delta_n S_{\boldsymbol{A}} \boldsymbol{f} = \boldsymbol{J}_{\boldsymbol{T}} S_{\boldsymbol{B}} \left[\boldsymbol{V}^T \boldsymbol{v}_j \ : \ j \in \mathbb{Z}_s \right] = \boldsymbol{J}_{\boldsymbol{T}} \left[\boldsymbol{V}^T \boldsymbol{v}_j \ : \ j \in \mathbb{Z}_s \right] = \Delta_n \boldsymbol{g},$$

that is, $S_{\boldsymbol{A}} \boldsymbol{f} - \boldsymbol{g} \in \ker \Delta_n = \mathcal{Q}_{\boldsymbol{A}}$. ■

The next result is of technical nature.

Lemma 4 *For* $r \in \mathbb{N}$ *and* $\boldsymbol{v}_\alpha \in \mathcal{Q}_{\boldsymbol{A}}$, $\alpha \in \Gamma_r$, *define the* homogeneous *polynomial*

$$\boldsymbol{f}(z) = \sum_{\alpha \in \Gamma_r} \boldsymbol{v}_\alpha \, z^\alpha \in [\mathcal{Q}_{\boldsymbol{A}}] \, \Pi_r^0$$

Then

$$\Delta_n^r \boldsymbol{f} = (-1)^r \left[\alpha! \, \boldsymbol{V}^T \boldsymbol{v}_\alpha \; : \; \alpha \in \Gamma_r \right]. \tag{43}$$

PROOF. Since

$$\delta_j \, (\cdot)^\alpha = (\cdot - e_j)^\alpha - (\cdot)^\alpha = \sum_{k=0}^{\alpha_j} (-1)^k \binom{\alpha_j}{k} (\cdot)^{\alpha - k \, e_j} - (\cdot)^\alpha = \sum_{k=1}^{\alpha_j} (-1)^k \binom{\alpha_j}{k} (\cdot)^{\alpha - k \, e_j},$$

it follows by iteration that

$$\delta^\beta \, (\cdot)^\alpha = (-1)^{|\alpha|} \alpha! \, \delta_{\alpha,\beta},$$

which yields (43). ■

Finally, we define for $r \in \mathbb{N}$ the permutation matrix

$$\boldsymbol{W}_r := \begin{bmatrix} \boldsymbol{I}_n & & & & & & \\ & & & \boldsymbol{I}_{N-n} & & & \\ & \boldsymbol{I}_n & & & & & \\ & & \ddots & & \boldsymbol{I}_{N-n} & & \\ & & & & & \ddots & \\ & & \boldsymbol{I}_n & & & & \\ & & & & & & \boldsymbol{I}_{N-n} \end{bmatrix} \in \mathbb{R}^{N d_r \times N d_r} \tag{44}$$

which is obviously orthogonal as all its rows or columns have disjoint support, respectively. We now have all tools available which are needed to complete and finish the proof of Theorem 3.

PROOF OF THEOREM 3 Suppose that Theorem 3 has been verified for some $r \geq 1$. We prove that it is also valid for $r+1$.

Suppose that, for all $k \in \mathbb{Z}_{r+1}$, we have that

$$S_{\boldsymbol{A}} \Pi_k \left[\mathcal{Q}_{\boldsymbol{A}} \right] = \Pi_k \left[\mathcal{Q}_{\boldsymbol{A}} \right]. \tag{45}$$

Since this means, in particular, that (45) holds true for $k = r$, we conclude that

$$\Delta_n^r S_{\boldsymbol{A}} = \boldsymbol{J}_{\boldsymbol{T}_r} \, S_{\boldsymbol{B}_r} \Delta_n^r, \qquad \mathcal{Q}_{\boldsymbol{B}_r} = \left(\boldsymbol{V}^T \mathcal{Q}_{\boldsymbol{A}} \right)^{d_r} = \left(\begin{bmatrix} \mathbb{R}^n \\ \boldsymbol{0}_{N-n} \end{bmatrix} \right)^{d_r}.$$

With $\boldsymbol{W} := \boldsymbol{W}_r$ we have that $\mathcal{Q}_{\boldsymbol{B}_r} = \boldsymbol{W} \begin{bmatrix} \mathbb{R}^{n d_r} \\ \boldsymbol{0}_{d_r(N-n)} \end{bmatrix}$, that is, $\boldsymbol{W}$ is $\mathcal{Q}_{\boldsymbol{B}_r}$–generating. Defining

$$\widetilde{\Delta} = \left[\widetilde{\boldsymbol{D}}_j \; : \; j \in \mathbb{Z}_s \right] \boldsymbol{W}^T, \qquad \widetilde{\boldsymbol{D}}_j := \begin{bmatrix} \delta_j \boldsymbol{I}_{d_r \, n} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{d_r \, (N-n)} \end{bmatrix}, \quad j \in \mathbb{Z}_s,$$

Theorem 2 implies that there exists $\widetilde{\boldsymbol{B}} \in \ell_{00}^{s d_r N \times s d_r N} (\mathbb{Z}^s)$ such that $\widetilde{\Delta} S_{\boldsymbol{B}_r} = S_{\widetilde{\boldsymbol{B}}} \widetilde{\Delta}$. Using the abbreviation $\boldsymbol{T} = \boldsymbol{T}_r$, we next compute

$$\begin{aligned} \widetilde{\Delta} \boldsymbol{J}_{\boldsymbol{T}} &= \widetilde{\boldsymbol{D}} \, \boldsymbol{W}^T \boldsymbol{J}_{\boldsymbol{T}} = \widetilde{\boldsymbol{D}} \begin{bmatrix} \boldsymbol{T}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{d_r(N-n)} \end{bmatrix} \boldsymbol{W}^T = \left[\begin{bmatrix} \delta_j \boldsymbol{T}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{d_r(N-n)} \end{bmatrix} \; : \; j \in \mathbb{Z}_s \right] \boldsymbol{W}^T \\ &= \left(\boldsymbol{I}_s \otimes \begin{bmatrix} \boldsymbol{T}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{d_r(N-n)} \end{bmatrix} \right) \widetilde{\boldsymbol{D}} \boldsymbol{W}^T = \left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T \boldsymbol{J}_{\boldsymbol{T}} \boldsymbol{W} \right) \widetilde{\Delta} \\ &= \left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T \right) \left(\boldsymbol{I}_s \otimes \boldsymbol{J}_{\boldsymbol{T}} \right) \left(\boldsymbol{I}_s \otimes \boldsymbol{W} \right) \widetilde{\Delta}, \end{aligned}$$

which yields that

$$\widetilde{\Delta}\,\Delta_n^r S_{\boldsymbol{A}} = \widetilde{\Delta}\,\boldsymbol{J}_{\boldsymbol{T}_r} S_{\boldsymbol{B}_r}\Delta_n^r = \left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T\right) (\boldsymbol{I}_s \otimes \boldsymbol{J}_{\boldsymbol{T}}) \,(\boldsymbol{I}_s \otimes \boldsymbol{W})\, S_{\widetilde{\boldsymbol{B}}}\widetilde{\Delta}\,\Delta_n^r. \tag{46}$$

With the help of the identity

$$\begin{aligned}
\widetilde{\Delta}\,\Delta_n^r &= \widetilde{\boldsymbol{D}}\boldsymbol{W}^T\boldsymbol{D}_n^r\boldsymbol{V}^T = \widetilde{\boldsymbol{D}}\begin{bmatrix} [\delta^\alpha \boldsymbol{I}_n \,:\, \alpha \in \Gamma_r] & \boldsymbol{0}_{d_r n, N-n} \\ \boldsymbol{0}_{d_r(N-n),n} & \boldsymbol{1}_{d_r}\otimes \boldsymbol{I}_{N-n}\end{bmatrix}\boldsymbol{V}^T \\
&= \left[\begin{bmatrix} [\delta^{\alpha+e_j} \boldsymbol{I}_n \,:\, \alpha \in \Gamma_r] & \boldsymbol{0}_{d_r n, N-n} \\ \boldsymbol{0}_{d_r(N-n),n} & \boldsymbol{1}_{d_r}\otimes \boldsymbol{I}_{N-n}\end{bmatrix} \,:\, j \in \mathbb{Z}_s\right]\boldsymbol{V}^T = \left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T\boldsymbol{D}_n^r\right)\boldsymbol{D}_n\boldsymbol{V}^T \\
&= \left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T\right)(\boldsymbol{I}_s \otimes \boldsymbol{D}_n^r)\,\boldsymbol{D}_n\boldsymbol{V}^T = \left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T\right)\boldsymbol{X}\,\Delta_n^{r+1}
\end{aligned}$$

which holds true with the *rearrangement matrix* $\boldsymbol{X} \in \{0,1\}^{sd_rN\times d_{r+1}N}$, defined as

$$\boldsymbol{X} = \left[\boldsymbol{X}_{(\alpha,j),\beta} \,:\, (\alpha,j) \in \Gamma_r \otimes \mathbb{Z}_s,\ \beta \in \Gamma_{r+1}\right], \qquad \boldsymbol{X}_{(\alpha,j),\beta} = \delta_{\alpha+e_j,\beta}\,\boldsymbol{I}_N \tag{47}$$

and satisfying $(\boldsymbol{I}_s \otimes \boldsymbol{D}_n^r)\,\boldsymbol{D}_n = \boldsymbol{X}\boldsymbol{D}_n^{r+1}$, we obtain from (46) that

$$\boldsymbol{X}\Delta_n^{r+1}S_{\boldsymbol{A}} = (\boldsymbol{I}_s \otimes \boldsymbol{J}_{\boldsymbol{T}})\,(\boldsymbol{I}_s \otimes \boldsymbol{W})\,S_{\widetilde{\boldsymbol{B}}}\left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T\right)\boldsymbol{X}\Delta_n^{r+1}. \tag{48}$$

It follows by a straightforward computation from the definition (47) of $\boldsymbol{X}$ that $\boldsymbol{X}^T\boldsymbol{X} = s\,\boldsymbol{I}_{d_{r+1}N}$, and setting

$$\boldsymbol{B} := \frac{1}{s}\boldsymbol{X}^T(\boldsymbol{I}_s \otimes \boldsymbol{J}_{\boldsymbol{T}})\,(\boldsymbol{I}_s \otimes \boldsymbol{W})\,\widetilde{\boldsymbol{B}}\left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T\right)\boldsymbol{X} \qquad \in \ell_{00}^{d_{r+1}N\times d_{r+1}N}(\mathbb{Z}^s), \tag{49}$$

we obtain from (48) that

$$S_{\boldsymbol{B}}\,\Delta_n^{r+1} = \frac{1}{s}\boldsymbol{X}^T\boldsymbol{X}\Delta_n^{r+1}S_{\boldsymbol{A}} = \Delta_n^{r+1}S_{\boldsymbol{A}},$$

and so the mask $\boldsymbol{B}_{r+1} := \boldsymbol{J}_{\boldsymbol{T}_{r+1}}^{-1}\boldsymbol{B}$ satisfies $\Delta_n^{r+1}S_{\boldsymbol{A}} = \boldsymbol{J}_{\boldsymbol{T}_{r+1}}S_{\boldsymbol{B}_{r+1}}\Delta_n^{r+1}$. The containment of $\mathcal{Q}_{\boldsymbol{B}_{r+1}}$ in $\left(\boldsymbol{V}^T\mathcal{Q}_{\boldsymbol{A}}\right)^{d_{r+1}}$ is now easily obtained by making use of the preservation of all homogeneous polynomial sequences

$$\boldsymbol{f}(z) = \sum_{\alpha\in\Gamma_{r+1}}\boldsymbol{v}_\alpha\,z^\alpha, \qquad \boldsymbol{v}_\alpha \in \mathcal{Q}_{\boldsymbol{A}}, \quad z \in \mathbb{C}^s,$$

of degree $r+1$ by $S_{\boldsymbol{A}}$ and Lemma 4. To show that $\mathcal{Q}_{\boldsymbol{B}_{r+1}} = \left(\boldsymbol{V}^T\mathcal{Q}_{\boldsymbol{A}}\right)^{d_{r+1}}$, we again assume the existence of $\boldsymbol{w} \in \left(\begin{bmatrix}\boldsymbol{0}_n \\ \mathbb{R}^{N-n}\end{bmatrix}\right)^{d_{r+1}}$ such that $\boldsymbol{w} = S_{\boldsymbol{B}_{r+1}}\boldsymbol{w} = S_{\boldsymbol{B}}\boldsymbol{w}$. By (49) this means that

$$\frac{1}{s}\boldsymbol{X}^T(\boldsymbol{I}_s \otimes \boldsymbol{J}_{\boldsymbol{T}})\,\boldsymbol{X}\boldsymbol{w} = \boldsymbol{w} = \frac{1}{s}\boldsymbol{X}^T(\boldsymbol{I}_s \otimes \boldsymbol{J}_{\boldsymbol{T}})\,(\boldsymbol{I}_s \otimes \boldsymbol{W})\,S_{\widetilde{\boldsymbol{B}}}\left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T\right)\boldsymbol{X}\,\boldsymbol{w},$$

hence, defining $\boldsymbol{x} = \boldsymbol{X}\boldsymbol{w}$,

$$\frac{1}{s}\boldsymbol{X}^T(\boldsymbol{I}_s \otimes \boldsymbol{J}_{\boldsymbol{T}})\left(\boldsymbol{x} - (\boldsymbol{I}_s \otimes \boldsymbol{W})\,S_{\widetilde{\boldsymbol{B}}}\left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T\right)\boldsymbol{x}\right) = 0.$$

By (30), $\boldsymbol{z} := (\boldsymbol{I}_s \otimes \boldsymbol{W})\,\boldsymbol{X}\boldsymbol{w}$ is an eigensequence of $S_{\widetilde{\boldsymbol{B}}}$ and by Proposition 3 it must be of the form $\boldsymbol{1}_s \otimes \boldsymbol{y}$, $\boldsymbol{y} \in \begin{bmatrix}\boldsymbol{0}_{d_r n} \\ \mathbb{R}^{d_r(N-n)}\end{bmatrix}$, hence $\boldsymbol{z} = \boldsymbol{1}_s \otimes \boldsymbol{W}^T\boldsymbol{y}$. Moreover, since $\widetilde{\Delta}S_{\boldsymbol{B}_r} = S_{\widetilde{\boldsymbol{B}}}\widetilde{\Delta}$, it follows that $\boldsymbol{W}^T\boldsymbol{y}$

is an eigensequence of $S_{\boldsymbol{B}_r}$, an inductive repetition of the above argument gives that $\boldsymbol{W}^T \boldsymbol{y} = \boldsymbol{1}_{d_r} \otimes \boldsymbol{w}'$, $\boldsymbol{w}' \in \begin{bmatrix} \boldsymbol{0}_n \\ \mathbb{R}^{N-n} \end{bmatrix}$, and thus

$$\boldsymbol{X}\boldsymbol{w} = \left(\boldsymbol{I}_s \otimes \boldsymbol{W}^T\right)(\boldsymbol{1}_s \otimes y) = \boldsymbol{1}_s \otimes (\boldsymbol{1}_{d_r} \otimes \boldsymbol{w}'), \qquad \text{i.e.,} \qquad \boldsymbol{w} = \boldsymbol{1}_{d_{r+1}} \otimes \boldsymbol{w}'.$$

But then

$$\boldsymbol{1}_{d_{r+1}} \boldsymbol{w}' = S_{\boldsymbol{B}_{r+1}} \boldsymbol{1}_{d_{r+1}} \boldsymbol{w}' = S_{\boldsymbol{B}_{r+1}} \Delta_n^{r+1} \boldsymbol{w}' = \Delta_n^{r+1} S_{\boldsymbol{A}} \boldsymbol{w}'$$

again yields the contradiction that $S_{\boldsymbol{A}} \boldsymbol{w}' = \boldsymbol{w}'$. This proves that approximation power implies the existence of a factorization with the described properties. The converse is proved just like in the case $r = 1$. ■

To end this section, we give an ideal theoretic interpretation of the factorization in Theorem 3.

Proposition 6 *If the mask $\boldsymbol{A}$ satisfies (35) and (36), then*

$$\begin{aligned} \boldsymbol{V}^T \boldsymbol{A}^* \boldsymbol{V} \in & \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \left(\langle z^M - 1\rangle : \langle z-1\rangle\right)^{r+1} + \begin{bmatrix} \boldsymbol{1}_{n,n} - \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \left\langle z^M - 1\right\rangle^{r+1} : \langle z-1\rangle^r \\ & + \begin{bmatrix} \boldsymbol{0}_{n,n} & \boldsymbol{1} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \left(\left\langle z^M - 1\right\rangle : \langle z-1\rangle^r\right) + \begin{bmatrix} \boldsymbol{0}_{n,n} & \boldsymbol{0} \\ \boldsymbol{1} & \boldsymbol{0} \end{bmatrix} \left\langle z^M - 1\right\rangle^{r+1} \\ & + \begin{bmatrix} \boldsymbol{0}_{n,n} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix} \Lambda. \end{aligned} \tag{50}$$

Conversely, if (50) is satisfied, then there exists a factorization of the form (35). □

The statement of Proposition 6 becomes particularly simple provided that we are considering a *full rank scheme* where $n := \dim \mathcal{Q}_{\boldsymbol{A}} = N$; the most important case of such schemes is of course when $N = 1$.

Corollary 2 *If $\boldsymbol{A} \in \ell_{00}^{N\times N}(\mathbb{Z}^s)$ is of full rank and has approximation power r, then*

$$\boldsymbol{A}^* \in \boldsymbol{I}_N \left\langle z^M - 1\right\rangle^{r+1} : \langle z-1\rangle^{r+1} + (\boldsymbol{1}_{N,N} - \boldsymbol{I}_N) \left\langle z^M - 1\right\rangle^{r+1} : \langle z-1\rangle^r.$$

PROOF. Let $\widetilde{\boldsymbol{B}} \in \ell_{00}^{Nd_r\times Nd_r}(\mathbb{Z}^s)$ be defined by $\Delta_n^r S_{\boldsymbol{A}} = S_{\widetilde{\boldsymbol{B}}} \Delta_n^r$. Since $\widetilde{\boldsymbol{B}}$ is only a renormalization of $\boldsymbol{B}$, i.e., all entries of the sequence $\boldsymbol{B}$ are multiplied with the same diagonal matrix, the existence of $\widetilde{\boldsymbol{B}}$ follows from the validity of (35). Using again the abbreviation $\boldsymbol{W} = \boldsymbol{W}_r$, the computation

$$\boldsymbol{W}^T \Delta_n^r = \begin{bmatrix} [\delta_\alpha \boldsymbol{I}_n \ : \ \alpha \in \Gamma_r] & \boldsymbol{0}_{d_r n, N-n} \\ \boldsymbol{0}_{d_r(N-n),n} & \boldsymbol{1}_{d_r} \otimes \boldsymbol{I}_{N-n} \end{bmatrix} \boldsymbol{V}^T =: \widetilde{\boldsymbol{D}} \boldsymbol{V}^T =: \widetilde{\Delta},$$

implies that

$$\widetilde{\Delta} S_{\boldsymbol{A}} = \boldsymbol{W}^T \Delta_n^r S_{\boldsymbol{A}} = \boldsymbol{W}^T S_{\widetilde{\boldsymbol{B}}} \Delta_n^r = \boldsymbol{W}^T S_{\widetilde{\boldsymbol{B}}} \boldsymbol{W}\boldsymbol{W}^T \Delta_n^r = \left(\boldsymbol{W}^T S_{\widetilde{\boldsymbol{B}}} \boldsymbol{W}\right) \widetilde{\Delta},$$

hence,

$$\widetilde{\boldsymbol{D}} \left(\boldsymbol{V}^T S_{\boldsymbol{A}} \boldsymbol{V}\right) = \left(\boldsymbol{W}^T S_{\boldsymbol{B}} \boldsymbol{W}\right) \widetilde{\boldsymbol{D}}, \tag{51}$$

or, in terms of symbols,

$$\widetilde{\boldsymbol{D}}^*(z)\, \boldsymbol{V}^T \boldsymbol{A}^* \boldsymbol{V} = \boldsymbol{W}^T \widetilde{\boldsymbol{B}}^*(z) \boldsymbol{W}\, \widetilde{\boldsymbol{D}}^*\left(z^M\right), \qquad z \in \mathbb{C} \setminus \{0\}. \tag{52}$$

By (36) and Corollary 1, we get that

$$\boldsymbol{W}^T \widetilde{\boldsymbol{B}}^*(z) \boldsymbol{W} = \begin{bmatrix} \boldsymbol{P}^*(z) & \boldsymbol{Q}_1^*(z) \\ \boldsymbol{Q}_2^*(z) & \boldsymbol{R}^*(z) \end{bmatrix}, \tag{53}$$

where

$$\boldsymbol{P}^* \in \boldsymbol{I}_{d_r n} \left(\left\langle z^M - 1 \right\rangle : \langle z - 1 \rangle \right) + \left(\mathbf{1}_{d_r n, d_r n} - \boldsymbol{I}_n \right) \left\langle z^M - 1 \right\rangle \tag{54}$$

and

$$\boldsymbol{Q}_1^* \in \left\langle z^M - 1 \right\rangle^{d_r n \times d_r (N-n)}, \quad \boldsymbol{Q}_2^* \in \left\langle z^M - 1 \right\rangle^{d_r (N-n) \times d_r n}, \quad \boldsymbol{R}^* \in \Lambda^{d_r (N-n) \times d_r (N-n)}. \tag{55}$$

Noting that

$$\widetilde{\boldsymbol{D}}^*(z) = \begin{bmatrix} \left[(z-1)^\alpha \, \boldsymbol{I}_n \; : \; \alpha \in \Gamma_r \right] & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{d_r} \otimes \boldsymbol{I}_{N-n} \end{bmatrix}, \qquad (z-1)^\alpha = \prod_{j \in \mathbb{Z}_s} (z_j - 1)^{\alpha_j}, \quad \alpha \in \Gamma_r,$$

and substituting this identity and the decomposition (27) from the proof of Theorem 2 into (51), we thus obtain that

$$\begin{bmatrix} \left[(z-1)^\alpha \, \boldsymbol{F}^*(z) \; : \; \alpha \in \Gamma_r \right] & \left[(z-1)^\alpha \, \boldsymbol{G}_1^*(z) \; : \; \alpha \in \Gamma_r \right] \\ \mathbf{1}_{d_r} \otimes \boldsymbol{G}_2^* & \mathbf{1}_{d_r} \otimes \boldsymbol{H}^* \end{bmatrix}$$
$$= \begin{bmatrix} \boldsymbol{P}^* \; \left[\left(z^M - 1 \right)^\alpha \, \boldsymbol{I}_n \; : \; \alpha \in \Gamma_r \right] & \boldsymbol{Q}_1^* \; \mathbf{1}_{d_r} \otimes \boldsymbol{I}_{N-n} \\ \boldsymbol{Q}_2^* \; \left[\left(z^M - 1 \right)^\alpha \, \boldsymbol{I}_n \; : \; \alpha \in \Gamma_r \right] & \boldsymbol{R}^* \; \mathbf{1}_{d_r} \otimes \boldsymbol{I}_{N-n} \end{bmatrix}$$

If we write the block components of (53) as

$$\boldsymbol{P}^* = \left[\boldsymbol{P}_{\alpha,\beta}^* \; : \; \alpha, \beta \in \Gamma_r \right], \qquad \boldsymbol{X}_{\alpha,\beta}^* \in \Lambda^{n \times n},$$

and respective representations for $\boldsymbol{Q}_1^*, \boldsymbol{Q}_2^*$ as well as $\boldsymbol{R}^*$, we first obtain that

$$(z-1)^\alpha \, \boldsymbol{F}^*(z) = \sum_{\beta \in \Gamma_r} \boldsymbol{P}_{\alpha,\beta}^*(z) \; \left(z^M - 1 \right)^\beta, \qquad \alpha \in \Gamma_r, \quad z \in \mathbb{C}_\times^s,$$

which yields, by (54), that

$$\langle z - 1 \rangle^r \; \boldsymbol{F}^* \in \left(\boldsymbol{I}_n \, \left\langle z^M - 1 \right\rangle : \langle z - 1 \rangle + \left(\mathbf{1}_{n,n} - \boldsymbol{I}_n \right) \left\langle z^M - 1 \right\rangle \right) \left\langle z^M - 1 \right\rangle^r,$$

that is, by the properties of the ideals $\left\langle z^M - 1 \right\rangle$, $\langle z - 1 \rangle$ and their quotients, cf. [15],

$$\boldsymbol{F}^* \in \boldsymbol{I}_n \left(\left\langle z^M - 1 \right\rangle : \langle z - 1 \rangle \right)^{r+1} + \left(\mathbf{1}_{n,n} - \boldsymbol{I}_n \right) \left(\left\langle z^M - 1 \right\rangle^{r+1} : \langle z - 1 \rangle^r \right). \tag{56}$$

In the same fashion we also obtain

$$\boldsymbol{Q}_1^* \in \left\langle z^M - 1 \right\rangle : \langle z - 1 \rangle, \qquad \boldsymbol{Q}_2^* \in \left\langle z^M - 1 \right\rangle^{r+1}. \tag{57}$$

Since (56) and (57) are precisely (50), this proves the necessity of that condition.

For the converse, assuming that (50) holds true, we take any representations of $\boldsymbol{V}^T \boldsymbol{A}^* \boldsymbol{V}$ with respect to the given ideals and follow the above argument in reverse order to arrive at (35). ■

References

[1] Cavaretta, A. S., Dahmen, W. and Micchelli, C. A. (1991). *Stationary Subdivision.* Amer. Math. Soc. Memoirs of the AMS **93**.

[2] Cox, D., Little, J. and O'Shea, D. (1996) *Ideals, Varieties and Algorithms*. 2. edition. Springer–Verlag. UTM. New York Berlin Heidelberg Tokyo.

[3] Dahmen, W. and Micchelli, C. A. (1990). Stationary subdivision, fractals and wavelets. In *Computation of Curves and Surfaces*, W. Dahmen et al. (eds.), Kluwer Academic Publishers, 3–26.

[4] Dahmen, W. and Micchelli, C. A. (1997). Biorthogonal wavelet expansion. *Constr. Approx.*, **13**, 294–328.

[5] Dyn, N. (1992). Subdivision schemes in Computer Aided Geometric Design. In *Advances in Numerical Analysis – Volume II, Wavelets, Subdivision Algorithms and Radial Basis Functions*, W. Light (ed.), Clarendon Press. Oxford, 36–104.

[6] Han, B. and Jia, R.–Q. (1998). Multivariate refinement equations and convergence of subdivision schemes. *SIAM J. Math. Anal.*, **29**, 1177–1199.

[7] Jetter, K. and Plonka, G. (2001). A survey on L_2–approximation order from shift–invariant spaces. In *Multivariate Approximation and Applications*, N. Dyn, D. Leviatan, D. Levin, and A. Pinkus (eds.), Cambridge University Press. Cambridge, 73–111.

[8] Jia, R.-Q. (1998). Convergence of vector subdivision schemes and construction of multiple biorthogonal wavelets. In *Advances in Wavelets*, K. S. Lau (ed.), 189–216. Springer-Verlag.

[9] Jia, R.-Q., Riemenschneider, S. D. and Zhou, D.-X. (1998). Vector subdivision schemes and multiple wavelets. *Math. Comput.*, **67**, 1533–1563.

[10] Latour, V., Müller, J. and Nickel, W. (1998). Stationary subdivision for general scaling matrices. *Math. Z.*, **227**, 645–661.

[11] Micchelli, C. A. and Prautzsch, H. (1989). Uniform refinement of curves. *Linear Algebra Appl.*, **114**, 841–870.

[12] Micchelli, C. A. and Sauer, T. (1997). Regularity of multiwavelets. *Advances Comput. Math.*, **7**, 455–545.

[13] Micchelli C. A. and Sauer, T. On vector subdivision. *Math. Z.*, **229**, 621–674.

[14] Micchelli, C. A. (1995). *Mathematical Aspects of Geometric Modeling*, SIAM. CBMS-NSF Regional Conference Series in Applied Mathematics **65**.

[15] Möller, H. M. and Sauer, T (2002). Multivariate refinable functions of high approximation order via quotient ideals of Laurent polynomials. *Advances Comput. Math.* To appear.

[16] Plonka, G. (1997). Approximation order provided by refinable function vectors. *Constr. Approx.*, **13**, 221–244.

[17] Sauer, T. (2002) Polynomial interpolation, ideals and approximation order of refinable functions. *Proc. Amer. Math. Soc.*. To appear.

[18] Steeb, W.-H. (1991). *Kronecker Product of Matrices and Applications*. BI–Wiss.–Ver.

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