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# Hermite Interpolation: a Survey of Univariate Computational Methods

**G. Mühlbach**

**Abstract.** Hermite interpolation of functions by univariate generalized polynomials is considered. Our aim is to show that computational techniques well known for polynomial interpolation quite naturally extend to interpolation by generalized polynomials. As an application of the general results interpolation by rational functions with prescribed poles is discussed in some detail. Here polynomial interpolation is contained iff all poles are prescribed at infinity.

**Interpolación de Hermite: una revisión de métodos computacionales en una variable**

**Resumen.** Se considera la interpolación de Hermite de funciones de una variable mediante polinomios generalizados. Se pretende mostrar que técnicas computacionales conocidas para interpolación polinómica se pueden aplicar también a interpolación mediante polinomios generalizados. Como aplicación se estudia con cierto detalle la interpolación mediante funciones racionales con polos prefijados. La interpolación polinómica corresponde al caso particular en que todos los polos prefijados están en el infinito.

## 1. Interpolation by generalized polynomials

### 1.1. Introduction

Generally speaking, the problem of *interpolation* arises if we want to replace a given ”complicated” function $f$ by a simpler one subject to the condition that the latter matches some given data of $f$. In this survey we understand interpolation in its classical sense due to Hermite. The simple function we are looking for, also called *interpolant*, has prescribed values of all its derivatives up to certain orders at certain points which are called *interpolation points* or *nodes*. We assume that their number is finite.

There are three questions to be discussed:

(i) What kind of ”simple functions” can serve as interpolants?

(ii) How to compute an interpolant?

(iii) How to control the interpolation error?

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A first choice for a class of simple functions well suited for interpolants are the algebraic polynomials. The particular interest in approximation by polynomials is easily understood: they are extremely smooth and they can be evaluated using only additions, subtractions and multiplications. Of course, this fact and that there is no need of a division has been really important in the pre–computer time. Today, computational difficulties are no longer a real obstacle. So in many problems it seems to be quite natural to ask for other classes of "simple functions" which can serve as interpolants as well as algebraic polynomials. Depending on the function $f$ to be interpolated and on the information we have about (for instance: $|f|$ is decaying to zero for $|x| \to \infty$, or $f(x)$ has an asymptotic expansion if $x$ approaches a certain point, etc.), there may be other classes of interpolants more natural than algebraic polynomials.

## 1.2. Ordinary and generalized polynomials

Algebraic polynomials of one real or complex variable, with real or complex coefficients, respectively, can be constructed by

(i) multiplication, or by

(ii) integration.

ad (i): If all zeros $x_1, \ldots, x_n$ counting multiplicities of a polynomial $p$ of one variable and its leading coefficient $c$ are known, by the Fundamental Theorem of Algebra it is completely determined: $p(x) = c \cdot (x - x_1) \cdot \ldots \cdot (x - x_n)$.

ad (ii): The polynomials of degree $n - 1$ at most form the solution space of the homogeneous linear differential equation

$$D^n y(x) = 0, \quad D := \frac{d}{dx},$$

and $\frac{1}{n!}\pi_n$ solves the initial value problem

$$D^n y(x) = 1, \quad D^\nu y(0) = 0 \quad (\nu = 0, \ldots, n-1).$$

We use the notation $\pi_j(x) := x^j$ for the monomials and

$$\Pi_n := \text{span}\{\pi_0, \ldots, \pi_n\}$$

for the linear space of polynomials of degree $n$ at most with coefficients in the field $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on the context.

Whereas the factorisation property is particular to the algebraic polynomials, it is the second property which is shared by many more families of functions.

We suppose for the whole paper that $\Omega$ is a real interval of positive length or a non void region in the complex plane. A system $(f_0, \ldots, f_n)$ of continuous functions $f_j : \Omega \to \mathbb{K}$ is referred to as a *Čebyšev–system of order* $n+1$ on $\Omega$, also called a *T–system* (from the German spelling of the name Tschebyscheff) provided every element different from the zero element of the linear space

$$F_n := \text{span}_{\mathbb{K}}\{f_0, \ldots, f_n\} \tag{1}$$

has at most $n$ zeros in $\Omega$. Equivalently, for every choice of distinct points $x_0, \ldots, x_n \in \Omega$ the *generalized Vandermonde matrix*

$$V\begin{pmatrix} f_0 & \cdots & f_n \\ L_0 & \cdots & L_n \end{pmatrix} := (\langle L_i, f_j \rangle)_{i=0,\ldots,n}^{j=0,\ldots,n}$$

where $L_i$ are the evaluation functionals $\langle L_i, f \rangle := f(x_i)$ is nonsingular. Evidently, by the Fundamental Theorem of Algebra $(\pi_0, \ldots, \pi_n)$ is a complex (resp. real) T–system on every region $\Omega \subset \mathbb{C}$ or on every interval $\Omega \subset \mathbb{R}$, respectively.

If $F_n \subset C^n(\Omega;\mathbb{K}), \Omega \subset \mathbb{K}$ and $\mathbb{K} = \mathbb{R}$ or $= \mathbb{C}$, respectively, then $(f_0, \ldots, f_n)$ is referred to as an *extended real or complex Čebyšev–system* or an ET–system of order $n+1$ on $\Omega$ provided every element different from the zero element of $F_n$ has at most $n$ zeros in $\Omega$, counting multiplicities. Equivalently, for every system $(x_0, \ldots, x_n)$ of possibly repeated nodes in $\Omega$ the *generalized confluent Vandermonde matrix*

$$V\begin{pmatrix} f_0 & \ldots & f_n \\ L_0 & \ldots & L_n \end{pmatrix} := (\langle L_i, f_j\rangle)_{i=0,\ldots,n}^{j=0,\ldots,n} \tag{2}$$

is nonsingular where $L_i$ are the Hermite functionals

$$\langle L_i, f\rangle := \left(\frac{d}{dx}\right)^{\mu_i(x_i)} f(x_i) \quad i = 0, \ldots, n \tag{3}$$

with

$$\mu_i(x) := \text{ multiplicity of } x \text{ in } (x_0, \ldots, x_{i-1}). \tag{4}$$

It should be noticed that $\mu_i(x)$ and the functionals $L_i$ essentially depend on the sequence of nodes which is not reflected by the notation. By

$$V\begin{vmatrix} f_0 & \ldots & f_n \\ L_0 & \ldots & L_n \end{vmatrix} := \det V\begin{pmatrix} f_0 & \ldots & f_n \\ L_0 & \ldots & L_n \end{pmatrix} \neq 0 \tag{5}$$

we denote the *generalized confluent Vandermonde determinant*. A T–system (resp. an ET–system) of order $n+1$ $(f_0, \ldots, f_n)$ on $\Omega$ such that for every $k = 0, \ldots, n$ also $(f_0, \ldots, f_k)$ is a T–system (resp. ET–system) of order $k+1$ on $\Omega$ is called a *complete T–system* or a *CT–system* (resp. an *extended complete* T–system or an *ECT–system*).

Again, as a consequence of the Fundamental Theorem of Algebra the best known example of a complex ECT–system is $\Pi_n$. In case of real functions defined on a real interval $\Omega$ every ECT–system $(f_0, \ldots, f_n)$ on $\Omega$ can be constructed from certain real "weight" functions $w_j \in C^{n-j}(\Omega;\mathbb{R}) (j = 0, \ldots, n)$ having no zeros in $\Omega$ such that for a fixed $a \in \Omega$

$$\begin{aligned}
f_0(x) &= w_0(x) \\
f_1(x) &= w_0(x)\int_a^x w_1(t)_1 dt_1 + a_{10}\cdot f_0(x) \\
f_2(x) &= w_0(x)\int_a^x w_1(t_1)\int_a^{t_1} w_2(t_2)dt_2 dt_1 + a_{20}f_0(x) + a_{21}f_1(x) \\
&\ldots \\
f_n(x) &= w_0(x)\int_a^x w_1(t_1)\int_a^{t_1} w_2(t_2)\ldots\int_a^{t_{n-1}} w_n(t_n)dt_n\ldots dt_1 \\
&\quad + \sum_{j=0}^{n-1} a_{nj}\cdot f_j(x)
\end{aligned} \tag{6}$$

where $a_{k,j}$ are certain real coefficients which are uniquely determined by the conditions that for every $k = 0, \ldots, n$

$$\begin{aligned}
v_k(x) &:= f_k(x) - \sum_{j=0}^{k-1} a_{kj}f_j(x) \\
&= w_0(x)\int_a^x w_1(t_1)\int_a^{t_1} w_2(t_2)\ldots\int_a^{t_{k-1}} w_k(t_k)dt_k\ldots dt_1
\end{aligned} \tag{7}$$

solves the initial value problem

$$L^k v_k = 1, \quad L^\mu v_k(a) = 0 \quad \mu = 0, \ldots, k-1 \tag{8}$$

and $F_{k-1} := \text{span}\{f_0, \dots, f_{k-1}\}$ is the solution space of the homogeneous linear differential equation

$$L^k v = 0, \tag{9}$$

where

$$\begin{aligned} L^0 v &:= \frac{v}{w_0}, \\ L^k v &:= \frac{1}{w_k}\frac{d}{dx}\left(\frac{1}{w_{k-1}}\cdots\frac{d}{dx}\left(\frac{1}{w_0}v\right)\cdots\right) \qquad k = 1, \dots, n. \end{aligned} \tag{10}$$

If all weight functions are positive then the system (7) is referred to as a *real ECT-system on* $\Omega$ *in its canonical initial form.* Then all Wronskians of subsystems $(v_0, \dots, v_j)$ for $j = 0, \dots, n$ of (7) are positive in $\Omega$

$$W(v_0, \dots, v_j)(x) := \det\left(D^\nu(v_i(x))\right)_{i=0,\dots,j}^{\nu=0,\dots,j} > 0 \qquad x \in \Omega.$$

Consequently, if $(f_0, \dots, f_n)$ is a real ECT-system on $\Omega$, then there are unique sign factors $\sigma_0, \dots, \sigma_n \in \{-1, 1\}$ such that every subsystem $(\tilde{f}_0, \dots, \tilde{f}_j)$ of

$$(\tilde{f}_0, \dots, \tilde{f}_n) = (\sigma_0 f_0, \dots, \sigma_n f_n)$$

has a positive Wronskian in $\Omega$. Moreover, the weight functions can be computed from the Wronskians

$$\begin{aligned} w_0(x) &= \tilde{f}_0(x), \\ w_1(x) &= \frac{W(\tilde{f}_0, \tilde{f}_1)(x)}{[\tilde{f}_0(x)]^2}, \\ w_k(x) &= \frac{W(\tilde{f}_0, \dots, \tilde{f}_{k-2})(x) W(\tilde{f}_0, \dots, \tilde{f}_k)(x)}{[W(\tilde{f}_0, \dots, \tilde{f}_{k-1})(x)]^2}, \quad k = 2, \dots, n. \end{aligned} \tag{11}$$

It should be noted that for complex ECT–systems nothing similar is known. If $(f_0, \dots, f_n)$ is an ET–system on $\Omega$, then the elements of its span (1) often are called *generalized polynomials of order* $n+1$ at most. Clearly, iff all weight functions $w_j$ are constant, then $F_n = \Pi_n$.

From its definition (10) we see that the differential operator $L^k$ is a linear combination of the operators $\left(\frac{d}{dx}\right)^j$ of ordinary $j$–fold differentiation for $j = 0, \dots, k$, with coefficients depending on the weight functions $w_0, \dots, w_k$. Clearly, the leading coefficient of $L^k$ is $\frac{1}{w_k}$. It has no zeros in $\Omega$.

Conversely, under certain conditions, each Hermite functional (3) can be written as a linear combination of some differential operators (10) evaluated at a node. More precisely, we now assume that the node system

$$(x_0, \dots, x_n) = (\underbrace{\xi_0, \dots, \xi_0}_{\nu_0}, \underbrace{\xi_1, \dots, \xi_1}_{\nu_1}, \dots, \xi_{p-1}, \underbrace{\xi_p, \dots, \xi_p}_{\nu_p}) \tag{12}$$

is *consistently ordered* with $\xi_0, \xi_1, \dots, \xi_p \in \Omega$ pairwise distinct and $\nu_0 + \cdots \nu_p = n+1$. Then we can switch from the one–index notation

$$\langle L_i, v\rangle := \left(\frac{d}{dx}\right)^{\mu_i(x_i)} v(x_i) = \left(\frac{d}{dx}\right)^{\rho} v(\xi_r) =: d_r^\rho v \tag{13}$$

to a two–index–notation where

$$\begin{aligned} \{0, \dots, n\} \ni i &= \varphi(r, \rho) = \nu_0 + \cdots + \nu_{r-1} + \rho \\ &\quad \rho = 0, \dots, \nu_r; r = 0, \dots, p, \end{aligned} \tag{14}$$

is injective, and conversely. Then $L^\rho v(\xi_r)$ according to (10) can be solved for $d_r^\rho v$ yielding

$$\langle L_i, v\rangle = d_r^\rho v = \sum_{\sigma=0}^{\rho} \beta_{\sigma,i} \underbrace{L^\sigma v(\xi_r)}_{=:L_r^\sigma v}, \tag{15}$$

with real coefficients $\beta_{\sigma,i}$ and $\beta_{\rho,i} = w_\rho(\xi_r) \neq 0$.

## 1.3. Unisolvency of ET–systems

Suppose that $(f_0, \ldots, f_n)$ is an ET–system on $\Omega$. Then, by definition, every *problem of Hermite interpolation*

- given nodes $x_0, \ldots, x_n \in \Omega$, possibly repeated
- given a function $f : \Omega \to \mathbb{K}$ which is sufficiently often differentiable at the multiple nodes
- find a generalized polynomial

$$p = \sum_{k=0}^{n} a_k \cdot f_k \in F_n \tag{16}$$

satisfying

$$\left(\frac{d}{dx}\right)^{\mu_i(x_i)} p(x_i) = \left(\frac{d}{dx}\right)^{\mu_i(x_i)} f(x_i) \quad i = 0, \ldots, n \tag{17}$$

where $\mu_i(x)$ is defined by (4), has a unique solution. We denote the solution of (17) by any of the following more and more detailed notations

$$p = pf = p_n f = pf[x_0, \ldots, x_n] = pf \begin{bmatrix} f_0 & \cdots & f_n \\ x_0 & \cdots & x_n \end{bmatrix}, \tag{18}$$

where the last one shows all data $p$ is depending on. Clearly, $p$ is a symmetric function of the nodes, i.e. for any permutation $(x_{\nu_0}, \ldots, x_{\nu_n})$ of $(x_0, \ldots, x_n)$

$$p \begin{bmatrix} f_0 & \cdots & f_n \\ x_0 & \cdots & x_n \end{bmatrix} = p \begin{bmatrix} f_0 & \cdots & f_n \\ x_{\nu_0} & \cdots & x_{\nu_n} \end{bmatrix}.$$

Moreover, $p$ does depend only on the space $F_n := \text{span}\,\{f_0, \ldots, f_n\}$ and not on the basis chosen. Nevertheless, its representation severely depends on the basis, and also the method of computation depends on the ordering of the basic elements. For these purposes the detailed notations are useful.

How to compute $p$?

If $(f_0, \ldots, f_n)$ actually is an ECT–system on $\Omega$, then every formula for interpolation by algebraic polynomials has a counterpart, a similar formula for interpolation by generalized polynomials. This will be shown and proved in the next sections.

## 1.4. Newton's procedure generalized

Suppose we are given an ECT–system $(f_0, \ldots, f_n)$ on $\Omega$. Given a system $(x_0, \ldots, x_n)$ of possibly repeated nodes in $\Omega$, then the interpolation remainders $(\varphi_0, \ldots, \varphi_n)$ where

$$\begin{aligned} \varphi_0 &:= f_0 \\ \varphi_j &:= rf_j \begin{bmatrix} f_0 & \cdots & f_{j-1} \\ x_0 & \cdots & x_{j-1} \end{bmatrix} := f_j - pf_j \begin{bmatrix} f_0 & \cdots & f_{j-1} \\ x_0 & \cdots & x_{j-1} \end{bmatrix} \qquad j = 1, \ldots, n \end{aligned} \tag{19}$$

also constitute a basis of $F_n$. It is called Newton's basis. $\varphi_j$ has zeros $x_0, \ldots, x_{j-1}$, counting multiplicities. It should be noticed that in case of interpolation by algebraic polynomials

$$\varphi_j(x) = \prod_{k=0}^{j-1} (x - x_k), \quad j = 0, \ldots, n. \tag{20}$$

The coefficients $c_j$ of the interpolant $p$ developed in the Newton basis

$$p = p_n f = pf \begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} = pf \begin{bmatrix} \varphi_0 & \dots & \varphi_n \\ x_0 & \dots & x_n \end{bmatrix} = \sum_{j=0}^{n} c_j \varphi_j \tag{21}$$

are called *(generalized) divided differences of* $f$ *with respect to* $(f_0, \dots, f_j)$ *and the nodes* $(x_0, \dots, x_j)$, $j = 0, \dots, n$. They are denoted by

$$c_j = c_j f = \begin{bmatrix} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{bmatrix} f = \begin{bmatrix} \varphi_0 & \dots & \varphi_j \\ x_0 & \dots & x_j \end{bmatrix} f \quad j = 0, \dots, n. \tag{22}$$

Evidently, $c_n = a_n$ in the normal representation (16) of $p$ in the basis $(f_0, \dots, f_n)$, but in general, $c_j \neq a_j$ for $j = 0, \dots, n-1$. Also, $c_n f$ is a symmetric function of its nodes as can be seen from its determinantal representation which results from Cramer's rule

$$c_n f = \begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f = \frac{V\begin{vmatrix} f_0 & \dots & f_{n-1} f \\ L_0 & \dots & L_{n-1} L_n \end{vmatrix}}{V\begin{vmatrix} f_0 & \dots & f_{n-1} f_n \\ L_0 & \dots & L_{n-1} L_n \end{vmatrix}}. \tag{23}$$

From (23), by elementary column operations, we get

$$\begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f = \frac{V\begin{vmatrix} \varphi_0 & \dots & \varphi_{n-1} rf \begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix} \\ L_0 & \dots & L_{n-1} L_n \end{vmatrix}}{V\begin{vmatrix} \varphi_0 & \dots & \varphi_{n-1} \varphi_n \\ L_0 & \dots & L_{n-1} L_n \end{vmatrix}}, \tag{24}$$

where we denote the interpolation remainders for $j = 1, \dots, n+1$ by

$$rf \begin{bmatrix} f_0 & \dots & f_{j-1} \\ x_0 & \dots & x_{j-1} \end{bmatrix} := f - pf \begin{bmatrix} f_0 & \dots & f_{j-1} \\ x_0 & \dots & x_{j-1} \end{bmatrix} = \frac{V\begin{vmatrix} f_0 & \dots & f_{j-1} & f \\ L_0 & \dots & L_{j-1} & L_j \end{vmatrix}}{V\begin{vmatrix} f_0 & \dots & f_{j-1} \\ L_0 & \dots & L_{j-1} \end{vmatrix}} \tag{25}$$

and by $rf[\,] := f$, for $j = 0$. Since the matrices whose determinants occur in the right hand side of (24) are lower triangular, we also have

$$\begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f = \frac{\left(\frac{d}{dx}\right)^{\mu_n(x_n)} rf \begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix} (x_n)}{\left(\frac{d}{dx}\right)^{\mu_n(x_n)} rf_n \begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix} (x_n)}. \tag{26}$$

Moreover, by applying the same argument to the permuted nodes $x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n, x_i$ we likewise get for every $i \in \{0, \dots, n\}$

$$\begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f = \frac{\left(\frac{d}{dx}\right)^{\mu_n(x_i)} rf \begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 \dots x_{i-1} & & x_{i+1} \dots x_n \end{bmatrix} (x_i)}{\left(\frac{d}{dx}\right)^{\mu_n(x_i)} rf_n \begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 \dots x_{i-1} & & x_{i+1} \dots x_n \end{bmatrix} (x_i)}. \tag{27}$$

If we are dealing with a real ECT–system as in (6) and if $(x_0, \dots, x_n)$ after being ordered consistently is identical with (12) we get from (26) and (27) in view of (15)

$$\begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f = \frac{L_r^\rho r f \begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0, \dots, x_{i-1} & & x_{i+1}, \dots, x_n \end{bmatrix}}{L_r^\rho r f_n \begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0, \dots, x_{i-1} & & x_{i+1}, \dots, x_n \end{bmatrix}} \quad \text{if } i = \varphi(r, \rho). \tag{28}$$

For an arbitrary ECT–system $(f_0, \dots, f_n)$ from (21) we have the recurrence

$$\begin{aligned} pf \begin{bmatrix} f_0 \\ x_0 \end{bmatrix} &= c_0 f \cdot \varphi_0 = \frac{f(x_0)}{f_0(x_0)} \cdot f_0 \\ pf \begin{bmatrix} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{bmatrix} &= pf \begin{bmatrix} f_0 & \dots & f_{j-1} \\ x_0 & \dots & x_{j-1} \end{bmatrix} + \begin{bmatrix} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{bmatrix} f \cdot \varphi_j, \quad j = 1, \dots, n. \end{aligned} \tag{29}$$

Therefore, what is said in (23)–(28) for $c_n f$ does hold accordingly for every divided difference $c_j f$, $j = 0, \dots, n$. Newton's generalized recurrence relation (29) constitutes a procedure to compute the interpolant $p_n f$ recursively since also the generalized divided differences (22) can be computed recursively, as we are going to show now. This is well known for interpolation by algebraic polynomials where

$$\begin{bmatrix} \pi_0 & \dots & \pi_j \\ x_0 & \dots & x_j \end{bmatrix} f = \begin{cases} \dfrac{f^{(j)}(x_0)}{j!} & \text{if } x_j = x_{j-1} = \dots = x_0 \\ \dfrac{\begin{bmatrix} \pi_0 & \dots & \pi_{j-1} \\ x_1 & \dots & x_j \end{bmatrix} f - \begin{bmatrix} \pi_0 & \dots & \pi_{j-1} \\ x_0 & \dots & x_{j-1} \end{bmatrix} f}{x_j - x_0} & \text{if } x_j \neq x_0, j = 1, \dots, n. \end{cases} \tag{30}$$

By the symmetry of divided differences as function of their nodes if not all nodes are identical we may and we do assume that $x_j \neq x_0$. The denominator actually is

$$x_j - x_0 = \begin{bmatrix} \pi_0 & \dots & \pi_{j-1} \\ x_1 & \dots & x_j \end{bmatrix} \pi_j - \begin{bmatrix} \pi_0 & \dots & \pi_{j-1} \\ x_0 & \dots & x_{j-1} \end{bmatrix} \pi_j, \tag{31}$$

since in view of (27) the difference on the right hand side equals

$$\begin{aligned} & \frac{\left(\dfrac{d}{dx}\right)^{\mu_{j-1}(x_j)} r\pi_j \begin{bmatrix} \pi_0 & \dots & \pi_{j-2} \\ x_1 & \dots & x_{j-1} \end{bmatrix} (x_j)}{\left(\dfrac{d}{dx}\right)^{\mu_{j-1}(x_j)} r\pi_{j-1} \begin{bmatrix} \pi_0 & \dots & \pi_{j-2} \\ x_1 & \dots & x_{j-1} \end{bmatrix} (x_j)} - \frac{\left(\dfrac{d}{dx}\right)^{\mu_{j-1}(x_0)} r\pi_j \begin{bmatrix} \pi_0 & \dots & \pi_{j-2} \\ x_1 & \dots & x_{j-1} \end{bmatrix} (x_0)}{\left(\dfrac{d}{dx}\right)^{\mu_{j-1}(x_0)} r\pi_{j-1} \begin{bmatrix} \pi_0 & \dots & \pi_{j-2} \\ x_1 & \dots & x_{j-1} \end{bmatrix} (x_0)} \\ = \; & (x_j - \alpha) - (x_0 - \alpha). \end{aligned}$$

This results by applying Leibniz' rule (32) to

$$r\pi_{j-1} \begin{bmatrix} \pi_0 & \dots & \pi_{j-2} \\ x_1 & \dots & x_{j-1} \end{bmatrix} (x) = \prod_{k=1}^{j-1} (x - x_k)$$

and to

$$r\pi_j \begin{bmatrix} \pi_0 & \dots & \pi_{j-2} \\ x_1 & \dots & x_{j-1} \end{bmatrix} (x) = (x - \alpha) \cdot \prod_{k=1}^{j-1} (x - x_k)$$

with a certain $\alpha \in \mathbb{K}$ which both follow from the Fundamental Theorem of Algebra.

Since it is a basic tool here and also later let us recall *Leibniz' rule for derivatives of higher orders*:

$$\left(\frac{d}{dx}\right)^N (u\cdot v)(x) = \sum_{k=0}^{N} \binom{N}{k} \left(\frac{d}{dx}\right)^k u(x) \left(\frac{d}{dx}\right)^{N-k} v(x). \tag{32}$$

There is also a *Leibniz rule for ordinary divided differences*

$$\left[\begin{array}{ccc} \pi_0 & \dots & \pi_n \\ x_0 & \dots & x_n \end{array}\right] (u\cdot v) = \sum_{j=0}^{n} \left[\begin{array}{ccc} \pi_0 & \dots & \pi_j \\ x_0 & \dots & x_j \end{array}\right] u \cdot \left[\begin{array}{ccc} \pi_0 & \dots & \pi_{n-j} \\ x_j & \dots & x_n \end{array}\right] v. \tag{33}$$

Both are easily proved by induction. Inserting (31) into (30) and replacing the system $(\pi_0,\dots,\pi_n)$ by $(f_0,\dots,f_n)$ we formally are lead to

$$\left[\begin{array}{ccc} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{array}\right] f = \begin{cases} \dfrac{\left(\frac{d}{dx}\right)^j rf \left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_0 & \dots & x_0 \end{array}\right](x_0)}{\left(\frac{d}{dx}\right)^j rf_j \left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_0 & \dots & x_0 \end{array}\right](x_0)} & \text{if } x_j = x_{j-1} = \dots = x_0 \\[2ex] \dfrac{\left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_1 & \dots & x_j \end{array}\right] f - \left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_0 & \dots & x_{j-1} \end{array}\right] f}{\left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_1 & \dots & x_j \end{array}\right] f_j - \left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_0 & \dots & x_{j-1} \end{array}\right] f_j} & \text{if } x_j \neq x_0, j = 1,\dots,n. \end{cases} \tag{34}$$

The first line of (34) is clear from (26) and the second line is proved by induction on $j$. The proof starts with

$$\begin{aligned} \left[\begin{array}{cc} f_0 & f_1 \\ x_0 & x_1 \end{array}\right] f &= \frac{V\left|\begin{array}{cc} f_0 & f \\ L_0 & L_1 \end{array}\right|}{V\left|\begin{array}{cc} f_0 & f_1 \\ L_0 & L_1 \end{array}\right|} = \frac{f_0(x_0)f(x_1) - f_0(x_1)f(x_0)}{f_0(x_0)f_1(x_1) - f_0(x_1)f_1(x_0)} \\ &= \frac{\dfrac{f(x_1)}{f_0(x_1)} - \dfrac{f(x_0)}{f_0(x_0)}}{\dfrac{f_1(x_1)}{f_0(x_1)} - \dfrac{f_1(x_0)}{f_0(x_0)}} = \frac{\left[\begin{array}{c} f_0 \\ x_1 \end{array}\right] f - \left[\begin{array}{c} f_0 \\ x_0 \end{array}\right] f}{\left[\begin{array}{c} f_0 \\ x_1 \end{array}\right] f_1 - \left[\begin{array}{c} f_0 \\ x_0 \end{array}\right] f_1} \quad \text{if } x_0 \neq x_1. \end{aligned}$$

Assume now that (34) has been proved for all systems of nodes $x_0,\dots,x_j$ in $\Omega$. Let $j \geq 1$ and $x_0,\dots,x_j,x_{j+1} \in \Omega$ be arbitrary, but $x_{j+1} \neq x_0$. According to (29) we write $p_{j+1}f$ in two different ways with respect to the unions $(x_1,\dots,x_j) \cup (x_{j+1}) \cup (x_0)$ and $(x_1,\dots,x_j) \cup (x_0) \cup (x_{j+1})$:

$$\begin{aligned} pf\left[\begin{array}{ccc} f_0 & \dots & f_{j+1} \\ x_0 & \dots & x_{j+1} \end{array}\right] = pf\left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_1 & \dots & x_j \end{array}\right] &+ \left[\begin{array}{ccc} f_0 & \dots & f_{j-1}f_j \\ x_1 & \dots & x_j, x_{j+1} \end{array}\right] f \cdot rf_j \left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_1 & \dots & x_j \end{array}\right] \\ &+ \left[\begin{array}{ccc} f_0 & \dots & f_{j+1} \\ x_0 & \dots & x_{j+1} \end{array}\right] f \cdot rf_{j+1} \left[\begin{array}{ccc} f_0 & \dots & f_j \\ x_1 & \dots & x_{j+1} \end{array}\right] \\ = pf\left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_1 & \dots & x_j \end{array}\right] &+ \left[\begin{array}{ccc} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{array}\right] f \cdot rf_j \left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_1 & \dots & x_j \end{array}\right] \\ &+ \left[\begin{array}{ccc} f_0, & \dots & f_{j+1} \\ x_0 & \dots & x_{j+1} \end{array}\right] f \cdot rf_{j+1} \left[\begin{array}{ccc} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{array}\right]. \end{aligned}$$

Subtraction yields the equation

$$\begin{aligned} &\left(\left[\begin{array}{ccc} f_0 & \dots & f_j \\ x_1 & \dots & x_{j+1} \end{array}\right] f - \left[\begin{array}{ccc} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{array}\right] f\right) \cdot rf_j \left[\begin{array}{ccc} f_0 & \dots & f_{j-1} \\ x_1 & \dots & x_j \end{array}\right] \\ + &\left[\begin{array}{ccc} f_0 & \dots & f_{j+1} \\ x_0, & \dots & x_{j+1} \end{array}\right] f \cdot \left(pf_{j+1} \left[\begin{array}{ccc} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{array}\right] - pf_{j+1} \left[\begin{array}{ccc} f_0 & \dots & f_j \\ x_1 & \dots & x_{j+1} \end{array}\right]\right) = 0. \end{aligned} \tag{35}$$

Comparing coefficients of $f_j$ gives the second line of (34) with $j$ replaced by $j+1$ provided it can be shown that when $x_{j+1} \neq x_0$ the denominator

$$\begin{bmatrix} f_0 & \dots & f_j \\ x_1 & \dots & x_{j+1} \end{bmatrix} f_{j+1} - \begin{bmatrix} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{bmatrix} f_{j+1} \neq 0 \tag{36}$$

is different from zero. If (36) fails to hold then by (35)

$$\begin{bmatrix} f_0 & \dots & f_j \\ x_1 & \dots & x_{j+1} \end{bmatrix} f - \begin{bmatrix} f_0 & \dots & f_j \\ x_0 & \dots & x_j \end{bmatrix} f = 0$$

for every $f \in C^j(\Omega; \mathbb{K})$ which obviously is false. For instance, take $f = P \in F_{j+1}$ solving the interpolation problem

$$\left(\frac{d}{dx}\right)^{\mu_{i-1}(x_i)} P(x_i) = 0 \quad \text{for } i = 1, \dots, j,$$

$$\left(\frac{d}{dx}\right)^{\mu_j(x_0)} P(x_0) = 0,$$

$\mu_k(x) :=$ multiplicity of $x$ in $(x_1, \dots, x_k)$, and

$$\left(\frac{d}{dx}\right)^{\mu_{j+1}(x_{j+1})} P(x_{j+1}) = 1,$$

with

$$\begin{aligned} \mu_{j+1}(x_{j+1}) &:= \text{multiplicity of } x_{j+1} \text{ in } (x_1, \dots, x_j, x_0), \\ &= \text{multiplicity of } x_{j+1} \text{ in } (x_1, \dots, x_j), \end{aligned}$$

since $x_{j+1} \neq x_0$ by assumption. Then according to (23)

$$\begin{bmatrix} f_0 & \dots & f_j \\ x_1 & \dots & x_{j+1} \end{bmatrix} P - \begin{bmatrix} f_0 & \dots & f_j \\ x_1 & \dots & x_j \end{bmatrix} P = \frac{V\begin{vmatrix} f_0 & \dots & f_{j-1} \\ L_1 & \dots & L_j \end{vmatrix}}{V\begin{vmatrix} f_0 & \dots & f_{j-1} f_j \\ L_1 & \dots & L_{j-1} L_j \end{vmatrix}} = 0,$$

which contradicts (5).

Since the denominators in the second line of (34) themselves can be computed recursively using the same recurrence relation (34) with $j$ replaced by $j-1$ and $f$ by $f_j$, Newton's generalized procedure has complexity $\mathcal{O}(n^3)$. It reduces to $\mathcal{O}(n^2)$ if the denominators of (34) are known explicitly in terms of the nodes as is true for interpolation by algebraic polynomials or by rational functions with prescribed poles.

We close this section by proving a mean value theorem for generalized divided differences with respect to a real ECT–system $(f_0, \dots, f_n)$ on a real interval $\Omega$ as in (6). Then for any function $f \in C^n(\Omega; \mathbb{R})$ in the convex hull $C := \text{con}\{x_0, \dots, x_n\}$ of $x_0, \dots, x_n$ there exists a point $\xi$ such that

$$\begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f = L^n f(\xi). \tag{37}$$

To prove (37) consider $r := rf \begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix}$ which by construction has zeros $x_0, \dots, x_n$ counting multiplicities. By application of Rolle's theorem to $\frac{1}{w_0} r$ one finds that $L^1 r$ has at least $n$ zeros in $C$, counting multiplicities. Applying Rolle's theorem again to $L^1 r$ one sees that $L^2 r$ has at least $n-1$ zeros in $C$, etc. Finally, $L^n r$ must have at least one zero $\xi$ in $C$. On the other hand from (8) and (29)

$$\begin{aligned} O = L^n r(\xi) &= L^n \left( f - pf \begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix} - \begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f \cdot r f_n \begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix} \right)(\xi) \\ &= L^n f(\xi) - \begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f \end{aligned}$$

as claimed.

For interpolation by algebraic polynomials (37) is Cauchy's mean value theorem

$$\begin{bmatrix} \pi_0 & \dots & \pi_n \\ x_0 & \dots & x_n \end{bmatrix} f = \frac{f^{(n)}(\xi)}{n!} \quad \text{for a certain } \xi \in \operatorname{con}\{x_0, \dots, x_n\}. \tag{38}$$

Also, (38) is an immediate consequence of the Hermite–Genocchi formula

$$\begin{bmatrix} \pi_0 & \dots & \pi_n \\ x_0 & \dots & x_n \end{bmatrix} f = \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{tn-1} f^{(n)} \underbrace{(x_0 + (x_1 - x_0)t_1 + \dots + (x_n - x_{n-1})t_n)}_{\in \operatorname{con}\{x_0, \dots, x_n\}} dt_n \tag{39}$$

which is valid also in the complex case. It is easily proved by induction on $n$. From (39) it is clear that ordinary divided differences are continuous functions of their nodes provided the function is sufficiently smooth.

If the real function $f$ is merely continuous and the nodes all are simple then Popoviciu's mean value theorem says that in con $\{x_0, \dots, x_n\}$ there exists a point $\xi$ such that in every neighbourhood of $\xi$ there are equidistant points $y_0, y_0 + h, \dots, y_0 + nh, h \neq 0$, such that

$$\begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f = \begin{bmatrix} f_0, f_1 & \dots & f_n \\ y_0, y_0 + h & \dots & y_0 + nh \end{bmatrix} f. \tag{40}$$

Nothing similar to (37) or (40) is known for complex ECT–systems.

## 1.5. Interpolants are continuous functions of their nodes

In this section we are going to show that also generalized divided differences of a sufficiently smooth function are continuous functions of their nodes: if $(f_0, \dots, f_n)$ is an ET–system on $\Omega \subset \mathbb{K}$ and $f \in C^n(\Omega; \mathbb{K})$, then in the natural topology of $\mathbb{K}^{n+1}$

$$\lim_{\Omega^{n+1} \ni (y_0, \dots, y_n) \to (x_0, \dots, x_n)} \begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f = \begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f. \tag{41}$$

As remarked above, for ordinary divided differences this follows from the Hermite–Genocchi formula. Unfortunately, for generalized divided differences no formula of Hermite–Genocchi type is known. Nevertheless, we can prove (41) by making use of the continuity of ordinary divided differences.

Since divided differences are invariant under permutations of the nodes we may assume that $(x_0, \dots, x_n)$ is consistently ordered as in (12). From (23) we see

$$\begin{aligned} \begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix} f &= \frac{V \begin{vmatrix} f_0 & & \dots & & & \dots & & \dots & f_{n-1} & f \\ d_0^0 & d_0^1 & \dots & d_0^{\nu_0 - 1} & d_1^0 & \dots & d_p^0 & \dots & d_p^{\nu_p - 2} & d_p^{\nu_p - 1} \end{vmatrix}}{V \begin{vmatrix} f_0 & & \dots & & & \dots & & \dots & f_{n-1} & f_n \\ d_0^0 & d_0^1 & \dots & d_0^{\nu_0 - 1} & d_1^0 & \dots & d_p^0 & \dots & d_p^{\nu_p - 2} & d_p^{\nu_p - 1} \end{vmatrix}} \\ &= \frac{V \begin{vmatrix} f_0 & \dots & & & & \dots & f_{n-1} & f \\ [\xi_0] & \dots & [\xi_0 \cdots \xi_0] & [\xi_1] & \dots & [\xi_p] & \dots & [\xi_p, \dots, \xi_p] \end{vmatrix}}{V \begin{vmatrix} f_0 & \dots & & & & \dots & f_{n-1} & f_n \\ [\xi_0] & \dots & [\xi_0 \cdots \xi_0] & [\xi_1] & \dots & [\xi_p] & \dots & [\xi_p, \dots, \xi_p] \end{vmatrix}}. \end{aligned}$$

Since $(y_0, \dots, y_n) \to (\underbrace{\xi_0, \dots, \xi_0}_{\nu_0}, \xi_1, \dots, \xi_{p-1}, \underbrace{\xi_p, \dots, \xi_p}_{\nu_p})$ in the natural topology of $\mathbb{K}^{n+1}$ we must have the limit relations

$$
\begin{aligned}
(y_0, \ldots, y_{\nu_0-1}) &\to (\underbrace{\xi_0, \ldots, \xi_0}_{\nu_0}) \\
(y_{\nu_0}, \ldots, y_{\nu_0+\nu_1-1}) &\to (\underbrace{\xi_1, \ldots, \xi_1}_{\nu_1}) \\
&\vdots \\
(y_{\nu_0+\ldots+\nu_{p-1}}, \ldots, y_n) &\to (\underbrace{\xi_p, \ldots, \xi_p}_{\nu_p}).
\end{aligned}
$$

Grouping the nodes according to their limits by elementary column operations it follows that

$$
\left[\begin{array}{ccc} f_0 & \cdots & f_n \\ y_0 & \cdots & y_n \end{array}\right] f
$$

$$
= \frac{V\left|\begin{array}{cccccccc} f_0 & \cdots & & & \cdots & f_{n-1} & & f \\ [y_0] & \cdots & [y_0, \ldots, y_{\nu_0-1}] & [y_{\nu_0}] & \cdots & [y_{\nu_0+\ldots+\nu_{p-1}}] & \cdots & [y_{\nu_0+\ldots+\nu_{p-1}}, \ldots, y_n] \end{array}\right|}{V\left|\begin{array}{cccccccc} f_0 & \cdots & & & \cdots & f_{n-1} & & f_n \\ [y_0] & \cdots & [y_0, \ldots, y_{\nu_0-1}] & [y_{\nu_0}] & \cdots & [y_{\nu_0+\ldots+\nu_{p-1}}] & \cdots & [y_{\nu_0+\ldots+\nu_{p-1}}, \ldots, y_n] \end{array}\right|}
$$

tends to

$$
\frac{V\left|\begin{array}{cccccccc} f_0 & \cdots & & & & \cdots f_{n-1} & f & \\ [\xi_0] & \cdots & [\xi_0, \ldots, \xi_0] & [\xi_1] & \cdots & [\xi_p] & \cdots & [\xi_p, \ldots, \xi_p] \end{array}\right|}{V\left|\begin{array}{cccccccc} f_0 & \cdots & & & & \cdots f_{n-1} & f_n & \\ [\xi_0] & \cdots & [\xi_0, \ldots, \xi_0] & [\xi_1] & \cdots & [\xi_p] & \cdots & [\xi_p, \ldots, \xi_p] \end{array}\right|} = \left[\begin{array}{ccc} f_0 & \cdots & f_n \\ x_0 & \cdots & x_n \end{array}\right] f.
$$

Here we use the facts that ordinary divided differences are continuous functions of their nodes, that determinants are continuous functions of their entries, and that all denominator determinants are nonzero since $(f_0, \ldots, f_n)$ is an ET–system on $\Omega$.

As a consequence of (41), (19) and (21) we have: if $(f_0, \ldots, f_n)$ is an ECT–system on $\Omega$ and if $f \in C^n(\Omega; \mathbb{K})$, then the interpolant $p_n f$ is a continuous function of its nodes:

$$
\lim_{\Omega^{n+1} \ni (y_0, \ldots, y_n) \to (x_0, \ldots, x_n)} pf \left[\begin{array}{ccc} f_0 & \cdots & f_n \\ y_0 & \cdots & y_n \end{array}\right] = pf \left[\begin{array}{ccc} f_0 & \cdots & f_n \\ x_0 & \cdots & x_n \end{array}\right]
$$

pointwise in $\Omega$ and uniformly on every compact subset of $\Omega$.

## 1.6. The interpolation error

As a function of $x \in \Omega$ the interpolation error is

$$
rf \left[\begin{array}{ccc} f_0 & \cdots & f_n \\ x_0 & \cdots & x_n \end{array}\right](x) := f(x) - pf \left[\begin{array}{ccc} f_0 & \cdots & f_n \\ x_0 & \cdots & x_n \end{array}\right](x) = \left[\begin{array}{ccc} f_0 & \cdots & f_n, f_{n+1} \\ x_0 & \cdots & x_n, x \end{array}\right] f \cdot \varphi_{n+1}(x) \tag{42}
$$

with $\varphi_{n+1}(x)$ defined as in (19), if $(f_0, \ldots, f_n, f_{n+1})$ is an ECT–system of order $n+1$ on $\Omega$. This is immediate for $x \in \Omega, x \neq \{x_0, \ldots, x_n\}$ from (29) if we take $j = n+1$ and $x_{n+1} = x$ as a new node. Of course, (42) trivially holds true if $x$ is a node.

As a corollary to (42) by applying (37) with $n$ replaced by $n+1$ if $(f_0, \ldots, f_{n+1})$ is a real ECT–system in canonical form with associated differential operators $L^j$

$$
\left| rf \left[\begin{array}{ccc} f_0 & \cdots & f_n \\ x_0 & \cdots & x_n \end{array}\right](x) \right| \leq \max_{\xi \in \operatorname{con}\{x_0, \ldots, x_n, x\}} \left| L^{n+1} f(\xi) \right| \cdot \left| rf_{n+1} \left[\begin{array}{ccc} f_0 & \cdots & f_n \\ x_0 & \cdots & x_n \end{array}\right](x) \right|, \quad x \in \Omega. \tag{43}
$$

Nothing similar is known for the case of interpolation by a complex ECT–system.

## 1.7. The generalized confluent Vandermonde determinant

The generalized confluent Vandermonde determinant has been considered already in (5). By elementary column operations it follows that

$$V\begin{vmatrix} f_0 & \dots & f_n \\ L_0 & \dots & L_n \end{vmatrix} = V\begin{vmatrix} \varphi_0 & \dots & \varphi_n \\ L_0 & \dots & L_n \end{vmatrix} \tag{44}$$

where $\varphi_0, \dots, \varphi_n$ is the Newton basis (19) of $F_n :=$ span $\{f_0, \dots, f_n\}$. Now the matrix

$$V\begin{pmatrix} \varphi_0 & \dots & \varphi_n \\ L_0 & \dots & L_n \end{pmatrix}$$

is lower triangular. Its diagonal entries are

$$\langle L_i, \varphi_i \rangle = \left(\frac{d}{dx}\right)^{\mu_i(x_i)} \varphi_i(x_i) \neq 0. \tag{45}$$

Hence

$$V\begin{vmatrix} f_0 & \dots & f_n \\ L_0 & \dots & L_n \end{vmatrix} = \prod_{i=0}^{n} \left(\frac{d}{dx}\right)^{\mu_i(x_i)} rf_i \begin{bmatrix} f_0 & \dots & f_{i-1} \\ x_0 & \dots & x_{i-1} \end{bmatrix}(x_i). \tag{46}$$

## 1.8. The general Neville–Aitken formula

It is easy to see that the well known Neville–Aitken formula for interpolation by algebraic polynomials can be written in the form

$$\begin{aligned} pf\begin{bmatrix} \pi_0 & \dots & \pi_n \\ x_0 & \dots & x_n \end{bmatrix}(x) = {} & \frac{r\pi_n\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix}(x) \cdot pf\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_1 & \dots & x_n \end{bmatrix}(x)}{r\pi_n\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix}(x) - r\pi_n\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_1 & \dots & x_n \end{bmatrix}(x)} \\ & - \frac{r\pi_n\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_1 & \dots & x_n \end{bmatrix}(x) \cdot pf\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix}(x)}{r\pi_n\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix}(x) - r\pi_n\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_1 & \dots & x_n \end{bmatrix}(x)} \end{aligned} \tag{47}$$

for $x \in \Omega\backslash\{x_0, \dots, x_n\}$ if $x_n \neq x_0$. Indeed, the right hand side of (47) results from the usual Neville–Aitken formula

$$\begin{aligned} pf\begin{bmatrix} \pi_0 & \dots & \pi_n \\ x_0 & \dots & x_n \end{bmatrix}(x) = {} & \frac{(x - x_0) \cdot pf\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_1 & \dots & x_n \end{bmatrix}(x)}{x_n - x_0} \\ & - \frac{(x - x_n) \cdot pf\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix}(x)}{x_n - x_0} \end{aligned} \tag{48}$$

for $x_n \neq x_0$ by multiplying its numerator and denominator by $(x - x_1) \dots (x - x_{n-1})$, because of

$$\begin{aligned} & r\pi_n\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix}(x) - r\pi_n\begin{bmatrix} \pi_0 & \dots & \pi_{n-1} \\ x_1 & \dots & x_n \end{bmatrix}(x) \\ & = (x - x_1) \dots (x - x_{n-1}) \cdot [(x - x_0) - (x - x_n)] \\ & = (x - x_1) \dots (x - x_{n-1}) \cdot (x_n - x_0). \end{aligned}$$

If in (48) formally the monomials $(\pi_0, \ldots, \pi_n)$ are replaced by an ECT–system $(f_0, \ldots, f_n)$ the following recurrence formula for $x \in \Omega \backslash \{x_0, \ldots, x_n\}$ and $x_n \neq x_0$ obtains:

$$pf \left[ \begin{array}{ccc} f_0 & \ldots & f_n \\ x_0 & \ldots & x_n \end{array} \right](x) = \frac{rf_n \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_0 & \ldots & x_{n-1} \end{array} \right](x) \cdot pf \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_1 & \ldots & x_n \end{array} \right](x)}{rf_n \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_0 & \ldots & x_{n-1} \end{array} \right](x) - rf_n \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_1 & \ldots & x_n \end{array} \right](x)} - \frac{rf_n \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_1 & \ldots & x_n \end{array} \right](x) pf \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_0 & \ldots & x_{n-1} \end{array} \right](x)}{rf_n \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_0 & \ldots & x_{n-1} \end{array} \right](x) - rf_n \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_1 & \ldots & x_n \end{array} \right](x)} \tag{49}$$

Of course, we have to prove (49). In contrast, if all nodes are identical, $x_n = x_{n-1} = \ldots = x_0$, then from (29) and the first line of (34) directly

$$pf \left[ \begin{array}{ccc} f_0 & \ldots & f_j \\ x_0 & \ldots & x_0 \end{array} \right] = \sum_{j=0}^{n} \left[ \begin{array}{ccc} f_0 & \ldots & f_j \\ x_0 & \ldots & x_0 \end{array} \right] f \cdot rf_j \left[ \begin{array}{ccc} f_0 & \ldots & f_{j-1} \\ x_0 & \ldots & x_0 \end{array} \right] \tag{50}$$

results.

Before showing (49) let us look for a simple proof of (48). Denote by $q(x)$ its right hand side. It suffices to show that if $x_n \neq x_0$ then

$$Q(x) := pf \left[ \begin{array}{ccc} \pi_0 & \ldots & \pi_n \\ x_0 & \ldots & x_n \end{array} \right](x) - q(x) \tag{51}$$

is a polynomial of degree $n$ at most having zeros $x_0, \ldots, x_n$, counting multiplicities. Obviously, $Q \in \Pi_n$ has zeros $x_1, \ldots, x_{n-1}$ counting multiplicities. Moreover by Leibniz' rule (32) it is easily shown $\left(\frac{d}{dx}\right)^{\mu_n(x_0)} Q(x_0) = 0$ and $\left(\frac{d}{dx}\right)^{\mu_n(x_n)} Q(x_n) = 0$. This proves (48). There is no easy way to use this idea in proving also the general result (49) for several reasons. Notice first, that the right hand side of (49) which again we call $q(x)$, when considered as a function of $x$, seems to be a rather complicated generalized rational function of $x$. It is well defined for $x \in \Omega \backslash \{x_0, \ldots, x_n\}$ since then its denominator certainly is different from zero. Indeed, the denominator equals

$$pf_n \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_1 & \ldots & x_n \end{array} \right](x) - pf_n \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_0 & \ldots & x_{n-1} \end{array} \right](x),$$

hence it is a linear combination of $f_0, \ldots, f_{n-1}$ that has $n-1$ zeros $x_1, \ldots, x_{n-1}$, counting multiplicities. Moreover, due to the assumption $x_n \neq x_0$ it is nontrivial, since the coefficient of $f_{n-1}$ is

$$\left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_1 & \ldots & x_n \end{array} \right] f_n - \left[ \begin{array}{ccc} f_0 & \ldots & f_{n-1} \\ x_0 & \ldots & x_{n-1} \end{array} \right] f_n \neq 0$$

as was shown in (36). Consequently, it has no other zeros. It is by no means obvious that $q \in F_n$ on $\Omega \backslash \{x_0, \ldots, x_n\}$. At least in general, it will not be possible to "factor out" the common zeros of denominator and numerator of (49). We are coming to see that this is possible in case of interpolation by algebraic polynomials; and we are going to see in the next chapter that this is likewise possible in case of interpolation by rational functions with prescribed poles. On the other side, application of Leibniz' rule to $q$ no longer gives a simple result. Nevertheless, we claim that $q(x) = pf(x)$ for every $x \in \Omega \backslash \{x_0, \ldots, x_n\}$ . Then, we can extend $q$ and its derivatives into the nodes continuously defining

$$\left(\frac{d}{dx}\right)^{\mu_i(x_i)} q(x_i) := \left(\frac{d}{dx}\right)^{\mu_i(x_i)} f(x_i) \quad i = 0, \ldots, n, \tag{52}$$

hence $q = pf$.

Proof of (49): Observe, that the right hand side of (49) is a generalized arithmetic mean

$$qf\begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix}(x) = \lambda(x)\cdot pf\begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_1 & \dots & x_n \end{bmatrix}(x) + (1-\lambda(x))pf\begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix}(x),$$

with weights adding to one, where

$$\lambda(x) = \frac{rf_n\begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix}(x)}{rf_n\begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_0 & \dots & x_{n-1} \end{bmatrix}(x) - rf_n\begin{bmatrix} f_0 & \dots & f_{n-1} \\ x_1 & \dots & x_n \end{bmatrix}(x)}, \tag{53}$$

depending on $x$ and on the nodes $x_0 \dots x_n$ is not necessarily nonnegative. For $x \in \Omega\backslash\{x_0,\dots,x_n\}$ arbitrary but fixed, consider the mappings

$$\begin{aligned} f \longmapsto S_x f &:= qf\begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix}(x) \\ f \longmapsto T_x f &:= pf\begin{bmatrix} f_0 & \dots & f_n \\ x_0 & \dots & x_n \end{bmatrix}(x). \end{aligned}$$

Obviously, as functions of $f$ both mappings are linear functionals that are linear combinations of the functionals $L_i (i = 0,\dots,n)$;

$$\begin{aligned} S_x f &= \sum_{i=0}^{n} A_i \left(\frac{d}{dx}\right)^{\mu_i(x_i)} f(x_i) \\ T_x f &= \sum_{i=0}^{n} B_i \left(\frac{d}{dx}\right)^{\mu_i(x_i)} f(x_i), \end{aligned}$$

where $\mu_i(x_i) \le n-1$ for all $i$ due to the assumption $x_n \ne x_0$, with coefficients $A_0,\dots,A_n$ (resp. $B_0,\dots,B_n$) depending on $x$ but independent of $f$. $S_x$ and $T_x$ agree on $F_n$, since for $f = f_j$ ($j = 0,\dots,n-1$)

$$T_x f_j = f_j(x) = S_x f_j$$

because the weights add to one. But $S_x$ and $T_x$ also agree for $f = f_n$ as is easily checked using the explicit representation (53) of the weights. Since $(f_0,\dots,f_n)$ is an ET–system on $\Omega$, necessarily $A_i = B_i$ for $i = 0,\dots,n$, and therefore $S_x f = T_x f$ for all $f \in C^n(\Omega;\mathbb{K})$. This proves $pf(x) = q(x)$ for all $f \in C^n(\Omega;\mathbb{K})$ and all $x \in \Omega\backslash\{x_0,\dots,x_n\}$ as claimed.

It should be noticed that both terms of the denominator of (49) can be computed recursively again by (49) with $x_0,\dots,x_n$ replaced by $x_0,\dots,x_{n-1}$ or $x_1,\dots,x_n$, respectively, and by replacing $f$ by $f_n$. The complexity of the whole procedure is $\mathcal{O}(n^3)$. It reduces to $\mathcal{O}(n^2)$ if the denominators of the right hand side in (49) are known explicitly as in the cases of interpolation by algebraic polynomials or by rational functions with prescribed poles.

## 1.9. The general Lagrange–Hermite formula and the inverse of the generalized confluent Vandermonde matrix

Given an ET–system $(f_0,\dots,f_n)$ on $\Omega$ and a system of nodes $(x_0,\dots,x_n)$ in $\Omega$, possibly repeated, with associated Hermite functionals (3), then the corresponding generalized Lagrange–Hermite *basic functions* $l_j \in F_n$ ($j = 0,\dots,n$) are uniquely determined by the biorthogonality conditions

$$\langle L_i, l_j\rangle = \delta_{i,j} \quad i,j = 0,\dots,n. \tag{54}$$

Accordingly, the basis $(l_0, \ldots, l_n)$ is defined to be the dual basis in $F_n^{**}$ of the basis $(L_0, \ldots, L_n)$ of $F_n^*$. Here, as usual, we denote by $E^*$ the dual space of a finite dimensional linear space $E$, and we use that $E^{**}$ is canonically isomorphic to $E$. Formally,

$$l_i = \frac{1}{\det V} \begin{vmatrix} \langle L_0, f_0 \rangle & \ldots & \langle L_0, f_n \rangle \\ \vdots & & \vdots \\ \langle L_{i-1}, f_0 \rangle & \ldots & \langle L_{i-1}, f_n \rangle \\ f_0 & \ldots & f_n \\ \langle L_{i+1}, f_0 \rangle & \ldots & \langle L_{i+1}, f_n \rangle \\ \vdots & & \vdots \\ \langle L_n, f_0 \rangle & \ldots & \langle L_n, f_n \rangle \end{vmatrix} \tag{55}$$

where $V$ is the generalized confluent Vandermonde matrix (2). The numerator determinant is defined by its formal Laplacian development along the row labelled $i$ $(i = 0, \ldots, n)$.

For later use let us introduce a more detailed notation for the basic Lagrange–Hermite functions. If $(x_j)_{j=0}^n$ is a sequence of possibly repeated nodes in $\Omega$ we associate with it the sequence of Hermite functionals (3). For $0 \le k \le n$ and $0 \le m \le n-k$ we define

$$l_i^{m,k}(x) := \begin{cases} \dfrac{V\left| \begin{matrix} f_0 & \ldots & & f_i & & \ldots & f_k \\ L_m & \ldots & L_{m+i-1} & L & L_{m+i+1} & \ldots & L_{m+k} \end{matrix} \right|}{V\left| \begin{matrix} f_0 & \ldots & & f_i & & \ldots & f_k \\ L_m & \ldots & L_{m+i-1} & L_{m+i} & L_{m+i+1} & \ldots & L_{m+k} \end{matrix} \right|}, & i = 0, \ldots, k \\ 0, & \text{for all other } i \end{cases} \tag{56}$$

if the denominator is different from zero. $L$ is the evaluation functional at the point $x$

$$f \longmapsto \langle L, f \rangle := f(x)$$

where $x \notin \{x_0, \ldots, x_n\}$ is assumed.

Evidently, the linear functionals $(L_{m+j})_{j=0,\ldots,k}$ and the functions $\left(l_i^{m,k}\right)_{i=0,\ldots,k}$ are biorthogonal:

$$\langle L_{m+j}, l_i^{m,k} \rangle = \delta_{i,j} \quad i, j = 0, \ldots, k. \tag{57}$$

Accordingly,

$$p = pf \begin{bmatrix} f_0 & \ldots & f_k \\ L_m & \ldots & L_{m+k} \end{bmatrix} := \sum_{i=0}^{k} \langle L_{m+i}, f \rangle l_i^{m,k} \tag{58}$$

solves the interpolation problem

$$\langle L_{m+i}, p \rangle = \langle L_{m+i}, f \rangle \quad i = 0, \ldots, k. \tag{59}$$

An immediate consequence of the biorthogonality relations (57) is the Lagrange–Hermite interpolation formula

$$pf \begin{bmatrix} f_0 & \ldots & f_n \\ x_0 & \ldots & x_n \end{bmatrix} = \sum_{j=0}^{n} \langle L_j, f \rangle \cdot l_j. \tag{60}$$

Obviously, knowing the coefficients $d_{j,k}$ of the expansion

$$l_j = \sum_{k=0}^{n} d_{jk} f_k \quad j = 0, \ldots, n \tag{61}$$

it is easy to get an explicit representation of the adjoint of $V$. In fact, by its definition the adjoint of $V$ is

$$V_{adj} = \det V \cdot D^T, \quad D = (d_{jk}). \tag{62}$$

As a consequence,

$$V^{-1} = D^T. \tag{63}$$

Remark: If another basis $u_0, \ldots, u_n$ of $F_n$ is used instead of $f_0, \ldots, f_n$, then the inverse of the corresponding generalized Vandermonde matrix

$$V = V \begin{pmatrix} u_0 & \ldots & u_n \\ L_0 & \ldots & L_n \end{pmatrix} = (\langle L_i, u_j \rangle)$$

in the same way is obtained from explicit representations of the basic Lagrange–Hermite functions

$$l_j = \sum_{k=0}^{n} e_{jk} u_k \quad j = 0, \ldots, n \tag{64}$$

according to

$$V^{-1} = E^T, \quad E = (e_{jk}). \tag{65}$$

## 1.10. A recurrence relation for the generalized basic Lagrange functions

It is not so well known that for simple nodes also the generalized basic Lagrange functions (55) can be computed recursively.

Let $(f_0, \ldots, f_n)$ be a CT–system on $\Omega$ and let $(x_0, \ldots, x_n)$ be a system of simple nodes in $\Omega$ with associated functionals $\langle L_i, f \rangle := f(x_i)$. Then the basic Lagrange functions

$$l_i^{0,n-1}(x) = \frac{V \begin{vmatrix} f_0 & \ldots & & & & \ldots & f_{n-1} \\ L_0 & \ldots & L_{i-1} & L & L_{i+1}, & \ldots, & L_{n-1} \end{vmatrix}}{V \begin{vmatrix} f_0 & \ldots & & & & \ldots & f_{n-1} \\ L_0, & \ldots, & L_{i-1} & L_i & L_{i+1} & \ldots & L_{n-1} \end{vmatrix}} \quad i = 0, \ldots, n-1$$

as well as

$$l_i^{1,n-1}(x) = \frac{V \begin{vmatrix} f_0 & \ldots & & & & \ldots & f_{n-1} \\ L_1 & \ldots & L_i & L & L_{i+2} & \ldots & L_n \end{vmatrix}}{V \begin{vmatrix} f_0 & \ldots & & & & \ldots & f_{n-1} \\ L_1 & \ldots & L_i & L_{i+1} & L_{i+2} & \ldots & L_n \end{vmatrix}} \quad i = 0, \ldots, n-1$$

are well defined, i.e. their denominators are nonzero. For every $x \in \Omega \backslash \{x_0, \ldots, x_n\}$

$$l_i^{0,n}(x) = \lambda(x) l_{i-1}^{1,n-1}(x) + (1 - \lambda(x)) l_i^{0,n-1}(x) \quad i = 0, \ldots, n, \tag{66}$$

where

$$l_{-1}^{1,n-1}(x) := 0, \quad l_n^{0,n-1}(x) := 0 \tag{67}$$

and where $\lambda(x)$ is defined by (53). The recursion (66) starts with $n = 1$ and $l_0^{i,0}(x) = \frac{f_0(x)}{f_0(x_i)}, i = 0, \ldots, n.$

Proof of (66): Consider the recurrence relation (49) for the simple nodes $x_0, \ldots, x_n \in \Omega$. We insert for the interpolants on both sides the generalized Lagrange interpolation formula, thus deriving

$$\begin{aligned} \sum_{i=0}^{n} \langle L_i, f \rangle l_i^{0,n}(x) &= \lambda(x) \sum_{i=0}^{n-1} \langle L_{1+i}, f \rangle l_i^{1,n-1}(x) + (1 - \lambda(x)) \cdot \sum_{i=0}^{n-1} \langle L_i, f \rangle \cdot l_i^{0,n-1}(x) \\ &= \sum_{i=0}^{n} \langle L_i, f \rangle \left( \lambda(x) l_{i-1}^{1,n-1}(x) + (1 - \lambda(x)) l_i^{0,n-1}(x) \right), \end{aligned}$$

due to the definition (67). Putting in here $f = l_k^{0,n}$ $(k = 0, \ldots, n)$, from the orthogonality relations

$$\langle L_i, l_k^{0,n} \rangle = \delta_{i,k} \quad i, k = 0, \ldots, n$$

we obtain that

$$l_k^{0,n}(x) = \lambda(x) l_{k-1}^{1,n-1}(x) + (1 - \lambda(x)) l_k^{0,n-1}(x) \quad k = 0, \ldots, n$$

as claimed.

It should be noted that the recursion (66) does not extend by continuity to multiple nodes, for the basic Lagrange functions $l_i^{0,n}$ are not continuous functions of the nodes, at least in general. This is easily seen from the Lagrange basic functions for polynomial interpolation.

### 1.11. Notes and Remarks

The various classes of Čebyšev-systems have been studied extensively in the monograph of S. Karlin and W.J. Studden [13]. The formulas (11) for the weight functions there are given on p. 380. An earlier treatment can be found in papers of T. Popoviciu, for instance in [25]. Newton's generalized interpolation formula (21) for pairwise distinct nodes is due to H.E. Salzer [28]. However, there is an earlier approach by the French astronomer and mathematician H. M. Andoyer dating back to 1906, cf. [6]. The remainder representation (25) and generalized divided differences were first discussed by T. Popoviciu [25]. The recurrence relation (34) for generalized divided differences with respect to ECT-systems and multiple nodes was given by G. Mühlbach, for real functions in [14] and in [15], for complex functions in [22]. A proof of (34) for simple nodes using Sylvester's determinantal identity is due to Cl. Brezinski [4], another one via Gaussian elimination can be found in [11].

The recurrence relation (34) can be generalized considerably to the situation where the Čebyšev-system has a subsystem that again is a Čebyšev-system [18]. This covers also trigonometric interpolation, a topic not treated in this paper.

The representation (27) of a generalized divided difference as a quotient of remainder terms was given in a more general context in [19] and later in [20]. The formula (28) seems to be new. The mean value formula (38) can be found in [15]. The classical Hermite-Genocchi formula is proved in every comprehensive textbook on Numerical Analysis, for instance in [12].

The material of section 1.5 is from [22]. The general Neville-Aitken formula is due to G. Mühlbach [16], see also [17]. Particular cases used for extrapolation were used earlier [29] and later [3], [10], see also [31]. Actually, Brezinski's E-algorithm which is a rather general extrapolation method can be derived directly from the general Neville-Aitken algorithm [5]. The recurrence relation for the general basic Lagrange functions can be found in [19], see also [20].

## 2. Interpolation by rational functions with prescribed poles

### 2.1. Cauchy–Vandermonde systems

Let $\mathcal{B} = (b_0, b_1 \ldots)$ be a sequence of not necessarily distinct points of the extended complex plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. They will be the prescribed poles. With $\mathcal{B}$ we associate a sequence $\mathcal{U} = (u_0, u_1 \ldots)$ of basic rational functions defined by

$$u_j(x) = \begin{cases} x^{\nu_j(b_j)} & \text{if } b_j = \infty \\ \dfrac{1}{(x - b_j)^{\nu_j(b_j)+1}} & \text{if } b_j \in \mathbb{C}, \quad j = 0, 1, \ldots \end{cases} \tag{68}$$

Here denotes

$$\nu_j(b) := \text{multiplicity of } b \text{ in the sequence } (b_0, \ldots, b_{j-1}). \tag{69}$$

The system $\mathcal{U}$ is called the *Cauchy–Vandermonde system (CV–system*, for short) associated with the pole sequence $\mathcal{B}$. For any fixed nonnegative integer $n$ with the initial section of $\mathcal{B}$

$$\mathcal{B}_n := (b_0, \dots, b_n) \tag{70}$$

there is associated the basis

$$\mathcal{U}_n := (u_0, \dots, u_n) \tag{71}$$

of the *Cauchy–Vandermonde space* (CV–space)

$$U_n := \text{span}\, \mathcal{U}_n. \tag{72}$$

Clearly, iff all poles $b_j$ are at infinity, then $U_n = \Pi_n$ for every $n$ . As an application of the general theory developed in chapter 1 we are going to show that CV–systems are complex ECT–systems hence well suited for Hermite interpolation. The following notation will be useful. For any $z \in \overline{\mathbb{C}}$ we denote by

$$z^* := \begin{cases} 1 & \text{if } z = \infty \text{ or } z = 0 \\ z & \text{if } z \in \mathbb{C}, z \neq 0. \end{cases} \tag{73}$$

If $z_0, \dots, z_n \in \overline{\mathbb{C}}$ then

$$\prod_{j=0}^{n}{}^* z_j := \prod_{j=0}^{n} z_j^*. \tag{74}$$

With the pole sequence (70) we associate the polynomial

$$\beta_n(x) := \prod_{j=0}^{n}{}^*(x - b_j), \quad x \in \mathbb{C} \tag{75}$$

and call it the *pole polynomial* associated with $\mathcal{B}_n$.

When $X = (x_0, x_1, \dots)$ is any sequence of not necessarily distinct points in the complex plane which will serve as nodes, with the section

$$X_n := (x_0, x_1, \dots, x_n) \tag{76}$$

we associate the polynomial

$$\alpha_n := \prod_{j=0}^{n}(x - x_j) \tag{77}$$

and call it the *node polynomial* associated with $X_n$.

## 2.2. Unisolvency of CV–systems

In order to prove that CV–systems are complex ECT–systems we reduce the interpolation problem with CV–systems to polynomial interpolation.

Consider the problem of Hermite interpolation with a CV–system:

- given a CV–system $\mathcal{U}_n = (u_0, \dots, u_n)$ with associated pole sequence $\mathcal{B}_n = (b_0, \dots, b_n)$,
- given a system $X_n = (x_0, \dots, x_n)$ of possibly repeated nodes $x_i \in \mathbb{C}\backslash\{b_0, \dots, b_n\}$,
- given a complex function $f$ which is defined and sufficiently often differentiable at the multiple nodes
- find a rational function $p$ in the CV–space $U_n$ satisfying

$$\left(\frac{d}{dx}\right)^{\mu_i(x_i)} p(x_i) = \left(\frac{d}{dx}\right)^{\mu_i(x_i)} f(x_i) =: \langle L_i, f\rangle \quad i = 0, \dots, n \tag{78}$$

where $\mu_i(x)$ is defined by (4).

We are going to show that this problem always has a unique solution

$$p = pf = p_n f = pf \begin{bmatrix} u_0 & \dots & u_n \\ x_0 & \dots & x_n \end{bmatrix}. \tag{79}$$

Therefore, every CV–system $\mathcal{U}_n$ is an ECT–system on every subset of $\mathbb{C}$ not containing a pole of the associated pole sequence $\mathcal{B}_n$.

In fact, there is a unique polynomial $q \in \Pi_n$ that interpolates $\beta_n \cdot f$ on $X_n$ in the sense of Hermite. Then a partial fraction decomposition yields $p := \frac{q}{\beta_n} \in U_n$. Moreover, $p$ interpolates $f$ on $X_n$, since

$$\left(\frac{d}{dx}\right)^{\mu_i(x_i)} (f-p)(x_i) = 0 \quad i = 0, \dots, n \tag{80}$$

is equivalent with

$$\left(\frac{d}{dx}\right)^{\mu_i(x_i)} (\beta_n \cdot (f-p))(x_i) = 0 \quad i = 0, \dots, n. \tag{81}$$

This is a consequence of Leibniz' rule because $\alpha_n$ and $\beta_n$ are prime. Thus, the *confluent Cauchy–Vandermonde* matrix

$$V := V \begin{pmatrix} u_0 & \dots & u_n \\ L_0 & \dots & L_n \end{pmatrix} := (\langle V_i, u_j \rangle)_{i=0,\dots,n}^{j=0,\dots,n} := \left( \left(\frac{d}{dx}\right)^{\mu_i(x_i)} u_j(x_i) \right) \tag{82}$$

is nonsingular. Moreover, the interpolation remainder is

$$f(x) - pf \begin{bmatrix} u_0 & \dots & u_n \\ x_o & \dots & x_n \end{bmatrix}(x) = \frac{\alpha_n(x)}{\beta_n(x)} \cdot \begin{bmatrix} \pi_0 & \dots & \pi_n, \pi_{n+1} \\ x_0 & \dots & x_n, x \end{bmatrix} (\beta_n \cdot f), \tag{83}$$

which results if (42) is applied to $(f_0, \dots, f_{n+1}) = (\pi_0, \dots, \pi_{n+1})$ and the function $\beta_n \cdot f$.

If it is assumed that all poles are real or at infinity every CV–system $\mathcal{U}_n$ associated with them naturally is also a real ECT–system. Then we may ask: What is its canonical form (6)? Which are the weights?

It is not a deep result to find the answers. It is more a technical exercise, for it is known from (11) that the weights can be expressed explicitely as quotients of Wronskian determinants of subsystems of $\mathcal{U}_n$. Once the confluent Cauchy–Vandermonde determinant

$$V \begin{vmatrix} u_0 & \dots & u_n \\ L_0 & \dots & L_n \end{vmatrix} := \det V \begin{pmatrix} u_0 & \dots & u_n \\ L_0 & \dots & L_n \end{pmatrix} \tag{84}$$

is known explicitly as a function of the poles and the nodes, the weights can be calculated. This will be done in section 2.5 when (84) will be available.

## 2.3. Newton's procedure and the interpolation error

Let $\mathcal{U}_n = (u_0, \dots, u_n)$ be the CV–system with associated pole sequence $\mathcal{B}_n = (b_0, \dots, b_n)$. If $X_n = (x_0, \dots, x_n)$ is any node system in $\mathbb{C} \backslash \{b_0, \dots, b_n\}$, the interpolation remainders

$$\begin{aligned} \varphi_0 &= u_0 \qquad \text{and for } j = 1, \dots, n \\ \varphi_j &= ru_j \begin{bmatrix} u_0 & \dots & u_{j-1} \\ x_0 & \dots & x_{j-1} \end{bmatrix} = u_j - pu_j \begin{bmatrix} u_0 & \dots & u_{j-1} \\ x_0 & \dots & x_{j-1} \end{bmatrix} = \frac{V \begin{vmatrix} u_0 & \dots & u_{j-1} u_j \\ x_0 & \dots & x_{j-1} \end{vmatrix}}{V \begin{vmatrix} u_0 & \dots & u_{j-1} \\ x_0 & \dots & x_{j-1} \end{vmatrix}} \end{aligned} \tag{85}$$

are

$$\varphi_j(x) = \frac{\alpha_{j-1}(x)}{\beta_j(x)} \cdot \frac{\beta_{j-1}(b_j)}{\alpha_{j-1}(b_j)} \qquad j = 0, \dots, n. \tag{86}$$

Indeed, by partial fraction decomposition of the right hand side of (86) one sees that it is a linear combination of $u_0, \ldots, u_j$ with leading coefficient 1 for $u_j$, having zeros $x_0, \ldots, x_{j-1}$, counting multiplicities. According to (85) this property characterises $\varphi_j$ completely.

The coefficients

$$c_j = c_j f = \left[ \begin{array}{ccc} u_0 & \ldots & u_j \\ x_0 & \ldots & x_j \end{array} \right] f \tag{87}$$

of the interpolant

$$p = pf \left[ \begin{array}{ccc} u_0 & \ldots & u_n \\ x_0 & \ldots & x_n \end{array} \right] = \sum_{j=0}^{n} c_j \cdot \varphi_j \tag{88}$$

are the *divided differences* of $f$ with respect to the ECT–system $(u_0, \ldots, u_j)$ and the nodes $(x_0, \ldots, x_j)$. If all nodes coincide, $x_j = x_{j-1} = \ldots = x_0$, then according to (34) we have

$$\left[ \begin{array}{ccc} u_0 & \ldots & u_j \\ x_0 & \ldots & x_0 \end{array} \right] f = \frac{\left(\frac{d}{dx}\right)^j rf \left[ \begin{array}{ccc} u_0 & \ldots & u_{j-1} \\ x_0 & \ldots & x_0 \end{array} \right] (x_0)}{\frac{j!}{\beta_j(x_0)} \cdot \frac{\beta_{j-1}(b_j)}{\alpha_{j-1}(b_j)}} \tag{89}$$

by Leibniz' rule (32). If not all nodes are identical we may assume that $x_j \neq x_0$. Then

$$\left[ \begin{array}{ccc} u_0 & \ldots & u_j \\ x_0 & \ldots & x_j \end{array} \right] f = \frac{\left[ \begin{array}{ccc} u_0 & \ldots & u_{j-1} \\ x_1 & \ldots & x_j \end{array} \right] f - \left[ \begin{array}{ccc} u_0 & \ldots & u_{j-1} \\ x_0 & \ldots & x_{j-1} \end{array} \right] f}{\frac{x_j - x_0}{(x_j - b_j)^*} \cdot \frac{\beta_{j-1}(b_j)}{\alpha_{j-1}(b_j)} \cdot \frac{\hat{\alpha}_{j-2}(b_{j-1})}{\beta_{j-2}(b_{j-1})}}, \tag{90}$$

where $\hat{\alpha}_{j-2}$ is the node polynomial associated with the node system $\hat{X}_{j-2} = (x_1, \ldots, x_{j-1})$. For $j = 0, 1$, $\hat{X}_{j-2} = \emptyset$ and $\hat{\alpha}_{j-2} := 1$, $\beta_{j-2} := 1$.

For $x \in \mathbb{C} \backslash \{b_0, \ldots, b_n\}$ the interpolation remainder is

$$\begin{aligned} rf \left[ \begin{array}{ccc} u_0 & \ldots & u_n \\ x_0 & \ldots & x_n \end{array} \right] (x) &:= f(x) - pf \left[ \begin{array}{ccc} u_0 & \ldots & u_n \\ x_0 & \ldots & x_n \end{array} \right] (x) \\ &= \left[ \begin{array}{ccc} \pi_0 & \ldots & \pi_n \pi_{n+1} \\ x_0 & \ldots & x_n, x \end{array} \right] (\beta_n \cdot f) \cdot \frac{\alpha_n(x)}{\beta_n(x)} \end{aligned} \tag{91}$$

with

$$\begin{aligned} \left[ \begin{array}{ccc} \pi_0 & \ldots & \pi_n \pi_{n+1} \\ x_0 & \ldots & x_n x \end{array} \right] (\beta_n \cdot f) = \sum_{i=0}^{n} \left[ \begin{array}{ccc} \pi_0 & \ldots & \pi_i \\ x_0 & \ldots & x_i \end{array} \right] \beta_n \left[ \begin{array}{ccc} \pi_0 & \ldots & \pi_n \pi_{n-i+1} \\ x_i & \ldots & x_n x \end{array} \right] f \\ + f(x) \cdot \left[ \begin{array}{ccc} \pi_0 & \ldots & \pi_n \pi_{n+1} \\ x_0 & \ldots & x_n x \end{array} \right] \beta_n, \end{aligned} \tag{92}$$

where the last summand is zero if at least one pole is at infinity. Moreover, if $b_{n+1} \in \overline{\mathbb{C}}$ is chosen arbitrarily,

$$rf \left[ \begin{array}{ccc} u_0 & \ldots & u_n \\ x_0 & \ldots & x_n \end{array} \right] (x) = \left[ \begin{array}{ccc} u_0 & \ldots & u_n u_{n+1} \\ x_0 & \ldots & x_n x \end{array} \right] f \cdot \frac{\alpha_n(x) \cdot \beta_n(b_{n+1})}{\beta_{n+1}(x) \cdot \alpha_n(b_{n+1})}. \tag{93}$$

Before giving the proofs we remark that Newton's whole procedure for computing the interpolant as a rational function with prescribed poles in $U_n$ has complexity $\mathcal{O}(n^2)$ just as for polynomial interpolation.

Observe, that (91) already has been derived in (83). Equation (92) results by application of Leibniz' rule (33). By comparison of (91) and (93) we find the following relation between ordinary and generalized divided differences with respect to a CV–system

$$\left[ \begin{array}{ccc} u_0 & \ldots & u_n u_{n+1} \\ x_0 & \ldots & x_n x \end{array} \right] f = (x - b_{n+1})^* \frac{\alpha_n(b_{n+1})}{\beta_n(b_{n+1})} \cdot \left[ \begin{array}{ccc} \pi_0 & \ldots & \pi_n \pi_{n+1} \\ x_0 & \ldots & x_n x \end{array} \right] (\beta_n \cdot f). \tag{94}$$

Clearly, (94) holding for all $x \in \mathbb{C}$ outside of $\{x_0, \ldots, x_n\} \cup \{b_0, \ldots, b_n\}$, it remains true for $x = x_{n+1} \in \mathbb{C}\backslash\{b_0, \ldots, b_n\}$ by continuity of divided differences as functions of their nodes. Using this from (94) with the replacement $n + 1 = j$ we find that for all $j = 0, 1, \ldots$

$$\begin{bmatrix} u_0 & \ldots & u_j \\ x_0 & \ldots & x_j \end{bmatrix} f = (x_j - b_j)^* \frac{\alpha_{j-1}(b_j)}{\beta_{j-1}(b_j)} \begin{bmatrix} \pi_0 & \ldots & \pi_j \\ x_0 & \ldots & x_j \end{bmatrix} (\beta_{j-1} f). \tag{95}$$

In order to derive (90) from (34) we have to show that for $j \geq 1$ and $x_j \neq x_0$

$$\begin{bmatrix} u_0 & \ldots & u_{j-1} \\ x_1 & \ldots & x_j \end{bmatrix} u_j - \begin{bmatrix} u_0 & \ldots & u_{j-1} \\ x_0 & \ldots & x_{j-1} \end{bmatrix} u_j = \frac{x_j - x_0}{(x_j - b_j)^*} \frac{\beta_{j-1}(b_j)}{\alpha_{j-1}(b_j)} \frac{\hat{\alpha}_{j-2}(b_{j-1})}{\beta_{j-2}(b_{j-1})}.$$

The ordinary divided difference on the right hand side of (94), if $x_j \neq x_0$, according to (30) can be written

$$\begin{bmatrix} \pi_0 & \ldots & \pi_j \\ x_0 & \ldots & x_j \end{bmatrix} (\beta_{j-1} f) = \frac{\begin{bmatrix} \pi_0 & \ldots & \pi_{j-1} \\ x_1 & \ldots & x_j \end{bmatrix} (\beta_{j-1} f) - \begin{bmatrix} \pi_0 & \ldots & \pi_{j-1} \\ x_0 & \ldots & x_{j-1} \end{bmatrix} (\beta_{j-1} f)}{x_j - x_0}.$$

If Leibniz' rule (33) is applied to each of the two terms of the numerator we get

$$\begin{aligned} \begin{bmatrix} \pi_0 & \ldots & \pi_{j-1} \\ x_1 & \ldots & x_j \end{bmatrix} (\beta_{j-1} f) &= \begin{bmatrix} \pi_0 & \ldots & \pi_{j-1} \\ x_1 & \ldots & x_j \end{bmatrix} ((x - b_{j-1})^* \beta_{j-2} \cdot f) \\ &= (x_j - b_{j-1})^* \begin{bmatrix} \pi_0 & \ldots & \pi_{j-1} \\ x_1 & \ldots & x_j \end{bmatrix} (\beta_{j-2} \cdot f) + 1 \cdot \begin{bmatrix} \pi_0 & \ldots & \pi_{j-2} \\ x_1 & \ldots & x_{j-1} \end{bmatrix} (\beta_{j-2} \cdot f), \end{aligned}$$

$$\begin{aligned} &\begin{bmatrix} \pi_0 & \ldots & \pi_{j-1} \\ x_0 & \ldots & x_{j-1} \end{bmatrix} (\beta_{j-1} f) = \\ &(x_0 - b_{j-1})^* \cdot \begin{bmatrix} \pi_0 & \ldots & \pi_{j-1} \\ x_0 & \ldots & x_{j-1} \end{bmatrix} (\beta_{j-2} f) + 1 \cdot \begin{bmatrix} \pi_0 & \ldots & \pi_{j-2} \\ x_1 & \ldots & x_{j-1} \end{bmatrix} (\beta_{j-2} f). \end{aligned}$$

In view of (95) which must be applied accordingly to each term, subtraction yields

$$\begin{aligned} &\begin{bmatrix} \pi_0 & \ldots & \pi_j \\ x_0 & \ldots & x_j \end{bmatrix} (\beta_{j-1} f) \\ = &\frac{(x_j - b_{j-1})^* \begin{bmatrix} \pi_0 & \ldots & \pi_{j-1} \\ x_1 & \ldots & x_j \end{bmatrix} (\beta_{j-2} f) - (x_0 - b_{j-1})^* \begin{bmatrix} \pi_0 & \ldots & \pi_{j-1} \\ x_0 & \ldots & x_{j-1} \end{bmatrix} (\beta_{j-2} f)}{x_j - x_0} \\ = &\frac{\frac{\beta_{j-2}(b_{j-1})}{\hat{\alpha}_{j-2}(b_{j-1})} \begin{bmatrix} u_0 & \ldots & u_{j-1} \\ x_1 & \ldots & x_j \end{bmatrix} f - \frac{\beta_{j-2}(b_{j-1})}{\hat{\alpha}_{j-2}(b_{j-1})} \begin{bmatrix} u_0 & \ldots & u_{j-1} \\ x_0 & \ldots & x_j \end{bmatrix}}{x_j - x_0}. \end{aligned}$$

Inserting this in (95) we finally arrive at

$$\begin{aligned} &\begin{bmatrix} u_0 & \ldots & u_j \\ x_0 & \ldots & x_j \end{bmatrix} f \\ = &\frac{(x_j - b_j)^*}{x_j - x_0} \frac{\alpha_{j-1}(b_j)}{\beta_{j-1}(b_j)} \frac{\beta_{j-2}(b_{j-1})}{\hat{\alpha}_{j-2}(b_{j-1})} \cdot \left( \begin{bmatrix} u_0 & \ldots & u_{j-1} \\ x_1 & \ldots & x_j \end{bmatrix} f - \begin{bmatrix} u_0 & \ldots & u_{j-1} \\ x_0 & \ldots & x_{j-1} \end{bmatrix} f \right) \end{aligned}$$

which is equivalent with (90).

## 2.4. The confluent Cauchy–Vandermonde determinant

In order to get simple sign factors in the determinantal formula we are going to derive we now assume that the nodes $X_n = (x_0, \dots, x_n)$ are consistently ordered as in (12). Also, we suppose that the poles are consistently ordered according to

$$\mathcal{B} = (b_0, \dots, b_n) = (\underbrace{\eta_0, \dots, \eta_0}_{\mu_0}, \eta_1, \dots, \eta_{q-1}, \underbrace{\eta_q, \dots, \eta_q}_{\mu_q}) \tag{96}$$

with $\eta_0 = \infty, \eta_1, \dots, \eta_q \in \mathbb{C}$ pairwise distinct and $\mu_0 + \dots + \mu_q = n+1$ with $\mathcal{B}_n \cap X_n = \emptyset$. Then the confluent Cauchy–Vandermonde determinant (84) can be expressed in terms of the nodes and poles involved

$$V \begin{vmatrix} u_0 & \dots & u_n \\ L_0 & \dots & L_n \end{vmatrix} = \det \left( \left(\frac{d}{dx}\right)^{\mu_i(x_i)} u_j(x_i) \right)_{i=0,\dots,n}^{j=0,\dots,n} = \tag{97}$$

$$= \operatorname{mult}(X_n) \cdot \frac{\prod\limits_{\substack{k,j=0\\k>j}}^{n} {}^*(x_k - x_i) \cdot \prod\limits_{\substack{k,j=0\\k>j}}^{n} {}^*(b_k - b_j)}{\prod\limits_{\substack{k,j=0\\k\geq j}}^{n} {}^*(x_k - b_j) \cdot \prod\limits_{\substack{k,j=0\\k>j}}^{n} {}^*(b_k - x_j)}$$

where

$$\operatorname{mult}(X_n) := \prod_{i=0}^{n} \mu_i(x_i)! \tag{98}$$

is a measure of multiplicities of the nodes.

Note that (97) readily follows from (46) observing

$$\left(\frac{d}{dx}\right)^{\mu_i(x_i)} ru_i \begin{bmatrix} u_0 & \dots & u_{i-1} \\ x_0 & \dots & x_{i-1} \end{bmatrix} (x_i) = \frac{\beta_{i-1}(b_i)}{\alpha_{i-1}(b_i)} \left(\frac{d}{dx}\right)^{\mu_i(x_i)} \frac{\alpha_{i-1}}{\beta_i}(x_i)$$

$$= \mu_i(x_i)! \frac{\prod\limits_{k=0}^{i-1} {}^*(x_i - x_k)}{\prod\limits_{k=0}^{i} {}^*(x_i - b_k)} \frac{\prod\limits_{k=0}^{i-1} {}^*(b_i - b_k)}{\prod\limits_{k=0}^{i-1} {}^*(b_i - x_k)}$$

by Leibniz' rule (32).

## 2.5. The canonical form of a real CV–system

Suppose that the poles are consistently ordered according to (96) such that

$$\dots < \eta_3 < \eta_1 < a \leq x \leq b < \eta_2 < \eta_4 \dots$$

where the prescribed poles $\eta_0 = \infty, \eta_1, \eta_2, \dots, \eta_q$ have multiplicities $\mu_0, \mu_1, \dots, \mu_q \geq 0$ adding up to $n+1$. Notice that this distribution of poles is not a particular one since the multiplicities are allowed to be equal to zero. If $\mathcal{U}_n = (u_0, \dots, u_n)$ is the CV-system associated with (96) then

$$\mathcal{V}_n = (v_0, \dots, v_n) = (\sigma_0 u_0, \dots, \sigma_n u_n)$$

has the canonical initial form (7) where the sign factors $\sigma_j$ and the positive weight functions $w_j$ for $j = 0, \ldots, n$ are given by

$$\begin{aligned}\sigma_i &= 1 & i &= 0,\ldots,\mu_0-1\\ \sigma_{N_{2\lambda}+i} &= (-1)^{\mu_0+\mu_2+\ldots+\mu_{2\lambda}+i} & i &= 0,\ldots,\mu_{2\lambda+1}-1\\ \sigma_{N_{2\lambda+1}+i} &= (-1)^{\mu_2+\ldots+\mu_{2\lambda}+i+1} & i &= 0,\ldots,\mu_{2\lambda+2}-1\end{aligned}$$

where for $s = 0, \ldots, q \quad N_s := \mu_0 + \mu_1 + \ldots + \mu_s$ and

$$\begin{aligned}
w_0 &= 1 & k &= 0\\
w_k &= k & k &= 1,\ldots,\mu_0-1\\
w_{N_{2\lambda}}(x) &= N_{2\lambda}\cdot\frac{\prod_{j=1}^{2\lambda}(\eta_j-\eta_{2\lambda+1})^{\mu_j}}{\prod_{j=1}^{2\lambda-1}(\eta_{2\lambda}-\eta_j)^{\mu_j}}\cdot\frac{(\eta_{2\lambda}-x)^{N_{2\lambda}-1}}{(x-\eta_{2\lambda+1})^{N_{2\lambda}+1}} & k &= N_{2\lambda}\\
w_{N_{2\lambda}+i}(x) &= (N_{2\lambda}+i)\cdot\frac{1}{(x-\eta_{2\lambda+1})^2} & i &= 1,\ldots,\mu_{2\lambda+1}-1\\
w_{N_{2\lambda+1}}(x) &= N_{2\lambda+1}\cdot\frac{\prod_{j=1}^{2\lambda+1}(\eta_{2\lambda+2}-\eta_j)^{\mu_j}}{\prod_{j=1}^{2\lambda}(\eta_j-\eta_{2\lambda+1})^{\mu_j}}\cdot\frac{(x-\eta_{2\lambda+1})^{N_{2\lambda+1}-1}}{(\eta_{2\lambda+2}-x)^{N_{2\lambda+1}+1}} & k &= N_{2\lambda+1}\\
w_{N_{2\lambda+1}+i}(x) &= (N_{2\lambda+1}+i)\cdot\frac{1}{(\eta_{2\lambda+2}-x)^2} & i &= 1,\ldots,\mu_{2\lambda+2}-1
\end{aligned}$$

This follows by an elementary but lengthy calculation from the explicit formulas (11) of the weight functions in terms of the Wronskians of the subsystems of $\mathcal{U}_n$ whose values are known from (97).

## 2.6. The Neville–Aitken formula

The Neville–Aitken formula for interpolation by linear combinations of CV–systems is easily derived from the general Neville–Aitken recursion (49) since we are able to compute the weight factors explicitly in terms of the nodes and poles. In fact, from (85) and (86) for $x_n \neq x_0$ we get

$$\lambda_n(x) = \frac{ru_n\begin{bmatrix}u_0 & \ldots & u_{n-1}\\ x_0 & \ldots & x_{n-1}\end{bmatrix}(x)}{ru_n\begin{bmatrix}u_0 & \ldots & u_{n-1}\\ x_0 & \ldots & x_{n-1}\end{bmatrix}(x) - ru_n\begin{bmatrix}u_0 & \ldots & u_{n-1}\\ x_1 & \ldots & x_n\end{bmatrix}(x)} = \frac{(x-x_0)(b_n-x_n)^*}{(x_n-x_0)(b_n-x)^*}, \tag{99}$$

hence if $x_n \neq x_0$

$$pf\begin{bmatrix}u_0 & \ldots & u_n\\ x_0 & \ldots & x_n\end{bmatrix}(x) = \tag{100}$$
$$\frac{(x-x_0)(b_n-x_n)^* pf\begin{bmatrix}u_0 & \ldots & u_{n-1}\\ x_1 & \ldots & x_n\end{bmatrix}(x) - (x-x_n)(b_n-x_0)^* pf\begin{bmatrix}u_0 & \ldots & u_{n-1}\\ x_0 & \ldots & x_{n-1}\end{bmatrix}(x)}{(x_n-x_0)(b_n-x)^*}.$$

If all nodes coincide, $x_n = x_{n-1} = \ldots = x_0$, then according to (86) –(89) we have the generalized Taylor's expansion of $f$ with respect to $(u_0, \ldots, u_n)$ at $x_0$

$$pf\begin{bmatrix}u_0 & \ldots & u_n\\ x_0 & \ldots & x_0\end{bmatrix}(x) = \sum_{j=0}^{n}\left(\frac{d}{dx}\right)^j rf\begin{bmatrix}u_0 & \ldots & u_{j-1}\\ x_0 & \ldots & x_0\end{bmatrix}(x_0)\cdot\frac{(x-x_0)^j}{j!}\cdot\frac{\beta_j(x_0)}{\beta_j(x)}. \tag{101}$$

(101) constitutes a mean to compute the Taylor interpolants

$$pf\begin{bmatrix} u_0 \\ x_o \end{bmatrix}, \ldots, pf\begin{bmatrix} u_0 & \ldots & u_n \\ x_0 & \ldots & x_0 \end{bmatrix}$$

recursively from the data $x_0, \left(\frac{d}{dx}\right)^j f(x_0)(j = 0, \ldots, n)$, and from the poles with arithmetical complexity $\mathcal{O}(n^2)$.

## 2.7. The Lagrange–Hermite formula and the inverse of the confluent Cauchy–Vandermonde matrix

Given a CV–system $\mathcal{U}_n = (u_0, \ldots, u_n)$ with associated pole sequence $\mathcal{B}_n = (b_0, \ldots, b_n)$, and a system $X_n = (x_0, \ldots, x_n)$ of possibly repeated nodes in $\mathbb{C}\backslash\mathcal{B}_n$, we want to find the basic Lagrange–Hermite functions $l_j \in U_n$ $(j = 0, \ldots, n)$ such that the interpolant $pf \in U_n$ of a function $f \in C^n(\Omega; \mathbb{K})$ is

$$pf = pf\begin{bmatrix} u_0 & \ldots & u_n \\ x_0 & \ldots & x_n \end{bmatrix} = \sum_{j=0}^{n} \langle L_j, f\rangle \cdot l_j. \tag{102}$$

The general theory was treated in section 1.9. In order to get simple formulas when it is applied to CV–systems we assume that the node system after being consistently ordered is given by (12). We claim that

$$l_i(x) = l_{\varphi(r,\rho)}(x) = \omega_r^\rho(x) := \frac{\omega_r(x)}{\beta_n(x)} \cdot P_{r,\rho}(x) \cdot v_{r,\rho}(x), \tag{103}$$

where $i = \varphi(r, \rho)$ and $\varphi$ is defined by (14) and

$$\omega_r(x) := \prod_{\substack{s=0 \\ s\neq r}}^{p} (x - \xi_s)^{\nu_s} \qquad r = 0, \ldots, p \tag{104}$$

$$v_{r,\rho}(x) = \frac{(x - \xi_r)^\rho}{\rho!} \qquad \rho = 0, \ldots, \nu_r - 1; r = 0, \ldots, p \tag{105}$$

$$P_{r,\rho}(x) = \sum_{\sigma=0}^{\nu_r-\rho-1} d_r^\sigma\left(\frac{\beta_n}{\omega_r}\right) \cdot \frac{(x - \xi_r)^\sigma}{\sigma!}, \quad d_r^\sigma(v) := \left(\frac{d}{dx}\right)^\sigma v(\xi_r), \tag{106}$$

the Taylor's polynomial of order $\nu_r - \rho - 1$ of the function $\frac{\beta_n}{\omega_r}$ developed at the point $\xi_r$.

If all nodes are simple the basic Lagrange functions (103) simplify to

$$l_i(x) = \frac{\omega_i(x)}{\omega_i(x_i)} \cdot \frac{\beta_n(x_i)}{\beta_n(x)} \quad i = 0, \ldots, n \tag{107}$$

which, when all poles are at infinity, are the well known basic Lagrange functions for polynomial interpolation.

To prove (103) it is sufficient to show that

$$\langle L_i, l_j\rangle = \langle d_r^\rho, \omega_l^\lambda\rangle = \delta_{(r,\rho),(l,\lambda)} = \delta_{r,l} \cdot \delta_{\rho,\lambda} \quad i, j = 0, \ldots, n, i = \varphi(r, \rho), j = \varphi(l, \lambda) \tag{108}$$

But (108) is easily verified by making repeatedly use of Leibniz' rule (32). If $r \neq l$, then according to (32) $d_r^\rho \omega_l^\lambda = 0$, since then $\omega_l^\lambda$ contains the factor $(x - \xi_r)^{\nu_r}$. Suppose now $r = l$. We must show $d_r^\rho \omega_r^\lambda = \delta_{\rho,\lambda}$. Again, this is clear if $\rho < \lambda$. When $\rho \geq \lambda$ this is equivalent with $d_r^\rho(u \cdot v) = \delta_{\rho,\lambda}$ where we have set $u := \frac{\omega_r \cdot P_{r,\lambda}}{\beta_n}$ and $v = v_{r,\lambda}$. Using Leibniz' rule (32) repeatedly we find

$$d_r^\rho(u \cdot v) = \binom{\rho}{\lambda} d_r^{\rho-\lambda} u \cdot 1 = \binom{\rho}{\lambda} \sum_{\mu=0}^{\rho-\lambda} \binom{\rho - \lambda}{\mu} d_r^{\rho-\lambda-\mu} P_{r,\lambda} d_r^\mu \left(\frac{\omega_r}{\beta_n}\right).$$

Clearly, $d_r^{\rho-\lambda-\mu}P_{r,\lambda} = d_r^{\rho-\lambda-\mu}\left(\frac{\beta_n}{\omega_r}\right)$. Hence,

$$d_r^\rho(u \cdot v) = \binom{\rho}{\lambda} d_r^{\rho-\lambda}\left(\frac{\beta_n}{\omega_r} \cdot \frac{\omega_r}{\beta_n}\right) = \delta_{\rho,\lambda}.$$

Suppose now that the pole system $\mathcal{B}_n$, after being consistently ordered, is as in (96). Then there is a one–to–one mapping

$$\begin{aligned}(s,\sigma) \mapsto j &= \psi(s,\sigma) = \mu_0 + \ldots + \mu_{s-1} + \sigma \\ \sigma &= 1,\ldots,\mu_s,\ s = 0,\ldots,q\end{aligned} \tag{109}$$

such that for every $j = 0,\ldots,n$ there is precisely one pair $(s,\sigma)$ such that

$$u_j(x) = u_{s,\sigma}(x) := \begin{cases} x^{\sigma-1} & s = 0, \sigma = 1,\ldots,\mu_0 \\ \dfrac{1}{(x-\eta_s)^\sigma} & s = 1,\ldots,q; \sigma = 1,\ldots,\mu_s \end{cases} \tag{110}$$

If the coefficients $d_{i,j}$ of the expansion

$$l_i = \sum_{j=0}^{n} d_{i,j} \cdot u_j \quad i = 0,\ldots,n \tag{111}$$

are known, then according to section 1.9 for the confluent Cauchy–Vandermonde matrix $V$ as in (82)

$$V_{adj} = \det V \cdot D^T, \quad D = (d_{ij}) \tag{112}$$

$$V^{-1} = D^T. \tag{113}$$

It is a simple but somewhat tedious exercise to determine the coefficients $A_{s,\sigma}^{r,\rho}$ of the partial fraction decomposition

$$\omega_r^\rho = \sum_{s=0}^{q}\sum_{\sigma=1}^{\mu_s} A_{s,\sigma}^{r,\rho} \cdot u_{s,\sigma} \tag{114}$$

expressed in terms of the nodes and poles. Then by (113)

$$V^{-1} = D^T, \quad D = (d_{ij}), \quad d_{i,j} = A_{\psi^{-1}(j)}^{\varphi^{-1}(i)}.$$

## 2.8. Recurrence relation for the basic Lagrange functions of CV–systems

In this short last section we give a recurrence relation to compute the basic Lagrange functions with respect to a CV–system and to simple nodes. From (66) and (100) we find

$$l_i^{0,n}(x) = \frac{(x-x_0)(b_n-x_n)^*}{(x_n-x_0)(b_n-x)^*} \cdot l_{i-1}^{1,n-1}(x) + \frac{(x_n-x)(b_n-x_0)^*}{(x_n-x_0)(b_n-x)^*} l_i^{0,n-1}(x) \quad i = 1,\ldots,n \tag{115}$$

where

$$l_{-1}^{1,n-1}(x) := 0, \quad l_n^{0,n-1}(x) := 0.$$

The recursion (115) starts with $n = 1$ and

$$l_0^{0,0}(x) = \frac{u_0(x)}{u_0(x_0)}, \quad l_0^{1,0}(x) = \frac{u_0(x)}{u_0(x_1)}.$$

## 2.9. Notes and Remarks

Cauchy–Vandermonde systems are in use since the times of Cauchy to whom the computation of the determinant (97) is credited when all poles are in $\mathbb{C}$ and simple and all nodes are simple [8]. In [26], see also [24], this determinant has been calculated for multiple poles and simple nodes. In [9] the determinant (97) was computed for multiple poles and multiple nodes. A different proof can be found in [7].

The proof of unisolvency of the Hermite interpolation problem with CV–systems is due to J. Walsh [30].

The recursion (90) of divided differences with respect to CV–systems is due to G. Mühlbach [23]. There also the remainder formula (93) was derived.

The Neville–Aitken formula (100) for interpolants with respect to CV–systems was first given in [9]. For different proofs see [7] and [23].

The Taylor formula (101) with respect to CV–systems seems to be new. Using this formula in combination with algorithm 1 given in [9], that calculates the derivatives

$$\left(\frac{d}{dx}\right)^j \left(\frac{1}{\beta_k(x)}\right)(x_0),$$

it is possible to compute the Taylor interpolant

$$pf\begin{bmatrix} u_0 & \dots & u_n \\ x_0 & \dots & x_0 \end{bmatrix}$$

with $\mathrm{O}(n^2)$ arithmetical operations.

The weight functions and the canonical representation of real CV–systems given in section 2.5 are new.

The basic Lagrange–Hermite functions with respect to a CV–system were derived in [21], see also [23]. In that paper also an explicit formula for the inverse of the confluent CV–matrix is given.

A method solving a system of linear equations with a confluent CV–matrix recursively can be found in [23].

The recursion (115) for the basic Lagrange functions with respect to CV–systems and simple nodes is new.

For a theory of convergence of rational interpolants with prescribed poles to analytic functions as $n \to \infty$ confer [1]. Recently, interpolants from CV-spaces have been proved useful for approximation of transfer functions of infinite-dimensional dynamical systems [27].

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G. Mühlbach
Institute of Applied Mathematics
University of Hannover
Germany