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When is the least degree solution of a Bézout identity nonnegative on the interval [-1,1]?

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Abstract. We give conditions such that the least degree solution of a Bézout identity is nonegative on the interval $[-1, 1]$.

¿Cuándo es no negativa en el intervalo [-1,1] la solución de mínimo grado de una identidad de Bézout?

Resumen. Se dan condiciones para que la solución de mínimo grado de una identidad de Bézout sea no negativa en el intervalo [-1,1].

1. Introduction

A function P satisfies the conjugate quadrature filter equation, if for all z on the unit circle, we have that

$$|P(z)|^2 + |P(-z)|^2 = 1. \tag{1}$$

Various applications to wavelet construction and filter design require solutions of equation (1) which are polynomials of some fixed degree. A particularly noteworthy solution of this equation, which is prominently used in [1] for wavelet construction, seeks a polynomial P of degree $2N-1$ which has an N-fold zero at minus one. To find this polynomial, we write it in the form

$$P_{2N-1}(z) = 2^{-N}(1+z)^N Q_{N-1}(z),$$

substitute this expression into (1), set $z = e^{i\theta}$ and $x = \sin^2\theta/2$, with $|\theta| \le \pi$ to obtain the equation

$$(1-x)^N q_{N-1}(x) + x^N q_{N-1}(1-x) = 1 \tag{2}$$

where

$$q_{N-1}(x) = |Q_{N-1}(e^{i\theta})|^2. \tag{3}$$

From this equation it follows that

$$q_{N-1}(z) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} z^k$$

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which is the *least* degree solution of this Bézout identity. Remarkably, the polynomial $q_{N-1}$is *strictly positive* on the interval [0,1] and so equation (3) can be solved for the polynomial Q_{N-1} by the Riesz Lemma, see[1]. The existence of this solution of the CQF equation is the foundation for the construction of an orthonormal wavelet of smallest support having N vanishing moments and, by adjusting N, any prescribed number of continuous derivatives.

In another direction, our effort to construct orthonormal *spline projectors* of *maximum accuracy* which are *local* led us to consider the Bézout identity

$$p_n(x)q_{n-1}(x) + p_n(1-x)q_{n-1}(1-x) = 1 \tag{4}$$

for the polynomial q_{n-1} of degree at most $n-1$ where the polynomial p_n of degree n is given with *all* its zeros in $[1,\infty)$ and normalized to be nonnegative on [0,1]. This equation, under these circumstances, also came up in [3] when we constructed a spline which has *both* orthonormal integer translates, is zero at all integers except zero where it is one, has knots at half integers and decays exponentially fast at infinity. Fortunately, even in this general circumstance, when p_n has all its zeros in $[1,\infty)$ and is normalized to be nonnegative on [0,1], which includes the first example mentioned above, we showed in [3] that the least degree solution of (4), is also nonnegative on the interval $[0,1]$, thereby allowing for the construction of the desired splines described above.

Later, one of us used this general fact about the Bézout identity (4) to extend the Deslauries and Dubuc [2] interpolatory subdivision by local symmetric polynomial interpolation to interpolatory subdivision using local symmetric interpolation by *exponentials* with *arbitrary real* frequencies (with at least one being zero), [6]. This result leads to the construction of orthonormal wavelets of *smallest* support with N arbitrarily prescribed zeros on the *imaginary* axis and of orthonormal wavelets of compact support, with any prescribed regularity, relative to any given *Sobolev norm*, [7].

All of these applications, including the first example, suggest the following problem. Given a polynomial R_n of degree n can we find a polynomial Q_{n-1} such that the polynomial $P_{2n-1} := R_nQ_{n-1}$ satisfies the CQF equation? Equivalently, the polynomials $p_n(\cos\theta) := |R_n(e^{i\theta})|^2$ and $q_{n-1}(\cos\theta) := |Q_{n-1}(e^{i\theta})|^2$ satisfy the Bézout identity

$$p_n(z)q_{n-1}(z) + p_n(-z)q_{n-1}(-z) = 1. \tag{5}$$

When the degree of Q_{n-1} is *not* constrained to be of (least) degree $n-1$ this was indeed shown to be the case for *some* polynomial Q in [4] and later estimates for the degree of Q were given in [5] as well as multivariate versions of these results.

The zeros of a CQF influence properties of wavelets built from it. The point of view of constructing a polynomial solution of the CQF equation with some prescribed zeros should be contrasted with the complete characterization of [8] for the polynomial solution of the CQF equation. Unfortunately, this characterization does not allow for the specification of zeros, even in the special case mentioned above.

In this paper, we provide necessary and sufficient conditions on a given polynomial R_n,with zeros constrained to be either on the real axis or on the unit circle, equivalently all the zeros of p_n are on the real axis, for which P_{2n-1} can be constructed to satisfy the CQF equation. That is, we shall provide conditions on the polynomial p_n so that the least degree solution of the Bézout identity (5) will be nonnegative on the interval $I := [-1,1]$.

2. Preliminaries and the case $n = 2, 3$

We start with a polynomial P written in the form $P = RS$. The polynomial R is given and assumed to be of degree n with real coefficients. We consider the square of its modulus and express it in the form

$$R(z)R(z^{-1}) := p_n(\frac{1}{2}(z+\frac{1}{z}))$$

where the polynomial p_n is likewise of degree n, in particular,

$$|R(e^{i\theta})|^2 = p_n(\cos\theta).$$

In what follows we will primarily be concerned with the case that the zeros of R are either real or on the unit circle. Note that the zeros of R on the unit circle correspond to zeros of p_n on the interval $I = [-1, 1]$, the nonnegative zeros of R correspond to zeros of p_n on $[1, \infty)$ and the nonpositive zeros of R to the zeros of p_n on the interval $(-\infty, -1]$. Let us now do the same reduction for the modulus of S, that is, write it in the form

$$|S(e^{i\theta})|^2 = q_{n-1}(\cos\theta)$$

where in this equation we demand that both S and q_{n-1} are of degree at most $n-1$ and the polynomial q_{n-1} satisfies the CQF equation (5). In other words, we have that

$$p_n(z)q_{n-1}(z) + p_n(-z)q_{n-1}(-z) = 1. \tag{6}$$

As stated in the introduction, we know when p_n only has zeros in the interval $(-\infty, -1]$ and is normalized to be nonnegative on I then the unique least degree solution q_{n-1} of (6) is also nonnegative on I and hence a polynomial S of degree $n-1$ exists so that P satisfies the CQF equation. We shall improve on this result and demonstrate that this conclusion still holds even when p_n has some of its zeros inside the interval I; how far inside will be determined below.

We let $\mathbb{P}_n$ be the class of all monic polynomials p_n of degree n which are nonnegative on the interval I . When $p_n(z)$ and $p_n(-z)$ do not have any common zeros in the complex plane, we let q_{n-1} be the unique polynomial of degree at most $n-1$ which satisfies the Bézout identity (6). We want to identify conditions on $p_n \in \mathbb{P}_n$ under which $q_{n-1} \in \mathbb{P}_{n-1}$. We recall the following fact proved in [3] and already referred to above.

Lemma GM If $p_n \in \mathbb{P}_n$ and the zeros of p_n are in the set $\{x : x \leq -1\}$ then $q_{n-1} \in \mathbb{P}_{n-1}$.

Our first observation deals with the case when $n = 2, 3$.

Lemma 1 *If $p_n \in \mathbb{P}_n$, $n = 2, 3$ and the zeros of p_n are in the set $\{z : Re\ z \leq -\frac{1}{2}\}$ then $q_{n-1} \in \mathbb{P}_{n-1}$.*

PROOF. The case $n = 2$ is easily handled. Indeed, if $p(z) = z^2 + az + b$ is in $\mathbb{P}_2$ then $q_1(z) = \frac{1}{2ab}(a - z)$ is nonnegative on the interval I when $a \geq 1$.

In general, since $p_n \in \mathbb{P}_n$ all of its odd multiplicity negative zeros must be ≤ -1. Therefore, in view of Lemma GM, for n=3, we need to consider only the case when p_3 has one real zero at some $-\alpha_1$ with $\alpha_1 \geq 1$ and two complex zeros at some $\alpha_2 = -\alpha - i\beta$ and $\alpha_3 = -\alpha + i\beta$, where $\alpha \geq \frac{1}{2}$. We observe for $p_3(z) = z^3 + az^2 + bz + c$ that

$$q_2(z) = \frac{1}{2c(b - \frac{c}{a})}(z^2 - az + b - \frac{c}{a})$$

and so it suffices to show when the roots of of the polynomial $z^2 - az + b - \frac{c}{a}$ are real, they lie outside $(-1, 1)$. Indeed, if the roots are real then $a^2 \geq 4(b - \frac{c}{a})$ and we have that

$$\begin{aligned} a^2 - 4(b - \frac{c}{a}) &= (\alpha_1 + 2\alpha)^2 - 4(2\alpha_1\alpha + \alpha^2 + \beta^2 - \frac{\alpha_1(\alpha^2+\beta^2)}{\alpha_1 + 2\alpha}) \\ &= (2\alpha - \alpha_1)^2 - \frac{8\alpha(\alpha^2+\beta^2)}{\alpha_1 + 2\alpha} < (2\alpha - \alpha_1)^2. \end{aligned}$$

Consequently, the smaller root of q_2 satisfies the inequality

$$\begin{aligned} \frac{1}{2}\{a - \sqrt{a^2 - 4(b - \frac{c}{a})}\} &> \frac{1}{2}\{\alpha_1 + 2\alpha - |2\alpha - \alpha_1|\} \\ &= \min\{\alpha_1, 2\alpha\} \geq 1. \quad \blacksquare \end{aligned}$$

The above result is sharp. Indeed, for $n = 2$ the proof above shows $q_{n-1} \in \mathbb{P}_n$ if and only if the zeros of p_n are in the set $\{z : Re\ z \leq -\frac{1}{2}\}$. For $n = 3$, if $q_{n-1} \in \mathbb{P}_n$ we let the real root $-\alpha_1$ of p_n go to $-\infty$ and fix its complex root to have negative real part, that is, $Re\alpha > 0$. In the limit, it follows that $Re\alpha \geq \frac{1}{2}$.

3. The general case

In this section we consider the case that $n \geq 4$. We begin by recalling the method of finding the polynomial q_{n-1} by interpolation. To this end, we suppose that the zeros of p_n are at $-\alpha_1, \ldots, -\alpha_n$. According to the Bézout equation (6), the polynomial $v_{2n-1} := p_n q_{n-1}$ of degree $2n-1$ is determined uniquely by conditions

$$v_{2n-1}(-\alpha_i) = 0, \;\; i = 1, \ldots, n \tag{7}$$

and

$$v_{2n-1}(\alpha_i) = 1, \;\; i = 1, \ldots, n, \tag{8}$$

where repetition of α_i means that we include the appropriate derivatives of v_{2n-1}. Note that the polynomial $v_{2n-1} - \frac{1}{2}$ is an odd function and so we may write it in the form

$$v_{2n-1}(z) - \frac{1}{2} = \frac{1}{2} z r_{n-1}(z^2), \tag{9}$$

where r_{n-1} is the unique polynomial of degree at most $n-1$ which interpolates the function $g(z) := z^{-\frac{1}{2}}$ at $\alpha_1^2, \ldots, \alpha_n^2$ (possibly with multiplicity). Here we take the branch of $z^{-\frac{1}{2}}$ which gives $(\alpha_i^2)^{-\frac{1}{2}} = \frac{1}{\alpha_i}, \;\; i = 1, \ldots, n$.

Let us consider the special case of this reduction for the polynomial $p_4(z) = (z + \frac{1}{2})^4$, i.e. $\alpha_1 = \cdots = \alpha_4 = \frac{1}{2}$. In this case, since the cubic polynomial r_3 is the Taylor polynomial of the function g about the point $\frac{1}{4}$ it is given by the formula

$$r_3(z) = 2 - 4(z - \frac{1}{4}) + 12(z - \frac{1}{4})^2 - 40(z - \frac{1}{4})^3$$

from which we conclude that $r_3(1) < -1$ and therefore $v_7(1) < 0$. In particular, we have that $q_3(1) < 0$, which shows that Lemma 1 fails in the case $n = 4$.

Henceforth, we assume that the roots of p_n are *all* negative, that is, $\alpha_j > 0, j = 1, \ldots, n$. Recall that if α_j has odd multiplicity then $\alpha_j \geq 1$. We now show that the addition of zeros ≤ -1 will preserve the conclusion that $q_{n-1} \in \mathbb{P}_{n-1}$

Proposition 1 *If* $p_n \in \mathbb{P}_n$ *with* $q_{n-1} \in \mathbb{P}_{n-1}$ *and* $p_{n+1} \in \mathbb{P}_{n+1}$ *with* $\alpha_{n+1} \geq 1$ *then* $q_n \in \mathbb{P}_n$.

PROOF. Recall that r_{n-1} interpolates the function g at $\alpha_1^2, \ldots, \alpha_n^2$ and r_n interpolates g at $\alpha_1^2, \ldots, \alpha_{n+1}^2$. Thus, we conclude that

$$r_n(x) = r_{n-1}(x) + (x - \alpha_1^2) \cdots (x - \alpha_n^2)[\alpha_1^2, \ldots, \alpha_{n+1}^2]g.$$

In this equation we use the standard bracket notation for the divided difference. The points within the brackets are the places at which the divided difference of the function g is computed. Since $[\alpha_1^2, \ldots, \alpha_{n+1}^2]g = \frac{1}{n!} g^{(n)}(t)$ for some $t > 0$ we obtain that $(-1)^n [\alpha_1^2, \ldots, \alpha_{n+1}^2]g > 0$. Remember, if α_j for some $j = 1, \ldots, n+1$ occurs with odd multiplicity then $\alpha_j \geq 1$. This fact allows us to conclude for $x \leq 1$ that $(-1)^n (x - \alpha_1^2) \cdots (x - \alpha_n^2) \geq 0$. Combining these two inequalities we have for $x \leq 1$ that $r_n(x) \geq r_{n-1}(x)$. Hence, making use of equation (9) for $0 \leq x \leq 1$ we obtain that

$$v_{2n+1}(x) = \frac{1}{2} + \frac{1}{2} x r_n(x) \geq \frac{1}{2} + \frac{1}{2} x r_{n-1}(x) = v_{2n-1}(x) \geq 0.$$

Now, equations (7), (8) and Rolle's Theorem, applied to the polynomial v_{2n+1} implies that its derivative vanishes precisely once between any two distinct adjacent elements of $\{-\alpha_1, \ldots, -\alpha_{n+1}\}$ and of $\{\alpha_1, \ldots, \alpha_{n+1}\}$. Therefore, we obtain that $v_{2n+1} \in \mathbb{P}_{2n-1}$ from which it follows that $q_n \in \mathbb{P}_n$. ∎

From Proposition 1 we see that the critical case occurs when *all* the zeros of a polynomial p_n in $\mathbb{P}_n$ are in the interval $(-1, 0)$. In this case, all its zeros have *even* multiplicity. To address this important case, we set $n = 2m$, choose points $\alpha_1, \ldots, \alpha_n$ such that $0 < \alpha_1 \leq \cdots \leq \alpha_n < 1$ and for each positive integer $k \leq m$ we define the monic polynomial p_{2k} by the formula

$$p_{2k}(z) := \prod_{j=1}^{k} (z + \alpha_j)^2. \tag{10}$$

We let q_{2k-1} be the least solution the Bézout identity (5) determined by the polynomial p_{2k} and corresponding to its zeros form the quantity $r_{2k-1}(1)$, as explained after equation (9).

Proposition 2 *If p_{2k} and q_{2k-1}, $k = 1, \ldots, m$ are the polynomials defined in equation* (10) *then $q_{2k-1} \in \mathbb{P}_{2k-1}$ for $k = 1, \ldots, m$ if and only if $r_{2k-1}(1) \geq -1$, for $k = 1, \ldots, m$.*

PROOF. From equation (9) we see that if $q_{n-1} \in \mathbb{P}_{n-1}$ then $r_{n-1}(1) \geq -1$.

The converse is prove by induction on m where $n = 2m$. To this end, for later use we observe that equations (7), (8) and Rolle's Theorem applied to the function v_{2n-1} imply that its derivative vanishes precisely once between any two distinct consecutive elements of $\{-\alpha_m, \ldots, -\alpha_1\}$ and of $\{\alpha_1, \ldots, \alpha_m\}$ and so $v_{2n-1}(x) \geq 0$, for $x \in [-1, \alpha_1]$ and v_{2n-1} is decreasing on the interval $[\alpha_n, \infty)$.

Let us first consider the case $m = 1$. If $r_1(1) \geq -1$ we conclude by equation (9) that $v_3(1) \geq 0$. Moreover, by our remarks above, we know that the cubic polynomial v_3 is nonnegative on the interval $[-1, \alpha_1]$ while it is decreasing on the interval $[\alpha_1, 1]$. Therefore , we have that $v_3(x) \geq v_3(1) \geq 0$ on this interval and thus $q_1 \in \mathbb{P}_1$.

Now, we assume that the result is true for some integer $n = 2m$ and suppose that $r_{n-1}(1) \geq -1$ for $n = 2k, k = 1, \ldots, m+1$. By our inductive hypothesis, we conclude that $q_{2k-1} \in \mathbb{P}_{2k-1}$ for $k = 1, \ldots, m$. On the other hand, using the definition of the polynomial r_{n-1} we obtain the formula

$$\begin{aligned} r_{2m+1}(x) = r_{2m-1}(x) &+ (x - \alpha_1^2)^2 \cdots (x - \alpha_m^2)^2 [\alpha_1^2, \alpha_1^2, \cdots, \alpha_m^2, \alpha_m^2, \alpha_{m+1}^2] g \\ &+ (x - \alpha_1^2)^2 \ldots (x - \alpha_{m+1}^2) [\alpha_1^2, \alpha_1^2, \ldots, \alpha_{m+1}^2, \alpha_{m+1}^2] g. \end{aligned}$$

Since $[\alpha_1^2, \alpha_1^2, \ldots, \alpha_m^2, \alpha_m^2, \alpha_{m+1}^2] g = \frac{1}{(2m)!} g^{(2m)}(t)$ for some $t > 0$, we conclude that

$$[\alpha_1^2, \alpha_1^2, \ldots, \alpha_m^2, \alpha_m^2, \alpha_{m+1}^2] g > 0.$$

Similarly, we obtain that the divided difference $[\alpha_1^2, \alpha_1^2, \ldots, \alpha_{m+1}^2, \alpha_{m+1}^2] g$ is negative. Therefore, we have shown for $x \leq \alpha_{m+1}^2$ that $r_{2m+1}(x) \geq r_{2m-1}(x)$. Hence equation (9) shows for x in the interval $[0, \alpha_{m+1}]$, that $v_{2n+1}(x) \geq v_{2n-1}(x)$. However, our induction hypothesis implies that $q_{2m-1} \in \mathbb{P}_{2m-1}$ and so we conclude that $q_{2m+1}$is nonnegative on the interval $[0, \alpha_{m+1}]$. Our induction hypothesis also insures that $r_{m+1}(1) \geq -1$ and so, by equation (9), we obtain that $v_{2n+1}(1) \geq 0$. But we already pointed out that the polynomial v_{2n+1} is decreasing for $x \geq \alpha_{m+1}$ and so $v_{2n+1}(x) \geq 0, \alpha_{m+1} \leq x \leq 1$. Thus $q_{2m+1} \in \mathbb{P}_{2m-1}$ and the induction step is complete. ■

The conditions $r_{n-1}(1) \geq -1$ is an explicit algebraic condition on $\alpha_1, \ldots, \alpha_n$. Below we provide simpler sufficient condition on $\alpha_1, \ldots, \alpha_n$ for which $r_{n-1}(1) \geq -1$. For this purpose, we use the recursion for divided differences

$$[\alpha_1^2, \alpha_1^2, \ldots, \alpha_{m+1}^2, \alpha_{m+1}^2] g = \frac{[\alpha_1^2, \ldots, \alpha_{m+1}^2, \alpha_{m+1}^2] g - [\alpha_1^2, \alpha_1^2, \ldots, \alpha_{m+1}^2] g}{\alpha_{m+1}^2 - \alpha_1^2}$$

from which we get that

$$[\alpha_1^2, \alpha_1^2, \ldots, \alpha_{m+1}^2] g + (\alpha_{m+1}^2 - \alpha_1^2) [\alpha_1^2, \alpha_1^2, \ldots, \alpha_{m+1}^2, \alpha_{m+1}^2] g > 0.$$

If $1 - \alpha_{m+1}^2 \leq \alpha_{m+1}^2 - \alpha_1^2$, we use this inequality to prove that

$$[\alpha_1^2, \alpha_1^2, \ldots, \alpha_{m+1}^2] g + (1 - \alpha_{m+1}^2) [\alpha_1^2, \alpha_1^2, \ldots, \alpha_{m+1}^2, \alpha_{m+1}^2] g > 0$$

and by the recursion above relating r_{2m+1} and r_{2m-1} we conclude that $r_{2m+1}(1) \geq r_{2m-1}(1)$. Therefore, $r_{2m-1}(1) \geq -1$ and $2\alpha_{m+1}^2 \geq 1+\alpha_1^2$, imply that $r_{2m+1}(1) \geq -1$.

These facts prove the next result.

Corollary 1 *If p_{2k}, $k = 1,\ldots,m$, are the polynomials defined in equation* (10) *with $\alpha_1 \geq \frac{1}{2}$ and for $j = 2,\ldots,m$, $2\alpha_j^2 \geq 1+\alpha_1^2$ then $q_{2k-1} \in \mathbb{P}_{2k-1}$ for $k = 1,\ldots,m$.* □

Our next lemma will lead us to our main result.

Lemma 2 *For any positive integer k and $\beta \in (0,1]$, let $r_{2k-1}(\cdot|\beta)$ denote the Taylor polynomial of degree $2k-1$ for the function g at β^2 . There exists a unique value $\alpha_k \in (0,1]$ such that $r_{2k-1}(\cdot|\alpha_k)(1) = -1$. Moreover, the sequence $\{\alpha_k : k \geq 1\}$ strictly increases , $\alpha_1 = \frac{1}{2}$, $\alpha_k^2 < \frac{4k-3}{8k-5}$ and $\lim_{k\to\infty} \alpha_k = \frac{1}{\sqrt{2}}$.*

PROOF. For $0 < \beta \leq 1$, (with $1\cdot 3\cdots(-1) := 1$) we have that

$$\begin{aligned} r_{2k-1}(1|\beta) &= \sum_{j=0}^{2k-1} \frac{1\cdot 3\cdot 5\cdots(2j-1)}{2^j}\frac{(-1)^j\beta^{-(2j+1)}}{j!}(1-\beta^2)^j \\ &= \sum_{i=0}^{k-1} \frac{1\cdot 3\cdot 5\cdots(4i-1)}{2^{2i}}\frac{\beta^{-(4i+1)}}{(2i)!}(1-\beta^2)^{2i} \\ &- \sum_{i=0}^{k-1} \frac{1\cdot 3\cdot 5\cdots(4i+1)}{2^{2i+1}}\frac{\beta^{-(4i+3)}}{(2i+1)!}(1-\beta^2)^{2i+1} \\ &= \sum_{i=0}^{k-1} \frac{1\cdot 3\cdot 5\cdots(4i-1)}{2^{2i+1}(2i+1)!}\frac{1}{\beta}\left(\frac{1}{\beta^2}-1\right)^{2i}\left\{8i+3-\frac{4i+1}{\beta^2}\right\}. \end{aligned}$$

Now, for all $i \geq 0$, $\frac{1}{\beta}\left(\frac{1}{\beta^2}-1\right)^{2i}\left\{\frac{4i+1}{\beta^2}-(8i+3)\right\}$ is a strictly decreasing function of β on $(0,1]$. Thus $r_{2k-1}(1|\beta)$ is a strictly increasing function of β on $(0,1]$.
Since $r_{2k-1}(1|1) = 1$ and $\lim_{\beta\to 0} r_{2k-1}(1|\beta) = -\infty$ there is a unique value $\alpha_k \in (0,1]$ such that $r_{2k-1}(1|\alpha_k) = -1$. Also, because $r_1(1|\beta) = \frac{1}{2\beta}(3-\frac{1}{\beta^2})$, we have $\alpha_1 = \frac{1}{2}$.

We shall prove by induction on k that $\alpha_k^2 < \frac{4k-3}{8k-5}$. This is valid for $k = 1$. Suppose it is true for some $k \geq 1$. We conclude that $\alpha_k^2 < \frac{4k-3}{8k-5} < \frac{4k+1}{8k+3}$ and by the equation for $r_{2k-1}(1|\beta)$ we obtain that

$$r_{2k+1}(1|\beta) = r_{2k-1}(1|\beta) + \frac{1\cdot 3\cdot 5\cdots(4k-1)}{2^{2k+1}(2k+1)!}\frac{1}{\beta}\left(\frac{1}{\beta^2}-1\right)^{2k}\left(8k+3-\frac{4k+1}{\beta^2}\right).$$

We evaluate this formula for $\beta = \alpha_k$ and obtain the inequality $r_{2k+1}(1|\alpha_k) < r_{2k-1}(1|\alpha_k) = -1$. Since $r_{2k+1}(1|\cdot)$ is a strictly increasing function on the interval $(0,1]$, we have that $\alpha_{k+1} > \alpha_k$, thereby establishing by induction that α_k strictly increases with k. On the other hand , for $\mu^2 := \frac{4k+1}{8k+3}$ we get that $r_{2k+1}(1|\mu) = r_{2k-1}(1|\mu) > -1$, since $\mu > \alpha_k$. Thus we obtain that $\mu > \alpha_{k+1}$. In other words, we have confirmed that $\alpha_{k+1}^2 < \frac{4k+1}{8k+3}$,which advances the induction step.

So far, we know that $\lim_{k\to\infty}\alpha_k = \gamma$ for some $\gamma \leq \frac{1}{\sqrt{2}}$. It remains to prove that $\gamma = \frac{1}{\sqrt{2}}$. To this end, we choose any $\beta < \frac{1}{\sqrt{2}}$ and note that the terms in the expansion for $r_{2k-1}(1|\beta)$ can be written, with $m = 2i$, as

$$a_i := \frac{1\cdot 3\cdot 5\cdots(2m+1)}{2^{m+1}(m+1)!}\frac{1}{\beta}\left(\frac{1}{\beta^2}-1\right)^m\left\{\frac{4m+3}{2m+1}-\frac{1}{\beta^2}\right\}.$$

From this formula it follows that $\lim_{i\to\infty} a_i = -\infty$ and so $\lim_{k\to\infty} r_{2k-1}(1|\beta) = -\infty$. So, for all large enough k, $r_{2k-1}(1|\beta) < -1 = r_{2k-1}(1|\alpha_k)$ and so $\beta < \alpha_k$. Thus $\beta < \gamma$ and so $\gamma = \frac{1}{\sqrt{2}}$. ■

Lemma 3 *If k is a positive integer, $\alpha \in (0,1)$ and $p_{2k}(x) = (x+\alpha)^{2k}$ then $q_{2k-1} \in \mathbb{P}_{2k-1}$ if and only if $\alpha \geq \alpha_k$.*

PROOF. If $\alpha < \alpha_k$, then $r_{2k-1}(1|\alpha) < -1$ and by Proposition 2 it follows that $q_{2k-1} \in \mathbb{P}_{2k-1}$ does not hold. If $\alpha \geq \alpha_k$, then for $i = 1, \ldots, k$, $\alpha \geq \alpha_i$. Therefore , we get that $r_{2i-1}(1|\alpha) \geq -1$ and by Proposition 2 we conclude that $q_{2k-1} \in \mathbb{P}_{2k-1}$. ■

Theorem 1 *For positive integers k, n such that $1 \leq k \leq \frac{1}{2}n$ and $p_n \in \mathbb{P}_n$ with all negative zeros which has at most $2k$ zeros in $(-1,0)$ that lie in $(-1,-\alpha_k]$ then $q_{n-1} \in \mathbb{P}_{n-1}$ holds. Conversely, if for some α, $q_{n-1} \in \mathbb{P}_{n-1}$ whenever $p_n \in \mathbb{P}_n$ has at most $2k$ zeros in $(-1,0)$ which are in $(-1,-\alpha]$ then $\alpha \geq \alpha_k$.*

PROOF. For any $n \geq 2$ and $t_1, \ldots, t_n$ in $(0,1)$, let r_{n-1} denote the polynomial of degree $n-1$ which interpolates g at $t_1, \ldots, t_n$ (possibly with multiplicity). As before we have that

$$r_{n-1}(x) = r_{n-2}(x) + (x-t_1)\cdots(x-t_{n-1})[t_1, \ldots, t_n]g$$

and so for some $\tau \in (0,1)$ it follows that

$$\begin{aligned}\frac{\partial}{\partial t_n} r_{n-1}(x) &= (x-t_1)\cdots(x-t_{m-1})[t_1, \ldots, t_{n-1}, t_n, t_n]g \\ &= (x-t_1)\cdots(x-t_{n-1})\frac{1}{n!}g^{(n)}(\tau).\end{aligned}$$

Thus, for n even, we have that $\frac{\partial}{\partial t_n} r_{n-1}(1) > 0$,. Since $r_{n-1}(1)$ is a symmetric function of $t_1, \ldots, t_n$ it is a strictly increasing function of all its variables $t_1, \ldots, t_n$, when n is even. Now, suppose for $1 \leq m \leq k$, that p_{2m} has $2m$ zeros in $(-1, -\alpha_k]$. Let r_{2m-1} of degree $2m-1$ interpolate g at the squares of the zeros of p_{2m} and $r_{2m-1}(1|\alpha_m)$ be as in Lemma 2. Since $-\alpha_k \leq -\alpha_m$, we have that $r_{2m-1}(1) \geq r_{2m-1}(1|\alpha_m) = -1$. It then follows from Proposition 2 that if $p_{2m} \in \mathbb{P}_{2m}$, then $q_{2m-1} \in \mathbb{P}_{2m-1}$. Next, suppose that p_n is as in the statement of the Theorem and has at most $2k$ zeros in $(-1, -\alpha_k]$. Then, from our above result and Proposition 1, we get that $q_{n-1} \in \mathbb{P}_{n-1}$.

Finally, take an $\alpha < \alpha_k$ so that by Lemma 3, $q_{2k-1} \in \mathbb{P}_{2k-1}$ does not hold for $p_{2k}(x) = (x+\alpha)^{2k}$, i.e. there is some z in $(-1,1)$ with $q_{2k-1}(z) < 0$. Now, for $\beta > 0$, consider $p_n(x) = (\frac{x}{\beta}+1)^{n-2k}(x+\alpha)^{2k}$, and let q_{n-1} be the corresponding polynomial as in (5). Then by continuity, $\lim_{\beta\to\infty} q_{n-1}(z) = q_{2k-1}(z) < 0$. So, for large enough β we have that $q_{2k-1} \in \mathbb{P}_{2k-1}$ does not hold . Thus, if $q_{n-1} \in \mathbb{P}_{n-1}$ holds whenever p_n has at most $2k$ zeros in $(-1, -\alpha]$, then $\alpha \geq \alpha_k$. ■

Recalling from Lemma 2 that α_k increases and $\lim_{k\to\infty} \alpha_k = \frac{1}{\sqrt{2}}$ gives immediately the following.

Corollary 2 *If that $p_n \in \mathbb{P}_n$ has all its zeros in $\{x : x \leq -\frac{1}{\sqrt{2}}\}$ then $q_{n-1} \in \mathbb{P}_{n-1}$. Conversely, if there is a positive constant c such that whenever $p_n \in \mathbb{P}_n$ has all its zeros in $\{x : x \leq c\}$ it follows that $q_{n-1} \in \mathbb{P}_{n-1}$ then $c \leq -\frac{1}{\sqrt{2}}$.*

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