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Properties of refinable measures

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Abstract. We give some new properties of refinable measures and survey results on their asymptotic normality. We also give a survey on the asymptotically optimal time-frequency localisation of refinable measures and associated wavelets.

Propiedades de las medidas refinables

Resumen. Se presentan algunas nuevas propiedades de las medidas refinables y se revisan algunos resultados recientes de su normalidad asintótica. También revisamos resultados sobre la localización tiempo-frecuencia asintóticamente óptima de las medidas refinables y de las ondículas asociadas.

1. Introduction

Let m be a probability measure on $\mathbb{R}$ with finite mean

$$\mu(m) := \int_R x\, dm(x).$$

We shall denote by $\mathcal{A}$ the set of all Lebesgue measurable sets in $\mathbb{R}$. The following result is proved in [2].

Theorem 1 [2] *For $\alpha > 1$, there is a unique probability measure ν on $\mathbb{R}$ satisfying*

$$\nu(A) = \int_{\mathbb{R}} \nu(\alpha A - x)\, dm(x), \quad A \in \mathcal{A}. \tag{1}$$

Moreover

$$\mu(\nu) = (\alpha - 1)^{-1}\mu(m).$$

Further, if m has finite standard deviation

$$\sigma(m) := \left\{ \int_{\mathbb{R}} (x - \mu(m))^2\, dm(x) \right\}^{\frac{1}{2}},$$

then ν has standard deviation

$$\sigma(\nu) = (\alpha^2 - 1)^{-\frac{1}{2}}\sigma(m). \quad \square$$

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Equation (1) is called a *refinement equation* (or scaling equation) and we call ν a *refinable measure*. Refinement equations have many applications such as signal processing, data compression and subdivision algorithms for computer aided design. In Section 2 we give some properties and examples of refinable measures and this material is mostly new. In Section 3 we give a survey of recent results concerning when a sequence of refinable measures is asymptotically normal, i.e. it converges to the normal distribution. There are also results on the order of convergence. This is of interest in application because, in particular, the normal distribution has optimal time-frequency localisation. This is discussed in Section 4 where we survey recent results on the time-frequency localisation of asymptotically normal measures and corresponding results for the associated wavelets.

2. Refinable measures

Equation (1) can be reformulated neatly by introducing random variables. Suppose that X is a random variable with measure m, $i.e.$ $P(X \in A) = m(A)$, for all $A \in \mathcal{A}$. If Z is a random variable with measure ν which is independent of X, then for any $A \in \mathcal{A}$,

$$P\left(\frac{X+Z}{\alpha} \in A\right) = P(X + Z \in \alpha A) = \int_{\mathbb{R}} \mu(\alpha A - x)\, dm(x).$$

Thus (1) is equivalent to

$$Y = \alpha^{-1}(X + Z), \tag{2}$$

where Y is a random variable with measure ν.

Iterating (2) gives for $n = 1, 2, \ldots$,

$$Y = \sum_{j=1}^{n} \frac{X_j}{\alpha^j} + \frac{Z}{\alpha^n}, \tag{3}$$

where $X_1, \ldots, X_n$ all have measure m and $X_1, \ldots, X_n, Z$ are independent. In order to take the limit as $n \to \infty$ in (3) we need to define a notion of convergence. Following [4, Chapter 8] we say $\lim_{n\to\infty} m_n = m$ for probability measures m_n, m on $\mathbb{R}$, if

$$\lim_{n\to\infty} m_n(I) = m(I) \tag{4}$$

for any finite interval $I = (a, b) \subset \mathbb{R}$ for which a, b are not atoms of m, i.e. $m(\{a\}) = m(\{b\}) = 0$. This implies that (4) holds for any interval whose end-points are not atoms of m. Moreover for $\lim_{n\to\infty} m_n = m$ to hold it is sufficient that

$$\lim_{n\to\infty} m_n((-\infty, x)) = m((-\infty, x))$$

for all $x \in \mathbb{R}$ which are not atoms of m. If $\lim_{n\to\infty} m_n = m$ and X, X_n, $n = 1, 2, \ldots$, are random variables with measures m, m_n, $n = 1, 2, \ldots$, respectively, then we say $\lim_{n\to\infty} X_n = X$.

Theorem 2 *Equation (1) holds if and only if*

$$Y = \sum_{j=1}^{\infty} \frac{X_j}{\alpha^j}, \tag{5}$$

where Y is a random variable with measure ν and X_j, $j = 1, 2, \ldots$, are independent random variables each with measure m.

PROOF. If (5) holds, then it follows easily that (2) holds and hence so does (1).

Now suppose that (1) holds. So (3) holds. Take $x \in \mathbb{R}$ which is not an atom of ν. Then for $n = 1, 2, \ldots$

$$P\left(Y - \frac{Z}{\alpha^n} < x\right) = \int_{\mathbb{R}} \nu\left(\left(-\infty, x + \frac{y}{\alpha^n}\right)\right) \, d\nu(y).$$

Choose $\epsilon > 0$ and take K with $\nu([-K, K]) > 1 - \epsilon$. Then for $n = 1, 2, \ldots$,

$$\left| \int_{\mathbb{R}} \nu\left(\left(-\infty, x + \frac{y}{\alpha^n}\right)\right) \, d\nu(y) - \int_{-K}^{K} \nu\left(\left(-\infty, x + \frac{y}{\alpha^n}\right)\right) \, d\nu(y) \right| < \epsilon.$$

Also

$$\lim_{n\to\infty} \int_{-K}^{K} \nu\left(\left(-\infty, x + \frac{y}{\alpha^n}\right)\right) \, d\nu(y) = \int_{-K}^{K} \nu((-\infty, x)) \, d\nu(y) = \nu((-\infty, x))\nu([-K, K]).$$

Thus

$$\lim_{n\to\infty} P\left(Y - \frac{Z}{\alpha^n} < x\right) = \nu((-\infty, x)),$$

$i.e.$

$$\lim_{n\to\infty} P\left(\sum_{j=1}^{n} \frac{X_j}{\alpha^j} < x\right) = P(Y < x),$$

which, by definition, is equivalent to (5). ■

Corollary 1 *If (1) is satisfied and m has support in $[a, b]$, then ν has support in $[\frac{a}{\alpha-1}, \frac{b}{\alpha-1}]$.*

PROOF. If m has support in $[a, b]$, then $a \le X_j \le b$, $j = 1, 2, \ldots$. Hence

$$a \sum_{j=1}^{\infty} \frac{1}{\alpha^j} \le \sum_{j=1}^{\infty} \frac{X_j}{\alpha^j} \le b \sum_{j=1}^{\infty} \frac{1}{\alpha^j},$$

$i.e.$ $\frac{a}{\alpha-1} \le Y \le \frac{b}{\alpha-1}$ and the result follows. ■

The most usual case of equation (1) is when $\alpha = 2$. In this case it can be interpreted as follows. Suppose that the random variable with measure m denotes the payback from some gambling game. Then from Theorem 2 we can see that the random variable with measure ν denotes the payback from putting half the stake on the game, half the remaining stake on a second play at the game and so on ad infinitum (assuming different plays at the game are independent). As seen from Theorem 1, the expected return from this strategy is the same as for a single play, but the standard deviation decreases by a factor of $\sqrt{3}$.

If m has support on a single point a, i.e. $m(\{a\}) = 1$, then clearly (1) is satisfied by the measure ν with support at the point $(\alpha - 1)^{-1}a$. The most usual case considered is when m has support on a finite number of integers. In contrast, we see in the next result that ν cannot have support on a finite number of points except in the trivial case above.

Theorem 3 *Suppose that (1) is satisfied and m is not supported on a single point. If ν has any atom, then it has an infinite number of atoms.*

PROOF. Let $X \ne \emptyset$ denote the set of atoms of ν and Y the set of atoms of m. By (1), for $x \in X$,

$$\begin{aligned} \nu(\{x\}) &= \sum_{y \in Y} m(\{y\})\nu(\{\alpha x - y\}) \\ &= \sum \{m(\{y\})\nu(\{z\}) : y \in Y, z \in X, y + z = \alpha x\}. \end{aligned}$$

Then

$$\nu(X) \leq \sum\{m(\{y\})\nu(\{z\}) : y \in Y, z \in X\} = m(Y)\nu(X) \leq \nu(X).$$

Since $\nu(X) > 0$, $m(Y) = 1$, *i.e.* m is discrete. Also for any $y \in Y$, $z \in X$, $y + z = \alpha x$ for some $x \in X$, *i.e.* $\alpha^{-1}(y + z) \in X$. Take any $z \in X$. Since Y contains at least two points, $\exists y \in Y$ with $z \neq \frac{y}{\alpha-1}$. Then the points $\frac{z}{\alpha^n} + \sum_{k=1}^{n} \frac{y}{\alpha^n}$, $n = 1, 2, \ldots$, lie in X and form a strictly increasing (respectively strictly decreasing) sequence if $x <$ (respectively $>$) $\frac{y}{\alpha-1}$. Thus X comprises an infinite number of points. ■

We now consider the simple case when $\alpha = 2$ and m has support on two points. By a shift and change of scale, there is no loss of generality in assuming that the support is $\{0, 1\}$, *i.e.*

$$m(\{0\}) = p, \quad m(\{1\}) = q, \quad p, q > 0, \quad p + q = 1.$$

Take $x \in [0, 1)$ of form

$$x = \sum_{j=1}^{n} a_j 2^{-j}, \quad a_j = 0 \text{ or } 1, \quad n \geq 1.$$

Then we define

$$\nu([x, x + 2^{-n})) = p^{n-r}q^r, \text{ where } r = \sum_{j=1}^{n} a_j. \tag{6}$$

Note that

$$\nu([x, x + 2^{-n-1})) = p^{n+1-r}q^r = pm([x, x + 2^{-n})),$$
$$\nu([x + 2^{-n-1}, x + 2^{-n})) = p^{n-r}q^{r+1} = qm([x, x + 2^{-n})).$$

Thus

$$\nu([x, x + 2^{-n})) = \nu([x + 2^{-n-1})) + \nu([x + 2^{-n-1}, x + 2^{-n})).$$

It can easily be deduced that ν can be extended to a probability measure on R with support on $[0, 1]$.

Now for $I = [x, x + 2^{-n})$ as in (6) with $I \subset [0, \frac{1}{2})$, $2I = [2x, 2x + 2^{-n+1})$, where

$$2x = \sum_{j=1}^{n} a_j 2^{-j+1} = \sum_{j=1}^{n-1} a_{j+1} 2^{-j},$$

since $a_1 = 0$. So

$$\nu(2I) = p^{n-1-r}q^r = p^{-1}\nu(I).$$

Similarly for $J = [x, x + 2^{-n}) \subset [\frac{1}{2}, 1)$, $2J - 1 = [2x - 1, 2x - 1 + 2^{-n+1})$, where $2x - 1 = \sum_{j=1}^{n-1} a_{j+1} 2^{-j}$, and so

$$\nu(2J - 1) = p^{n-r}q^{r-1} = q^{-1}\nu(J).$$

Thus for any Lebesgue measurable sets $U \subset [0, \frac{1}{2})$, $V \subset [\frac{1}{2}, 1)$,

$$\nu(U) = p\nu(2U), \quad \nu(V) = q\nu(2V - 1),$$

and since $2U - 1 \subset (-\infty, 0]$, $2V \subset [1, \infty)$,

$$\nu(U \cup V) = p\nu(2U \cup 2V) + q\nu((2U - 1) \cup (2V - 1)).$$

Hence for any $A \in \mathcal{A}$,

$$\nu(A) = p\nu(2A) + q\nu(2A - 1),$$

i.e. ν satisfies (1).

We note that ν has no atoms. Also for $n = 1, 2, \ldots$

$$\nu([0, 2^{-n})) = p^n, \quad \nu([1 - 2^{-n}, 1)) = q^n,$$

and hence ν is not absolutely continuous (with respect to Lebesgue measure) except when $p = q = \frac{1}{2}$, when it reduces to Lebesgue measure on $[0, 1]$.

Now take $x \in [0, 1]$ and write

$$x = \sum_{j=1}^{\infty} a_j 2^{-j}, \quad a_j = 0 \text{ or } 1.$$

Put $x_0 = 0$ and for $n \geq 1$,

$$x_n = \sum_{j=1}^{n} a_j 2^{-j}.$$

Take $n \geq 0$. If $a_{n+1} = 0$, then $x_{n+1} = x_n$ and $\nu([x_n, x_{n+1})) = 0$. If $a_{n+1} = 1$, then

$$\nu([x_n, x_{n+1})) = \nu([x_n, x_n + 2^{-n-1})) = p^{n+1-r} q^r,$$

where $r = \sum_{j=1}^{n} a_j$.

So for all $n \geq 0$,

$$\nu([x_n, x_{n+1})) = a_{n+1} p^{n+1-r} q^r.$$

Thus for $n \geq 1$,

$$\nu([0, x_n)) = \sum_{k=1}^{n} \nu([x_{k-1}, x_k)) = \sum_{k=1}^{n} a_k p^{k-r_k} q^{r_k},$$

where $r_k = \sum_{j=1}^{k-1} a_j$. Thus

$$\nu((-\infty, x)) = \sum_{k=1}^{\infty} a_k p^{k-r_k} q^{r_k}. \tag{7}$$

If we denote the above measure by ν_p, for $0 < p < 1$, then we see that for any $x \in [0, 1]$,

$$\lim_{p \to \frac{1}{2}} \nu_p((-\infty, x)) = x,$$

i.e. ν_p converges to Lebesgue measure on $[0, 1]$ as $p \to \frac{1}{2}$.

The above example shows that refinable measures need not be absolutely continuous (with respect to Lebegue measure). However when the measure ν in (1) *is* absolutely continuous, with density ϕ, then (1) can be rewritten as

$$\phi(y) = \int_{\mathbb{R}} \alpha \phi(\alpha y - x) \, dm(x), \quad y \in \mathbb{R}.$$

3. Asymptotic normality

For $n = 1, 2, \ldots$ let m_n denote the binomial distribution

$$m_n(\{j\}) = 2^{-n} \binom{n}{j}, \quad j = 0, \ldots, n. \tag{8}$$

Then it is well-known that the corresponding solution of the refinement equation (1) with $\alpha = 2$ is the measure with density the uniform B-spline B_n of degree n, *i.e.*

$$B_n(x) = \sum_{j=0}^{n} 2^{-n+1} \binom{n}{j} B_n(2x - j), \quad x \in \mathbb{R}.$$

It is well-known that, when suitably scaled, the binomial distribution converges as $n \to \infty$ to the normal distribution, $i.e.$ the measure N with density the Gaussian

$$G(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

As a special case of a result of Schoenberg [8], the B-splines B_n, when suitably scaled, also converge as $n \to \infty$ to the Gaussian. This result is useful in practice because the Gaussian arises naturally in physical situations and has optimal time-frequency localisation (as we discuss in Section 4), while the B-splines inherit many of the rich properties of the Gaussian and also provide fast algorithms for practical computation. We shall study more generally the convergence of scaled refinable measures to the normal distribution. Firstly we make precise the notion of this convergence.

For any probability measure m on R with mean μ and standard deviation σ, we define a probability measure $\tilde{m}$ on $\mathbb{R}$ by

$$\tilde{m}(S) = m(\sigma S + \mu), \quad S \in \mathcal{A}.$$

Thus $\tilde{m}$ has mean 0 and standard deviation 1. If m has density ϕ, then $\tilde{m}$ has density

$$\tilde{\phi}(x) := \sigma\phi(\sigma x + \mu), \quad x \in \mathbb{R}.$$

We say a sequence (m_n) of probability measures on $\mathbb{R}$ is *asymptotically normal* if

$$\lim_{n\to\infty} \tilde{m}_n = N.$$

Our example above of the binomial distribution and the B-splines extends to the following result.

Theorem 4 [2] *For $n = 1, 2, \ldots$ let m_n be a discrete probability measure on Z with finite mean and standard deviation, and let ν_n be the corresponding solution of (1). Then (m_n) is asymptotically normal if and only if (ν_n) is asymptotically normal.* □

In order to extend this to more general measures m_n, we need a condition which is most easily expressed in terms of the Fourier transform $\hat{m}$ of a measure m, i.e.

$$\hat{m}(u) := \int_{\mathbb{R}} e^{-iux}\, dm(x), \quad u \in \mathbb{R}.$$

Indeed the Fourier transform is a crucial tool in proving the results of this and the next section.

Theorem 5 [2] *For $n = 1, 2, \ldots$ let m_n be a probability measure on R with finite mean and standard deviation, and let ν_n be the corresponding solution of (1). If $(\hat{\tilde{m}}'_n)$ is uniformly bounded on a neighbourhood of 0, then (m_n) is asymptotically normal if and only if (ν_n) is asymptotically normal.* □

We now consider conditions under which (m_n), and hence (ν_n), is asymptotically normal. We shall assume that for $n = 1, 2, \ldots$, m_n is a discrete probability measure on $\{0, 1, \ldots, n\}$ given by

$$m_n(\{k\}) = a_{n,k}, \quad k = 0, 1, \ldots, n. \tag{9}$$

Thus (1) takes the form

$$\nu_n(A) = \sum_{k=0}^{n} a_{n,k}\nu_n(\alpha A - k), \quad A \in \mathcal{A}. \tag{10}$$

We define $r_{n,j}$, $j = 1, \ldots, n$ so that for $z \in \mathbb{C}$,

$$\sum_{k=0}^{n} a_{n,k} z^k = \prod_{j=1}^{n} \frac{z + r_{n,j}}{1 + r_{n,j}}. \tag{11}$$

It is easily seen that m_n has mean μ_n and standard deviation σ_n given by

$$\mu_n = \sum_{j=1}^{n} \frac{1}{1+r_{n,j}}, \quad \sigma_n^2 = \sum_{j=1}^{n} \frac{r_{n,j}}{(1+r_{n,j})^2}. \tag{12}$$

We shall assume that all the values $r_{n,j}$ lie in a region D_γ for some $\gamma \in [0, \frac{\pi}{2})$, where D_γ is the set of all $z \in \mathbb{C}$ satisfying

$$\left| \mathrm{Im} \left\{ \frac{z}{(1+z)^2} \right\} \right| \leq \tan\gamma \ \mathrm{Re} \left\{ \frac{z}{(1+z)^2} \right\}.$$

It can be seen that D_γ contains the sector $|\arg z| \leq \gamma$, and for $z = \pm r e^{i\theta}$, $r > 0$, $\gamma \leq \theta \leq \pi$, z lies in D_γ if and only if

$$\frac{\sin(\frac{\theta-\gamma}{2})}{\sin(\frac{\theta+\gamma}{2})} \leq r \leq \frac{\sin(\frac{\theta+\gamma}{2})}{\sin(\frac{\theta-\gamma}{2})}.$$

In particular D_γ contains the unit circle $|z| = 1$.

Theorem 6 [2] *For $n = 1, 2, \ldots$, let m_n be a probability measure given by (9). Suppose that $r_{n,j}$, $j = 1, \ldots, n$, $n = 1, 2, \ldots$, given by (11) lie in D_γ for some $\gamma \in [0, \frac{\pi}{2})$, and are bounded away from -1. Suppose further that σ_n in (12) satisfies $\sigma_n \to \infty$ as $n \to \infty$. Then (m_n) is asymptotically normal.* □

A special case of this result, when all $r_{n,j} > 0$ was proved by probabilistic methods in [1], [9]. The completely different analytic techniques used in [2] not only prove asymptotic normality for a much larger class of measures m_n but give some results on the order of convergence, which we proceed to discuss. First we give a result on the convergence of the scaled refinable measures $\tilde{\nu}$ to N in the frequency domain. We note that

$$\widehat{N}(u) = e^{-\frac{u^2}{2}}, \quad u \in \mathbb{R}.$$

Theorem 7 [2] *Assume that the conditions of Theorem 6 hold and that for $n = 1, 2, \ldots$, ν_n is the probabilty measure satisfying the refinement equation (10). Then as $n \to \infty$,*

$$\|\hat{\tilde{\nu}}_n - \hat{N}\|_\infty = O(\sigma_n^{-1}).$$

Moreover if $\sum_{k=0}^{n} a_{n,k} z^k$ is a reciprocal polynomial, i.e. $a_{n,0} \neq 0$ and $a_{n,k} = a_{n,n-k}$, $k = 0, 1, \ldots, n$, then

$$\|\hat{\tilde{\nu}}_n - \hat{N}\|_\infty = O(\sigma_n^{-2}).$$

If in addition,

$$\sigma_n^{-1} \sum_{j=1}^{n} r_{n,j}(r_{n,j}^2 - 4r_{n,j} + 1)(1 + r_{n,j})^{-4} \text{ is bounded}, \tag{13}$$

then

$$\|\hat{\tilde{\nu}}_n - \hat{N}\|_\infty = O(\sigma_n^{-3}). \quad \square$$

We note that $r^2 - 4r + 1 = 0$ when $r = 2 \pm \sqrt{3}$ and so (13) requires that in some sense the roots of $\sum_{k=0}^{n} a_{n,k} z^k$ are close to $-2 \pm \sqrt{3}$. In particular (13) will be satisfied if

$$P_n(z) = Q_{l_n}(z)(z^2 + 4z + 1)^{k_n},$$

where Q_{l_n} is a reciprocal polynomial of degree $l_n = n - 2k_n$ and $n^{-1/2} l_n$ is bounded over n. In this case Theorem 7 gives order of convergence $O(\sigma_n^{-3}) = O(n^{-\frac{3}{2}})$.

Also note that for the case (8), when ν_n has density B_n, $\sum_{k=0}^{n} z^k = 2^{-n}(1+z)^n$, all $r_{n,j} = 1$, and so Theorem 7 gives order of convergence $O(\sigma_n^{-2})$ but not $O(\sigma_n^{-3})$.

The next result gives estimates of the error between the measure m_n in Theorem 6 and the scaled Gaussian.

Theorem 8 [2] *Assume that the conditions of Theorem 6 hold and that all $r_{n,j}$ lie in the sector $|\arg z| \le \frac{\pi}{3}$. Then as $n \to \infty$,*

$$\max_{k=0,\dots,n} |\sigma_n a_{n,k} - G(\frac{k-\mu_n}{\sigma_n})| = O(\sigma_n^{-\frac{1}{2}}),$$

while if $\sum_{k=0}^{n} a_{n,k} z^k$ is reciprocal, then

$$\max_{k=0,\dots,n} |\sigma_n a_{n,k} - G(\frac{k-\mu_n}{\sigma_n})| = O(\sigma_n^{-\frac{2}{3}}). \quad \square$$

We remark that in [1] this problem is considered, using probabilistic techniques, for the special case when $a_{n,0}, \dots, a_{n,n}$ are the Eulerian numbers. In this case $\sigma_n = \sqrt{\pi(n+1)/6}$. Thus our result gives order of convergence $O(\sigma_n^{-\frac{2}{3}}) = O(n^{-\frac{1}{3}})$, while [1] shows only convergence $O(n^{-\frac{1}{4}})$.

To finish the section we give a result on the uniform convergence of $\tilde{\nu}_n$ to N in the time domain. To prove this we need some results from [7] which were given only for $\alpha = 2$. Here it is shown that if the numbers $r_{n,j}$, $j = 1, \dots n$ in (11) include 1 and all have non-negative real part, then for $n \ge 2$ the refinable measure ν_n in (10) has a continuous density ϕ_n, say. Then (10) takes the form

$$\phi_n(x) = \sum_{k=0}^{n} 2a_{n,k}\phi_n(2x-k), \quad x \in \mathbb{R}. \tag{14}$$

It is further shown in [7] that ϕ_n has nice 'shape properties' which are used to prove the following.

Theorem 9 [2] *Assume that the conditions of Theorem 6 hold and that for $n = 2, 3, \dots$, the numbers $r_{n,j}$, $j = 1, \dots, n$ include 1 and have non-negative real part. Then for the density function ϕ_n in (14),*

$$\|\tilde{\phi}_n - G\|_\infty = O(\sigma_n^{-\frac{1}{2}}),$$

while if $\sum_{k=0}^{n} a_{n,k} z^k$ is reciprocal, then

$$\|\tilde{\phi}_n - G\|_\infty = O(\sigma_n^{-1}).$$

If, in addition, (13) is satisfied, then

$$\|\tilde{\phi}_n - G\|_\infty = O(\sigma_n^{-\frac{3}{2}}). \quad \square$$

4. Uncertainty products

One of the main reasons why the asymptotic normality of refinable measures is useful is that the normal distribution has optimal time-frequency localisation in the sense we shall now describe. For an L^2 function ϕ for which $\int_{\mathbb{R}} x^j |\phi(x)|^2\, dx$ exists, $j = 1, 2$, we write

$$\mu_\phi := \|\phi\|_2^{-2} \int_{\mathbb{R}} x|\phi(x)|^2\, dx,$$

$$\Delta_\phi := \|\phi\|_2^{-1} \left\{ \int_{\mathbb{R}} (x-\mu_\phi)^2 |\phi(x)|^2\, dx \right\}^{\frac{1}{2}}.$$

Similarly if $\int_{\mathbb{R}} u^j |\hat{\phi}(u)|^2\, dx$ exists, $j = 1, 2$, we may define $\mu_{\hat{\phi}}$ and $\Delta_{\hat{\phi}}$. Thus Δ_ϕ is the standard deviation of the density function $\|\phi\|_2^{-2}|\phi|^2$ and gives a measure of the localisation of ϕ in the time domain. Similarly $\Delta_{\hat{\phi}}$ is the standard deviation of $\|\hat{\phi}\|_2^{-2}|\hat{\phi}|^2$ and measures the localisation of ϕ in the frequency domain. The *uncertainty product* $\Delta_\phi \Delta_{\hat{\phi}}$ gives an overall measure of the time-frequency localisation of ϕ. Clearly for any $k, \mu \in \mathbb{R}, \sigma > 0$, the function $k\phi(\sigma. - \mu)$ has the same uncertainty product as ϕ.

Heisenberg's Uncertainty Principle states that for any ϕ as above,

$$\Delta_\phi \Delta_{\hat{\phi}} \geq \frac{1}{2}$$

and equality holds if and only if $\phi = kG(\sigma. - \mu)$ for some $k, \mu \in \mathbb{R}, \sigma > 0$, see [5] for a general discussion. We see in the next result that, under a mild extra assumption, the refinable functions ϕ_n in Theorem 9 approach optimal time-frequency localisation as $n \to \infty$.

Theorem 10 [6] *Assume the conditions of Theorem 9 and that for $n = 2, 3, \ldots$, the numbers $r_{n,j}$, $j = 1, \ldots, n$ include 1 twice. Then*

$$\lim_{n\to\infty} \Delta_\phi \Delta_{\hat{\phi}} = \frac{1}{2}. \quad \Box$$

This result was proved earlier in [3] for the special case when the polynomial in (11) is reciprocal, has negative roots, and has a factor of $(z+1)^{m_n}$, where $m_n \geq Cn$ for a constant $C > 0$. Under the same conditions, they also proved a similar result for the corresponding wavelets ψ_n, extending work in [10] for the B-spline wavelets. For completeness we recall the definition of a wavelet ψ_n corresponding to a refinable function ϕ_n satisfying (14). Let V_0 denote the subspace of $L^2(\mathbb{R})$ spanned by $\phi_n(. - k)$, $k \in Z$, and let $V_1 = \{f(2.) : f \in V_0\}$. Then ψ_n is a wavelet corresponding to ϕ_n if $\psi_n(. - k)$, $k \in Z$, span the orthogonal complement of V_0 in V_1.

Now $\int \psi_n = 0$ and for a function with this property (called a bandpass filter), the definition of the uncertainty product is modified to reflect the fact that $\hat{\psi}$ treats positive and negative frequency bands separately. Let

$$\mu_{\hat{\psi}}^+ := \frac{\int_0^\infty u|\hat{\psi}(u)|^2\,du}{\int_0^\infty |\hat{\psi}(u)|^2\,du},$$

$$\Delta_{\hat{\psi}}^+ := \left\{ \frac{\int_0^\infty (u - \mu_{\hat{\psi}}^+)^2 |\hat{\psi}(u)|^2\,du}{\int_0^\infty |\hat{\psi}(u)|^2\,du} \right\}^{\frac{1}{2}},$$

where we assume that these are well-defined. (Note that for a real-valued function ψ, $|\hat{\psi}|$ is even and so $\int_0^\infty |\hat{\psi}(u)|^2\,du = \frac{1}{2}\|\hat{\psi}\|_2^2$ and the definition of $\Delta_{\hat{\psi}}^+$ is unaltered by replacing $\int_0^\infty$ by $\int_{-\infty}^0$ in the above definitions.) Then a measure of the time-frequency localisation of ψ is given by $\Delta_\psi \Delta_{\hat{\psi}}^+$.

It is shown in [3] that if $\psi \in L^2 \cap L^1$ is a real-valued symmetric or anti-symmetric function for which the above definitions are well-defined and $\int \psi = 0$, then

$$\Delta_\psi \Delta_{\hat{\psi}}^+ > \frac{1}{2}$$

and the lower bound cannot be improved or attained. Next we see that under appropriate conditions, the wavelets ψ_n approach optimal time-frequency localisation as $n \to \infty$.

Theorem 11 [6] *For $n = 1, 2, \ldots$, let ϕ_n be the density satisfying (14) and ψ_n a corresponding wavelet, where the polynomial (11) is reciprocal and $r_{n,j}$, $j = 1, \ldots, n$ are positive and include 1 twice. Then*

$$\lim_{n\to\infty} \Delta_{\psi_n} \Delta_{\hat{\psi}_n}^+ = \frac{1}{2}. \quad \Box$$

When the polynomial (11) is reciprocal we can choose the wavelet ψ_n to be symmetric or anti-symmetric as n is even or odd. Theorem 11 follows from the fact that a suitable normalisation of ψ_n behaves like a 'modulated Gaussian' as $n \to \infty$, as is made precise in our final result.

Theorem 12 [6] *Under the conditions of Theorem 11, there are wavelets* ψ_n *and numbers* $\alpha_n \in (\frac{2\pi}{3}, \pi)$ *such that the following hold as* $n \to \infty$*, where* σ_n *are given by (12). For even* n*,*

$$\psi_n(\sigma_n x) - \cos(\sigma_n \alpha_n x) G(x) \to 0,$$

and for odd n

$$\psi_n(\sigma_n x) - \sin(\sigma_n \alpha_n x) G(x) \to 0,$$

where the convergence is in $L^p(\mathbb{R})$*,* $2 \leq p \leq \infty$*.* □

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