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The distance of a curve to its control polygon

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Abstract. Recently, Nairn, Peters, and Lutterkort bounded the distance between a Bézier curve and its control polygon purely in terms of differences in the control points. We show how such bounds easily extend to many of the curve types used in Geometric Modelling.

La distancia entre una curva y su polígono de control

Resumen. Recientemente, Nairn, Peters y Lutterkort han acotado la distancia entre una curva de Bézier y su polígono de control en términos de las diferencias entre los puntos de control. Mostramos cómo extender dichas cotas a muchos tipos de curvas utilizadas en el Diseño Geométrico.

1. Introduction

In [5], Nairn, Peters, and Lutterkort derived a new bound on the distance between a Bézier curve and its control polygon. The new aspect of the error bound is that it depends purely on second order differences of control points, rather than on derivatives, typical in previously derived error bounds. As one might expect, the bound tends to zero as the polygon tends to a straight line, reflecting the fact that the Bernstein basis has linear precision. Later, Reif [6] extended some of the results of [5], via a different approach, to bound the distance between a *spline* curve and its control polygon.

The purpose of this paper is to further generalize the work of [5] to deal with much larger classes of curves suitable for computer aided design. Drawing on some earlier work on linear precision and convexity preservation of [3], we show that under very mild and sensible assumptions on the curve type, the main results of [5] are extendable, namely to error bounds in terms of divided differences, or scaled divided differences. Our tools can be used, not only for bounding the distance of a curve to its control polygon, but also for comparing different curves. This analysis allows us to compare different representations associated to different bases.

2. Auxiliary results and a first example

Let $(u_0, \dots, u_n)$ be a system of functions defined on $[a, b]$. We may define the *collocation matrix* of the system $(u_0, \dots, u_n)$ at any sequence of points $x_0 < x_1 < \cdots < x_m$ by

$$M\binom{u_0, \dots, u_n}{x_0, \dots, x_m} := (u_j(x_i))_{i=0,\dots,n;j=0,\dots,m}.$$

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We say that $(u_0, \ldots, u_n)$ is a *blending system* if all functions u_i, $i = 0, \ldots, n$, are nonnegative and they sum up to 1. So, a system is blending if and only if all its collocation matrices are stochastic.

In computer-aided geometric design we use blending systems of functions to obtain curves

$$\gamma(x) = \sum_{i=0}^{n} P_i u_i(x), \quad x \in [a, b]. \tag{1}$$

The polygon $P_0 \cdots P_n$ formed by the ordered sequence of points $P_i \in \mathbb{R}^k$, $i = 0, \ldots, n$, is called the *control polygon*. It is well known that a curve (1) generated by a blending system satisfies the *convex hull property*, that is, the curve lies in the convex hull of its control polygon. A common property required by the designers is *linear precision*, which means that the parametric representation of a line segment with uniform speed is a curve γ generated by some control polygon. More precisely, $(u_0, \ldots, u_n)$ are said to have linear precision if there exists a sequence of real numbers $\alpha_0 < \cdots < \alpha_n$ such that

$$\sum_{i=0}^{n} \alpha_i u_i(x) = x. \tag{2}$$

Under this assumption, the graph $(x, u(x))$ of a given function $u(x) = \sum_{i=0}^{n} c_i u_i(x)$ is itself the curve $\gamma(x)$ in (1) generated by the control polygon $(\alpha_0, c_0) \cdots (\alpha_n, c_n)$. Both Bernstein polynomials and B-splines have the property of linear precision.

Another useful property is *convexity preservation*. We say that the system $(u_0, \ldots, u_n)$ is *convexity preserving* if any function

$$f(x) = \sum_{i=0}^{n} c_i u_i(x)$$

is convex whenever its control polygon $(\alpha_0, c_0) \cdots (\alpha_n, c_n)$ is convex, i.e. when

$$\delta c_i := \frac{c_i - c_{i-1}}{\alpha_i - \alpha_{i-1}} - \frac{c_{i-1} - c_{i-2}}{\alpha_{i-1} - \alpha_{i-2}} \geq 0, \qquad i = 2, \ldots, n. \tag{3}$$

A further usual property is that of *endpoint interpolation*, which is equivalent to the property that $\alpha_0 = a$ and $\alpha_n = b$, provided that linear precision holds. In Theorem 3.1 of [3] it was proved that a blending system $(u_0, \ldots, u_n)$ which has both linear precision and endpoint interpolation is convexity preserving if and only if all the functions

$$v_i(x) := \sum_{j=i}^{n} (\alpha_j - \alpha_{j-1}) u_j(x), \qquad i = 2, \ldots, n, \tag{4}$$

are convex.

We are interested in bounding the distance between a curve γ and its control polygon, and the simplest way to do this is to represent the polygon parametrically. A natural way to parameterize the polygon is as the parametric curve $\pi : [a, b] \to \mathbb{R}^k$, defined by

$$\pi(x) = \frac{\alpha_i - x}{\alpha_i - \alpha_{i-1}} P_{i-1} + \frac{x - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}} P_i, \quad \alpha_{i-1} \leq x \leq \alpha_i,$$

for $i = 1, \ldots, n$, and we will find ways to bound the distance $\|\gamma(x) - \pi(x)\|$, for an arbitrary norm $\|\cdot\|$ of $\mathbb{R}^k$. Note that we can further represent π using the basis functions

$$N_i(x) := \begin{cases} (x - \alpha_{i-1})/(\alpha_i - \alpha_{i-1}), & \text{if } \alpha_{i-1} \leq x \leq \alpha_i, \\ (\alpha_{i+1} - x)/(\alpha_{i+1} - \alpha_i), & \text{if } \alpha_i \leq x \leq \alpha_{i+1}, \\ 0, & \text{elsewhere,} \end{cases} \tag{5}$$

for $i = 0, \dots, n$, where $\alpha_{-1} := a$ and $\alpha_{n+1} := b$, so that

$$\pi(x) = \sum_{i=0}^{n} P_i N_i(x). \tag{6}$$

Thus the curve γ and its polygon π are two curves generated by the same control points, but different systems of functions, namely $(u_0, \dots, u_n)$, and $(N_0, \dots, N_n)$ respectively. This motivates the following proposition.

Proposition 1 *Let $(u_0, \dots, u_n)$ and $(\bar{u}_0, \dots, \bar{u}_n)$ be two blending systems defined on $[a, b]$ and let γ and $\bar{\gamma}$ be the two corresponding curves of the form* (1) *generated by given points $P_0, \dots, P_n$ in $\mathbb{R}^k$. Suppose further that $\sum_{i=0}^n \alpha_i u_i(x) = x$ and $\sum_{i=0}^n \alpha_i \bar{u}_i(x) = x$ for some common sequence of values $\alpha_0, \dots, \alpha_n$ with $\alpha_i \neq \alpha_{i-1}$, $i = 2, \dots, n$. Define*

$$v_i := \sum_{j=i}^{n} (\alpha_j - \alpha_{i-1}) u_j(x), \quad \bar{v}_i := \sum_{j=i}^{n} (\alpha_j - \alpha_{i-1}) \bar{u}_j(x), \qquad i = 2, \dots, n. \tag{7}$$

and

$$\delta P_i := \frac{P_i - P_{i-1}}{\alpha_i - \alpha_{i-1}} - \frac{P_{i-1} - P_{i-2}}{\alpha_{i-1} - \alpha_{i-2}}, \qquad i = 2, \dots, n. \tag{8}$$

Then for any sequence $d_2, \dots, d_n$ of positive numbers we have

(i)

$$\left\| \gamma(x) - \bar{\gamma}(x) \right\| \leq \max_{i \in \{2, \dots, n\}} \left\| \frac{\delta P_i}{d_i} \right\| \times \sum_{i=2}^{n} d_i |v_i(x) - \bar{v}_i(x)| \tag{9}$$

(ii) *If, in addition, $v_i(x) \geq \bar{v}_i(x)$, for all $x \in I$, then*

$$\left\| \gamma(x) - \bar{\gamma}(x) \right\| \leq \max_{i \in \{2, \dots, n\}} \left\| \frac{\delta P_i}{d_i} \right\| (\bar{s}(x) - s(x)), \tag{10}$$

where

$$s(x) = \sum_{i=0}^{n} \sigma_i u_i(x), \quad \bar{s}(x) = \sum_{i=0}^{n} \sigma_i \bar{u}_i(x), \tag{11}$$

and the sequence $\sigma_0, \dots, \sigma_n$ is uniquely defined by

$$\sigma_0 = \sigma_n = 0, \qquad \delta\sigma_i = -d_i, \quad i = 2, \dots, n. \tag{12}$$

PROOF. We use the fact that a sum $\sum_{i=0}^n A_i B_i$ with $\sum_{i=0}^n B_i = 0$ can be rewritten as

$$\sum_{i=0}^{n} A_i B_i = \sum_{i=1}^{n} (A_i - A_{i-1}) \sum_{k=i}^{n} B_k. \tag{13}$$

By (13), we can write

$$\sum_{i=0}^{n} P_i (u_i - \bar{u}_i) = \sum_{i=1}^{n} (P_i - P_{i-1}) \sum_{k=i}^{n} (u_k - \bar{u}_k) = \sum_{i=1}^{n} \frac{P_i - P_{i-1}}{\alpha_i - \alpha_{i-1}} \Big((\alpha_i - \alpha_{i-1}) \sum_{k=i}^{n} (u_k - \bar{u}_k) \Big).$$

We again apply (13) to the last formula and obtain

$$\sum_{i=0}^{n} P_i (u_i - \bar{u}_i) = \sum_{i=2}^{n} \delta P_i \sum_{j=i}^{n} (\alpha_j - \alpha_{j-1}) \sum_{k=j}^{n} (u_k - \bar{u}_k) = \sum_{i=2}^{n} \frac{\delta P_i}{d_i} d_i (v_i - \bar{v}_i).$$

Taking norms, formula (9) follows.

If $v_i(x) \geq \bar{v}_i(x)$ for all $x \in I$ we can write

$$\sum_{i=2}^n d_i|v_i(x) - \bar{v}_i(x)| = \Big(\beta(x-\alpha_0) - \sum_{i=2}^n d_i\bar{v}_i(x)\Big) - \Big(\beta(x-\alpha_0) - \sum_{i=2}^n d_i v_i(x)\Big),$$

where $\beta = (\alpha_n - \alpha_0)^{-1}\sum_{l=2}^n d_l(\alpha_n - \alpha_{l-1})$. Let us show that $s(x) = \beta(x-\alpha_0) - \sum_{i=2}^n d_i v_i(x)$. In fact,

$$\begin{aligned}\beta(x-\alpha_0) - \sum_{i=2}^n d_i v_i(x) &= \beta\sum_{i=1}^n(\alpha_i - \alpha_0)u_i(x) - \sum_{i=2}^n d_i \sum_{j=i}^n(\alpha_j - \alpha_{i-1})u_j \\ &= \beta\sum_{i=1}^n(\alpha_i-\alpha_0)u_i(x) - \sum_{i=2}^n\sum_{l=2}^i d_l(\alpha_i - \alpha_{l-1})u_i.\end{aligned}$$

Clearly the coefficient of u_0 is 0 and the coefficient of u_n is $\beta(\alpha_n - \alpha_0) - \sum_{l=2}^n d_l(\alpha_n - \alpha_{l-1}) = 0$. Now, it can be checked that $\delta[\beta(\alpha_i - \alpha_0) - \sum_{l=2}^i d_l(\alpha_i - \alpha_{l-1})] = -d_i$. Therefore $\beta(\alpha_i - \alpha_0) - \sum_{l=2}^i d_l(\alpha_i - \alpha_{l-1}) = \sigma_i$, $i = 0, \ldots, n$. Analogously $\bar{s}(x) = \beta(x-\alpha_0) - \sum_{i=2}^n d_i\bar{v}_i(x)$ and (ii) follows. ■

Remark 1 If we take in Proposition 1 $\bar{u}_i := N_i$ defined in (5), then we have $v_i(x) \geq \bar{v}_i(x)$. In fact, $(x, v_i(x))^T$ and $(x, \bar{v}_i(x))^T$ can be regarded as curves generated by two different blending systems and the same control polygon $W_0 \cdots W_n$, where

$$W_j := \begin{cases} (\alpha_j, 0)^T, & \text{if } j < i, \\ (\alpha_j, \alpha_j - \alpha_{i-1})^T, & \text{if } j \geq i. \end{cases}$$

The points $W_0, \ldots, W_{i-1}$ are collinear and $W_{i-1}, W_i, \ldots, W_n$ are also collinear, and then $W_0, \ldots, W_n$ lie in a two-sided polygon: $W_0W_{i-1}W_n$. The curve $(x, \bar{v}_i(x))^T$ describes the polygon $W_0W_{i-1}W_n$. On the other hand, the curve $(x, v_i(x))^T$ is included in the triangle with vertices W_0, W_{i-1}, W_n by the convex hull property. So we have seen that $v_i \geq \bar{v}_i$ and we can apply Proposition 1 (ii). ■

3. The distance of a blending curve to its control polygon

Let us first apply Proposition 1 to the problem of bounding the distance of a Bernstein-Bézier curve

$$b(x) = \sum_{i=0}^n P_i\binom{n}{i}(1-x)^{n-i}x^i, \quad x \in [0,1],$$

to its control polygon, parameterized by

$$\pi(x) = \sum_{i=0}^n P_i N_i(x), \quad x \in [0,1],$$

where $N_i(x)$ is defined by (5). In this case $\alpha_i = i/n$, $i = 0, \ldots, n$. We choose $d_i = 1/n$, $i = 2, \ldots, n$ and then $\delta P_i = n\Delta^2 P_{i-2}$, $i = 2, \ldots, n$, where $\Delta^2 P_i := P_{i+2} - 2P_{i+1} + P_i$ is the usual second order forward difference. We also have that the σ_i's defined in (12) take the form

$$\sigma_i = \frac{i(n-i)}{2n^2} = \frac{\alpha_i(1-\alpha_i)}{2}. \tag{14}$$

Let us define the Bernstein operator

$$B[f](x) := \sum_{i=0}^{n} f(\alpha_i) \binom{n}{i} (1-x)^{n-i} x^i, \quad x \in [0,1],$$

and the operator

$$P[f](x) := \sum_{i=0}^{n} f(\alpha_i) N_i(x), \quad x \in [0,1].$$

Due to Remark 1, we can apply the bound (10) of Proposition 1, which, from the form of the σ_i, reduces to

$$\|b(x) - \pi(x)\| \le n^2 \max_{i=0,\dots,n-2} \|\Delta^2 P_i\| \times \bigl(P\bigl[x(1-x)/2\bigr] - B\bigl[x(1-x)/2\bigr]\bigr). \tag{15}$$

Hence our problem is reduced to the problem of bounding the function

$$D(x) := P\bigl[x(1-x)/2\bigr] - B\bigl[x(1-x)/2\bigr].$$

Since $D(x)$ is piecewise convex, we can apply the maximum principle to each interval and deduce that

$$\max_{x\in[0,1]} D(x) = \max_{i=0,\dots,n} D(\alpha_i).$$

Since $B[x(1-x)/2] = \lambda x(1-x)/2$, with $\lambda = 1 - 1/n$, we may write

$$D(\alpha_i) = \alpha_i(1-\alpha_i)/2 - \lambda\alpha_i(1-\alpha_i)/2 = (1-\lambda)\alpha_i(1-\alpha_i)/2,$$

and so

$$\max_{x\in[0,1]} D(x) = (1-\lambda) \max_{i=0,\dots,n} \frac{\alpha_i(1-\alpha_i)}{2} \le \frac{1-\lambda}{8} = \frac{1}{8n}.$$

If the degree n is even, then this bound for $D(x)$ is attained at $x = \alpha_{n/2} = 1/2$. Otherwise for odd degree $n = 2k+1$, it is attained at $x = \alpha_k$, and then

$$\max_{x\in[0,1]} D(x) = \frac{1}{n} \max_{i=0,\dots,n} \frac{i(n-i)}{2n^2} = \frac{1}{2n}\frac{k(k+1)}{(2k+1)^2} < \frac{1}{8n}.$$

Thus through (15), we have shown that for any n,

$$\max \|b(x) - \pi(x)\| \le \frac{n}{8} \max_{i=0,\dots,n-2} \|\Delta^2 P_i\|. \tag{16}$$

This is the error bound established in [5]. However, the point is that the arguments we have used are more general and apply to types of curves other than just Bézier curves. Indeed, consider next the extension of the above analysis to curves generated by more general blending systems. Let $(u_0, \dots, u_n)$ be a blending system satisfying (2) and assume that the endpoint interpolation property holds. Given a control polygon $P_0 \cdots P_n$, we consider its parameterization π (6) and the curve γ (1) generated by the blending system. Taking into account (10) and Remark 1 we have

$$\bigl\|\gamma(x) - \pi(x)\bigr\| \le \max_{i\in\{2,\dots,n\}} \Bigl\|\frac{\delta P_i}{d_i}\Bigr\| \, \Bigl|\sum_{j=0}^{n} \sigma_j N_j(x) - \sum_{j=0}^{n} \sigma_j u_j(x)\Bigr|, \tag{17}$$

where the σ_j are defined in (12) and the functions N_j in (5). Defining

$$D(x) := \sum_{i=0}^{n} \sigma_i N_i(x) - \sum_{i=0}^{n} \sigma_i u_i(x) \tag{18}$$

and taking into account that $(\alpha_0, \sigma_0)^T \cdots (\alpha_n, \sigma_n)^T$ is a concave polygon, we derive from the convex hull property that

$$\sum_{i=0}^{n} \sigma_i u_i(x) \leq \sum_{i=0}^{n} \sigma_i N_i(x).$$

So (17) becomes

$$\left\|\gamma(x) - \pi(x)\right\| \leq \max_{i \in \{2,\ldots,n\}} \left\| \frac{\delta P_i}{d_i} \right\| D(x). \tag{19}$$

Formula (19) provides a pointwise bound for the distance between the points of a curve and the parameterization (6) of its control polygon. Let us interprete formula (19) from another point of view. The magnitude $\max_{i \in \{2,\ldots,n\}} \|\delta P_i / d_i\|$ can be seen as the maximum of a *weighted discrete curvature*. When the maximum weighted discrete curvature is zero, that is, $\delta P_i = 0$ for all i we say that the polygon is *linear*; in this case the curve and the control polygon coincide. When the maximum weighted discrete curvature increases the distance of the curve and the control polygon may grow. Formula (19) means that for nonlinear polygons the ratio

$$\frac{\left\|\gamma(x) - \pi(x)\right\|}{\max_{i \in \{2,\ldots,n\}} \|\delta P_i / d_i\|} \tag{20}$$

is bounded by the function $D(x)$ defined in (18). This definition shows that $D(x)$ is precisely the distance betwen the control polygon $\sigma_0 \cdots \sigma_n$ and the corresponding curve. Therefore, the ratio (20) equals $D(x)$ for the particular control polygon $\sigma_0 \cdots \sigma_n$. So we can write

$$\max_{P_0 \cdots P_n \in NL} \frac{\left\|\gamma(x) - \pi(x)\right\|}{\max_{i \in \{2,\ldots,n\}} \|\delta P_i / d_i\|} = D(x), \tag{21}$$

where NL stands for the set of nonlinear polygons.

The next result provides a global bound for convexity preserving systems.

Proposition 2 *Let $(u_0, \ldots, u_n)$ be a blending system satisfying* (2), *the endpoint interpolation and the convexity preserving properties. Let $\gamma(x)$, $\pi(x)$ be given by* (1) *and* (6). *Then*

$$\max_{x \in [a,b]} \left\|\gamma(x) - \pi(x)\right\| \leq \max_{i \in \{2,\ldots,n\}} \left\| \frac{\delta P_i}{d_i} \right\| \max_{i \in \{1,\ldots,n-1\}} D(\alpha_i), \tag{22}$$

where δP_i and $D(x)$ are defined in formulae (8) *and* (18).

PROOF. By (19), it is sufficient to show that

$$\max_{x \in [a,b]} D(x) = \max_{i \in \{0,\ldots,n\}} D(\alpha_i). \tag{23}$$

Since $(u_0, \ldots, u_n)$ is convexity preserving, the function $-\sum_{i=0}^{n} \sigma_i u_i(x)$ is convex and $D(x)$ is therefore a convex function in each subinterval $[\alpha_i, \alpha_{i+1}]$, $i = 0, \ldots, n-1$. Since a convex function defined on $[\alpha_i, \alpha_{i+1}]$ must attain its maximum at the endpoints α_i, α_{i+1}, (23) is confirmed. Finally, the end interpolation property implies $D(\alpha_0) = D(\alpha_n) = 0$. ■

Analogously to the derivation of (21) from (17) and (18), formula (22) can be interpreted as follows. The maximum value in (20) can be seen as the maximum of the discrete set of values $D(\alpha_1), \ldots, D(\alpha_{n-1})$, that is,

$$\max_{P_0 \cdots P_n \in NL} \frac{\max_{x \in [a,b]} \left\|\gamma(x) - \pi(x)\right\|}{\max_{i \in \{2,\ldots,n\}} \|\delta P_i / d_i\|} = \max_{i \in \{1,\ldots,n-1\}} D(\alpha_i), \tag{24}$$

It is not clear in general how to choose the weights d_i, $i = 2, \ldots, n$, in order to obtain sharp bounds. In the abscence of any special information on the basis, one could simply set $d_i = d$, for some constant $d > 0$. This was our choice for analyzing the Bernstein case at the beginning of this section. Another natural choice would be $d_i = \alpha_i - \alpha_{i-2}$, which makes $\delta P_i / d_i$ in (19) a second order divided difference.

4. Comparing different bases

This section is devoted to the problem of comparing the bounds of the distance of a curve to its control polygon corresponding to different blending bases. Let us start by measuring the distances for concave control polygons $\sigma_0 \cdots \sigma_n$, because these distances provide bounds associated to any other control polygon.

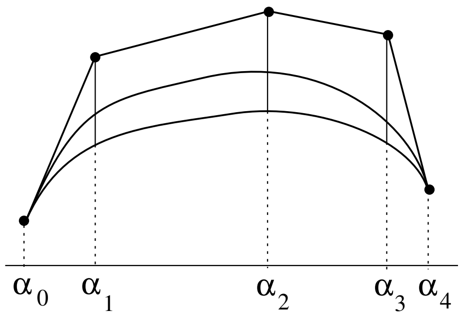

Figure 1. Comparing the distance of a concave polygon with two different curves

Proposition 3 *Let $\alpha_0 < \cdots < \alpha_n$ and let $K = (k_{ij})_{i,j=0,\dots,n}$ be a stochastic matrix such that $K\alpha = \alpha$, where $\alpha = (\alpha_0, \dots, \alpha_n)^T$.*

(i) *For any concave vector $\sigma = (\sigma_0, \dots, \sigma_n)^T$, that is, $\delta\sigma_i \le 0$, $i = 2, \dots, n$, we have $K\sigma \le \sigma$.*

(ii) *Let $(u_0, \dots, u_n)$, $(\bar{u}_0, \dots, \bar{u}_n)$ be two blending systems of functions on $[a, b]$ such that $\sum_{i=0}^n \alpha_i u_i(x) = x$ and $\sum_{i=0}^n \alpha_i \bar{u}_i(x) = x$. Let us assume that $M\binom{\bar{u}_0,\dots,\bar{u}_n}{\alpha_0,\dots,\alpha_n} = M\binom{u_0,\dots,u_n}{\alpha_0,\dots,\alpha_n}K$. Then*

$$\sum_{i=0}^n \sigma_i \bar{u}_i(\alpha_j) \le \sum_{i=0}^n \sigma_i u_i(\alpha_j) \le \sigma_j, \quad j = 0, \dots, n.$$

(iii) *Let $(u_0, \dots, u_n)$, $(\bar{u}_0, \dots, \bar{u}_n)$ be two blending bases of a space of functions defined on $[a, b]$ such that $\sum_{i=0}^n \alpha_i u_i(x) = x$, $\sum_{i=0}^n \alpha_i \bar{u}_i(x) = x$ and satisfying the endpoint interpolation property. Let us assume that $(\bar{u}_0, \dots, \bar{u}_n) = (u_0, \dots, u_n)K$. Then*

$$\sum_{i=0}^n \sigma_i \bar{u}_i \le \sum_{i=0}^n \sigma_i u_i \le \sum_{i=0}^n \sigma_i N_i.$$

PROOF. (i) Let us consider the control polygon $S_0 \cdots S_n$, $S_i = (\alpha_i, \sigma_i)^T$. Let $\bar{S}_i := \sum_{j=0}^n k_{ij} S_j$, $j = 0, \dots, n$. Since K is a stochastic matrix all the points $\bar{S}_i$ are in the convex hull of the polygon $S_0, \dots, S_n$. Using the fact that $K\alpha = \alpha$, we see that $\bar{S}_i = (\alpha_i, \bar{\sigma}_i)$, $i = 0, \dots, n$ where $(\bar{\sigma}_0, \dots, \bar{\sigma}_n)^T = K\sigma$. Since $S_0 \cdots S_n$ is a concave polygon, it follows that $\bar{\sigma}_i \le \sigma_i$.

(ii) Let $M := M\binom{u_0,\dots,u_n}{\alpha_0,\dots,\alpha_n}$. Then M is stochastic with $M\alpha = \alpha$. From (i) we obtain $K\sigma \le \sigma$. Taking into account that M is a nonnegative matrix $MK\sigma \le M\sigma$. Applying again (i), $M\sigma \le \sigma$. Taking components in $MK\sigma \le M\sigma \le \sigma$, (ii) follows.

(iii) For any points $x_0 < \cdots < x_n$ in $[a, b]$, let $M := M\binom{u_0,\dots,u_n}{x_0,\dots,x_n}$. Then M is an stochastic matrix. From (i) and the nonnegativity of the matrix M we see that, similarly as in the proof of (ii), we see that $MK\sigma \le M\sigma$. Taking components we obtain $\sum_{i=0}^n \sigma_i \bar{u}_i(x_j) \le \sum_{i=0}^n \sigma_i u_i(x_j)$. Since the points x_j are arbitrary we have $\sum_{i=0}^n \sigma_i \bar{u}_i \le \sum_{i=0}^n \sigma_i u_i$. On the other hand, taking into account that the curve $(x, \sum_{i=0}^n \sigma_i u_i(x))^T$ must be contained in the convex hull of its control polygon $S_0, \dots, S_n$ we have $\sum_{i=0}^n \sigma_i u_i(x) \le \sum_{i=0}^n \sigma_i N_i(x)$ for all $x \in [\alpha_0, \alpha_n]$. By the endpoint interpolation property $\alpha_0 = a$, $\alpha_n = b$. ∎

Let us derive some consequences of the previous result. If we have two different curves $\gamma(x)$, $\bar{\gamma}(x)$ generated by different systems $(u_0, \ldots, u_n)$, $(\bar{u}_0, \ldots, \bar{u}_n)$ with the same control polygon $\pi(x)$. Then we can bound the maximum distance of each of the curves to its common control polygon using formulae (21) and (24). Proposition 3 (ii) implies that the bound of the maximal distance corresponding to the curve γ is lower than the bound corresponding to $\bar{\gamma}$.

Theorem 1 *Let $\alpha_0 < \cdots < \alpha_n$ and $(u_0, \ldots, u_n)$, $(\bar{u}_0, \ldots, \bar{u}_n)$ be two blending convexity preserving systems of functions defined on $[a,b]$ such that $\sum_{i=0}^n \alpha_i u_i(x) = x$ and $\sum_{i=0}^n \alpha_i \bar{u}_i(x) = x$ satisfying the endpoint interpolation property. Let us assume that $M\begin{pmatrix} \bar{u}_0,\ldots,\bar{u}_n \\ \alpha_0,\ldots,\alpha_n \end{pmatrix} = M\begin{pmatrix} u_0,\ldots,u_n \\ \alpha_0,\ldots,\alpha_n \end{pmatrix} K$ for some stochastic matrix K. Then*

$$\max_{P_0\cdots P_n \in NL} \frac{\max_{x\in[a,b]} \|\gamma(x) - \pi(x)\|}{\max_{i\in\{2,\ldots,n\}} \|\delta P_i/d_i\|} \leq \max_{P_0\cdots P_n \in NL} \frac{\max_{x\in[a,b]} \|\bar{\gamma}(x) - \pi(x)\|}{\max_{i\in\{2,\ldots,n\}} \|\delta P_i/d_i\|}. \quad \Box$$

If we have two different bases $(u_0, \ldots, u_n)$, $(\bar{u}_0, \ldots, \bar{u}_n)$ of the same space related by an stochastic matrix K, we can bound the pointwise distance of each of the corresponding curves γ, $\bar{\gamma}$ to its common control polygon using formulae (19), (21). Proposition 3 (iii) implies that the pointwise distance bound corresponding to the curve γ is lower than the corresponding to $\bar{\gamma}$.

Theorem 2 *Let $\alpha_0 < \cdots < \alpha_n$ and $(u_0, \ldots, u_n)$, $(\bar{u}_0, \ldots, \bar{u}_n)$ be two blending bases of a space of functions defined on $[a,b]$ such that $\sum_{i=0}^n \alpha_i u_i(x) = x$ and $\sum_{i=0}^n \alpha_i \bar{u}_i(x) = x$ satisfying the endpoint interpolation property. Let us assume that $(\bar{u}_0, \ldots, \bar{u}_n) = (u_0, \ldots, u_n)K$, for some stochastic matrix K. Then*

$$\max_{P_0\cdots P_n \in NL} \frac{\|\gamma(x) - \pi(x)\|}{\max_{i\in\{2,\ldots,n\}} \|\delta P_i/d_i\|} \leq \max_{P_0\cdots P_n \in NL} \frac{\|\bar{\gamma}(x) - \pi(x)\|}{\max_{i\in\{2,\ldots,n\}} \|\delta P_i/d_i\|}, \quad \forall x \in [a,b]. \quad \Box$$

A common source of examples for the situation described in the hypotheses of the previous result is provided by totally positive bases of blending functions, usually called NTP bases ([4]).

Definition 1 *A basis of functions $(u_0, \ldots, u_n)$ is normalized totally positive (NTP) if it is blending and all its collocation matrices $M\begin{pmatrix} u_0,\ldots,u_n \\ x_0,\ldots,x_n \end{pmatrix}$ are totally positive, that is, all their minors are nonnegative.*

This concept corresponds to a usual requirement in computer-aided geometric design: the curves generated by a NTP basis from a control polygon $P_0 \cdots P_n$ preserve many shape properties of it. A space with a normalized totally positive basis has a special basis called a normalized B-basis, which has optimal shape preserving properties (see [1], [2]). A normalized B-basis $(b_0, \ldots, b_n)$ is the unique NTP basis of a space such that for any other NTP basis $(u_0, \ldots, u_n)$ the matrix K such that $(u_0, \ldots, u_n) = (b_0, \ldots, b_n)K$ is stochastic and totally positive. Examples of normalized B-bases with linear precision properties are the Bernstein basis of the space of polynomials and the B-spline basis of the corresponding spline space. As a consequence of Theorem 2 we may say that the Bézier curve is the closest to a given control polygon among all curves generated by shape preserving representations in the space of polynomials. A similar consequence can be derived for B-spline curves in the corresponding spline spaces.

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