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Multivector fields and connections. Applications to field theories

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Abstract. We study the integrability of multivector fields in a differentiable manifold, and the relation between some kinds of multivector fields in a jet bundle and connections in this bundle. As a particular case, integrable multivector fields and connections whose integral manifolds are holonomic sections are related. As an application, these results allow us to set the field equations for first-order classical field theories in several equivalent geometrical ways.

Campos multivectoriales y conexiones. Aplicaciones a las teorías de campos.

Resumen. Se estudia la integrabilidad de campos multivectoriales en variedades diferenciables y la relación entre algunos tipos de campos multivectoriales en un fibrado de jets y conexiones en dicho fibrado. Como caso particular se relacionan los campos multivectoriales integrables y las conexiones cuyas secciones integrales son holonómicas. Como aplicación de todo ello, estos resultados permiten escribir las ecuaciones de campo de las teorías clásicas de campos de primer orden en varias formas equivalentes.

1. Introduction

As it is well-known, the geometric description of systems of ordinary differential equations involve vector fields in differentiable manifolds in general. For systems of partial differental equations the analogous description can be made by using different geometrical objects. For instance, we can take *multivector fields* in differentiable manifolds in general (for instance, *Hamiltonian multivector fields* in *multisymplectic manifolds* [2], [1], [9]); or also *connections* in jet bundles [13]. In both cases, their contraction with differential forms gives the intrinsic formulation of a system of partial differential equations locally describing the corresponding multivector field or connection.

Therefore, two questions arise as a matter of interest. First, the analysis of the integrability of such equations; that is, of the corresponding multivector fields and connections. Second, the study on the relation between multivector fields and connections (in jet bundles). These questions constitute the first goal in this work.

As an application, in the jet bundle description of classical field theories, the field equations are partial differential equations which are usually obtained using the multisymplectic form in order to characterize the

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critical sections which are solutions of some suitable variational problem [7], [8]. Nevertheless, there are also other attempts in order to obtain these equations in a more geometric-algebraic way (in a similar form as in mechanics, when use is made of the contraction of vector fields with the Lagrangian and the Hamiltonian forms). So, we can use *Ehresmann connections* in a jet bundle [11], [12], [13] or, what is equivalent, their associated *jet fields* [5]. Moreover, an approach to set the field equations in the Hamiltonian formalism using multivector fields is given in [10] (see also [8] as a first reference on the use of multivector fields in the realm of field theories). The final part of this work is devoted to show these procedures.

This article is based on a recent work of the authors [6], which we give as the main reference for the proofs and details of the constructions that we present here.

All maps are C^∞. All manifolds are real, paracompact, connected and C^∞. Sum over repeated indices is understood.

2. Multivector fields in differentiable manifolds

Let E be a n-dimensional differentiable manifold. The sections of $\Lambda^m(\mathrm{T}E)$ are the m-*multivector fields* in E. We will denote by $\mathcal{X}^m(E)$ the set of m-multivector fields in E. In general, for every $Y \in \mathcal{X}^m(E)$ and $p \in E$, there exists an open neighbourhood $U_p \subset E$ and $Y_1, \ldots, Y_r \in \mathcal{X}(U_p)$ such that

$$Y \underset{U_p}{=} \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} Y_{i_1} \wedge \ldots \wedge Y_{i_m} \tag{1}$$

with $f^{i_1 \ldots i_m} \in \mathrm{C}^\infty(U_p)$ and $m \leq r \leq \dim E$. Then, every multivector field $Y \in \mathcal{X}^m(E)$ defines an antiderivation of degree $-m$ of the exterior algebra $\Omega(E)$ as follows: if $\Omega \in \Omega^k(E)$ we have

$$\begin{aligned} i(Y)\Omega &\underset{U_p}{=} \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m}\, i(Y_1 \wedge \ldots \wedge Y_m)\Omega \\ &= \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m}\, i(Y_1) \ldots i(Y_m)\Omega \end{aligned}$$

if $k \geq m$, and it is obviously equal to zero if $k < m$.

Definition 1 *A m-multivector field $Y \in \mathcal{X}^m(E)$ is said to be* decomposable *iff there are $Y_1, \ldots, Y_m \in \mathcal{X}(E)$ such that $Y = Y_1 \wedge \ldots \wedge Y_m$.*

The multivector field $Y \in \mathcal{X}^m(E)$ is said to be locally decomposable *iff, for every $p \in E$, there exist an open neighbourhood $U_p \subset E$ and $Y_1, \ldots, Y_m \in \mathcal{X}(U_p)$ such that $Y \underset{U_p}{=} Y_1 \wedge \ldots \wedge Y_m$.*

Let D be a m-dimensional distribution in E, that is, a m-dimensional subbundle of $\mathrm{T}E$. Obviously sections of $\Lambda^m D$ are m-multivector fields in E. The existence of a non-vanishing global section of $\Lambda^m D$ is equivalent to the orientability of D. We set:

Definition 2 *A non-vanishing m-multivector field $Y \in \mathcal{X}^m(E)$ and a m-dimensional distribution $D \subset \mathrm{T}E$ are* locally associated *iff there exists a connected open set $U \subseteq E$ such that $Y|_U$ is a section of $\Lambda^m D|_U$.*

If $Y, Y' \in \mathcal{X}^m(E)$ are non-vanishing multivector fields locally associated, on the same connected open set U, with the same distribution D, then there exists a non-vanishing function $f \in \mathrm{C}^\infty(U)$ such that $Y' \underset{U}{=} fY$. This fact defines an equivalence relation in the set of non-vanishing m-multivector fields in E, whose equivalence classes will be denoted by $\{Y\}_U$. Therefore:

Theorem 1 *There is a bijective correspondence between the set of orientable m-dimensional distributions D in $\mathrm{T}E$ and the set of the equivalence classes $\{Y\}_E$ of non-vanishing, locally decomposable m-multivector fields in E.*

If $Y \in \mathcal{X}^m(E)$ is a non-vanishing m-multivector field and $U \subseteq E$ is a connected open set, the distribution associated with the class $\{Y\}_U$ will be denoted by $\mathcal{D}_U(Y)$. If $U = E$ we will write simply $\mathcal{D}(Y)$.

Definition 3 *Let $Y \in \mathcal{X}^m(E)$ a multivector field.*

1. *A submanifold $S \hookrightarrow E$, with* $\dim S = m$*, is said to be an* integral manifold *of Y iff, for every $p \in S$, Y_p spans $\Lambda^m \mathrm{T}_p S$.*
2. *Y is said to be an* integrable multivector field *on an open set $U \subseteq E$ iff, for every $p \in U$, there exists an integral manifold $S \hookrightarrow U$ of Y, with $p \in S$.*

   Y is said to be integrable *iff it is integrable in E.*

Obviously, every integrable multivector field is non-vanishing.

Definition 4 *Let $Y \in \mathcal{X}^m(E)$ be a multivector field.*

1. *Y is said to be* involutive *on a connected open set $U \subseteq E$ iff it is locally decomposable in U and its associated distribution $\mathcal{D}_U(Y)$ is involutive.*
2. *Y is said to be* involutive *iff it is involutive on E.*
3. *Y is said to be* locally involutive *iff, for every $p \in E$, there is a connected open neighbourhood $U_p \subset E$ such that Y is involutive on U_p.*

The *Frobenius' theorem* can be reformulated in this context as follows:

Proposition 1 *A non-vanishing and locally decomposable multivector field $Y \in \mathcal{X}^m(E)$ is integrable on a connected open set $U \subseteq E$ if, and only if, it is involutive on U.*

Note that if a multivector field Y is integrable, then so is every other in its equivalence class $\{Y\}$, and all of them have the same integral manifolds.

Definition 5 *A multivector field $Y \in \mathcal{X}^m(E)$ is said to be a* dynamical multivector field *iff*

1. *Y is integrable.*
2. *For every $p \in E$, there exist an open neighbourhood $U_p \subset E$ and $Y_1, \ldots, Y_m \in \mathcal{X}(U_p)$ such that $[Y_\mu, Y_\nu] = 0$, for every pair Y_μ, Y_ν, and $Y|_{U_p} = Y_1 \wedge \ldots \wedge Y_m$.*

In this case, using the local one-parameter groups of diffeomorphisms of Y_μ around $p \in E$, we can construct a map τ which is called the *m-flow* associated with the multivector field Y (see [4] for the terminology and notation). Then we have that:

Proposition 2 *Let $\{Y\}$ be a class of integrable m-multivector fields. Then there is a representative Y of the class which is a dynamical multivector field.*

The results here obtained can be summarized in the following table:

$$
\begin{array}{ccccc}
\begin{array}{c}\text{Integrable}\\ \text{Orientable}\\ \text{distribution}\end{array} & \Leftrightarrow & \begin{array}{c}\text{Integrable m.v.f.}\\ (class)\end{array} & \left\{\begin{array}{c}\left.\left\{\begin{array}{c}\text{Non-vanish. m.v.f.}\\ \text{Loc. decom. m.v.f.}\end{array}\right.\right\}(class)\\ \text{Involutive m.v.f. } (class)\end{array}\right. & \Leftrightarrow \quad \begin{array}{c}\text{Orientable}\\ \text{distribution}\end{array}\\
 & & \downarrow & & \\
 & & \begin{array}{c}\text{Dynamical m.v.f.}\\ (representative)\end{array} & &
\end{array}
$$

3. Connections in first-order jet bundles

Let $\pi\colon E \to M$ be a fiber bundle, $\pi^1\colon J^1E \to E$ the corresponding first-order jet bundle, and $\bar{\pi}^1 = \pi \circ \pi^1\colon J^1E \longrightarrow M$. Denote $\mathrm{V}(\pi) = \mathrm{Ker}\,\mathrm{T}\pi$, $\mathrm{V}(\pi^1) = \mathrm{Ker}\,\mathrm{T}\pi^1$, and $\mathcal{X}^{\mathrm{V}(\pi)}(E)$, $\mathcal{X}^{\mathrm{V}(\pi^1)}(J^1E)$ the corresponding sections or vertical vector fields. Let (x^μ, y^A, v^A_μ) be a natural local system of coordinates in J^1E ($\mu = 1,\ldots,m$, $A = 1,\ldots,N$).

Definition 6 *A* connection *in J^1E is one of the following equivalent elements:*

1. *A global section of $\pi^1\colon J^1E \to E$, (that is, a mapping $\Psi\colon E \to J^1E$ such that $\pi^1 \circ \Psi = \mathrm{Id}_E$). It is called a* jet field.
2. *A subbundle $\mathrm{H}(E)$ of $\mathrm{T}E$ such that $\mathrm{T}E = \mathrm{V}(\pi) \oplus \mathrm{H}(E)$. It is called a* horizontal subbundle*, and it is also denoted $\mathcal{D}(\Psi)$ when is considered as the distribution associated with Ψ.*
3. *A π-semibasic 1-form ∇ on E with values in $\mathrm{T}E$, such that $\nabla^*\alpha = \alpha$, for every π-semibasic form $\alpha \in \Omega^1(E)$. It is called the* connection form *or* Ehresmann connection.

Let (x^μ, y^A) be a local system of coordinates in an open set $U \subset E$, then the local expressions of these elements are

$$\begin{aligned} \Psi &= (x^\mu, y^A, \Gamma^A_\rho(x^\mu, y^A)) \\ \mathrm{H}(E) &= \mathrm{span}\left\{\frac{\partial}{\partial x^\mu} + \Gamma^A_\mu \frac{\partial}{\partial y^A}\right\} \\ \nabla &= \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^A_\mu \frac{\partial}{\partial y^A}\right) \end{aligned} \tag{2}$$

Definition 7 *Let $\Psi\colon E \to J^1E$ be a jet field. A section $\phi\colon M \to E$ is said to be an* integral section *of Ψ iff $\Psi \circ \phi = j^1\phi$ (where $j^1\phi\colon M \to J^1E$ denotes the canonical lifting of ϕ). Ψ is said to be an* integrable jet field *iff it admits integral sections through every point of E.*

If (x^μ, y^A, v^A_μ) is a natural local system in J^1E and, in this system, $\Psi = (x^\mu, y^A, \Gamma^A_\rho(x^\mu, y^A))$ and $\phi = (x^\mu, f^A(x^\nu))$, then ϕ is an integral section of Ψ if, and only if, ϕ is a solution of the following system of partial differential equations

$$\frac{\partial f^A}{\partial x^\mu} = \Gamma^A_\mu \circ \phi \tag{3}$$

Proposition 3 *A jet field $\Psi\colon E \to J^1E$ is integrable if, and only if, $\mathcal{D}(\Psi)$ is an involutive distribution (that is, $\mathcal{D}(\Psi)$ is integrable).*

Now consider the second jet bundle $J^1J^1E \xrightarrow{\pi^1_1} J^1E \xrightarrow{\bar{\pi}^1} M$. Let $\mathcal{Y}\colon J^1E \to J^1J^1E$ be a jet field in J^1J^1E. which is associated with a connection form ∇ on J^1E and a horizontal m-subbundle $\mathrm{H}(J^1E)$. If $(x^\mu, y^A, v^A_\mu, a^A_\rho, b^A_{\rho\mu})$ is a natural local system in J^1J^1E, we have the following local expressions for these elements

$$\begin{aligned} \mathcal{Y} &= (x^\mu, y^A, v^A_\mu, F^A_\rho(x^\mu, y^A, v^A_\mu), G^A_{\nu\rho}(x^\mu, y^A, v^A_\mu)) \\ \nabla &= \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + F^A_\mu \frac{\partial}{\partial y^A} + G^A_{\mu\rho} \frac{\partial}{\partial v^A_\rho}\right) \\ \mathrm{H}(J^1E) &= \mathrm{span}\left\{\frac{\partial}{\partial x^\mu} + F^A_\mu \frac{\partial}{\partial y^A} + G^A_{\mu\rho} \frac{\partial}{\partial v^A_\rho}\right\} \end{aligned} \tag{4}$$

If $\mathcal{Y}\colon J^1E \to J^1J^1E$ is a jet field then a section $\psi\colon M \to J^1E$ is said to be an *integral section* of $\mathcal{Y}$ iff $\mathcal{Y} \circ \psi = j^1\psi$. $\mathcal{Y}$ is said to be an *integrable jet field* iff it admits integral sections. In a natural local system of coordinates in J^1J^1E, $\psi = (x^\mu, f^A(x^\nu), g^A_\mu(x^\nu))$ is an integral section of $\mathcal{Y}$ if, and only if, ψ is a solution of the following system of differential equations

$$\frac{\partial f^A}{\partial x^\mu} = F^A_\mu \circ \psi \qquad \frac{\partial g^A_\rho}{\partial x^\mu} = G^A_{\rho\mu} \circ \psi \tag{5}$$

Now we want to characterize the integrable jet fields in J^1J^1E whose integral sections are *holonomic*, that is, canonical prolongations of sections $\phi\colon M \to E$. Let $\mathbf{y} \in J^1J^1E$ with $\mathbf{y} \overset{\pi^1_1}{\mapsto} \bar{y} \overset{\pi^1}{\mapsto} y \overset{\pi}{\mapsto} x$, and $\psi\colon M \to J^1E$ a representative of $\mathbf{y}$. Consider now the section $\phi = \pi^1 \circ \psi\colon M \to E$ and let $j^1\phi$ be its canonical prolongation. Then we can define another natural projection

$$\begin{array}{cccc} j^1\pi^1 & : & J^1J^1E & \longrightarrow & J^1E \\ & & \mathbf{y} & \mapsto & j^1(\pi^1 \circ \psi)(\bar{\pi}^1_1(\mathbf{y})) \\ & & (x^\mu, y^A, v^A_\mu, a^A_\rho, b^A_{\mu\rho}) & \mapsto & (x^\mu, y^A, a^A_\rho) \end{array}$$

Definition 8 *A jet field $\mathcal{Y}\colon J^1E \to J^1J^1E$ is said to be a* Second Order Partial Differential Equation (SOPDE)*, or also that it verifies the* SOPDE condition*, iff it is a section of the projection $j^1\pi^1$ or, what is equivalent, $j^1\pi^1 \circ \mathcal{Y} = \mathrm{Id}_{J^1E}$.*

Proposition 4 *An integrable jet field $\mathcal{Y}\colon J^1E \to J^1J^1E$ is a SOPDE if, and only if, its integral sections are canonical prolongations of sections $\phi\colon M \to E$.*

Definition 9 *A jet field $\mathcal{Y}\colon J^1E \to J^1J^1E$ is called* holonomic *iff it is integrable and SOPDE.*

In coordinates, the condition $j^1\pi^1 \circ \mathcal{Y} = \mathrm{Id}_{J^1E}$ is expressed in the following way: the jet field $\mathcal{Y} = (x^\mu, y^A, v^A_\mu, F^A_\rho, G^A_{\nu\rho})$ is a SOPDE if, and only if, $F^A_\rho = v^A_\rho$. On the other hand, if $\mathcal{Y}$ is a SOPDE then $j^1\phi = \left(x^\mu, f^A, \dfrac{\partial f^A}{\partial x^\mu}\right)$ is an integral section of it if, and only if ϕ is solution of the following system of (second order) partial differential equations

$$G^A_{\nu\rho}\left(x^\mu, f^A, \frac{\partial f^A}{\partial x^\mu}\right) = \frac{\partial^2 f^A}{\partial x^\rho \partial x^\nu} \tag{6}$$

which justifies the nomenclature.

4. Multivector fields and jet fields in jet bundles

Let $J^1E \xrightarrow{\pi^1} E \xrightarrow{\pi} M$. The multivector fields in J^1E which we are going to be interested in are those verifying the condition of *transversality* with respect to the projection π. They can be characterized as follows:

Definition 10 *A non-vanishing multivector field $Y \in \mathcal{X}^m(E)$ is said to be* transverse to the projection π *(or π-*transverse*) iff, at every point $y \in E$, $(i(Y)(\pi^*\omega))_y \neq 0$,for every $\omega \in \Omega^m(M)$ with $\omega(\pi(y)) \neq 0$.*

Theorem 2 *Let $Y \in \mathcal{X}^m(E)$ be integrable. Then Y is π-transverse if, and only if, its integral manifolds are local sections of $\pi\colon E \to M$.*

Definition 11 *A jet field $\Psi\colon E \to J^1E$ is said to be* orientable *iff $\mathcal{D}(\Psi)$ is an orientable distribution on E.*

The relation between multivector fields and jet fields is:

Theorem 3 *Every orientable jet field $\Psi\colon E \to J^1E$ defines a class of locally decomposable and π-transverse multivector fields $\{Y\} \subset \mathcal{X}^m(E)$, and conversely. They are characterized by the fact that $\mathcal{D}(\Psi) = \mathcal{D}(Y)$.*

In addition, the orientable jet field Ψ is integrable if, and only if, so is Y, for every $Y \in \{Y\}$.

As is obvious, recalling the local expression (2), we obtain the following local expression for a representative multivector field Y of the class $\{Y\}$ associated to the jet vector field Ψ

$$Y \equiv \bigwedge_{\mu=1}^{m} Y_\mu = \bigwedge_{\mu=1}^{m}\left(\frac{\partial}{\partial x^\mu} + \Gamma_\mu^A \frac{\partial}{\partial y^A}\right)$$

and, in the same way as stated in section 3., $\phi = (x^\mu, f^\mu(x^\nu))$ is an integral section of Y if, and only if, ϕ is a solution of the system of partial differential equations (3).

Next we apply the considerations in the above section to the second jet bundle $J^1J^1E \to J^1E \to M$. So we have a bijective correspondence between the set of m-dimensional orientable distributions D in J^1E and the set of equivalence classes $\{X\}$ of locally decomposable m-multivector fields in J^1E. Then a locally decomposable multivector field $X \in \mathcal{X}^m(J^1E)$ is *integrable* iff the distribution $\mathcal{D}(X)$ is also. Thus, Theorem 2 adapted to the present situation states that, if $X \in \mathcal{X}^m(J^1E)$ is an integrable multivector field, X is $\bar{\pi}^1$-transverse if, and only if, its integral manifolds are local sections of $\bar{\pi}^1 : J^1E \to M$.

The following step is to characterize the integrable multivector fields in J^1E whose integral manifolds are canonical prolongations of sections of π. In order to achieve this, define the vector bundle projection $\kappa\colon \mathrm{T}J^1E \to \mathrm{T}E$ by

$$\kappa(\bar{y}, \bar{u}) := \mathrm{T}_{\bar{\pi}^1(\bar{y})}\phi(\mathrm{T}_{\bar{y}}\bar{\pi}^1(\bar{u}))$$

where $(\bar{y}, \bar{u}) \in \mathrm{T}J^1E$ and $\phi \in \bar{y}$. If $(W; x^\mu, y^A, v_\mu^A)$ is a local natural chart in J^1E and $\bar{y} \in J^1E$ with $\bar{y} \xrightarrow{\pi^1} y \xrightarrow{\pi} x$, then

$$\kappa\left(\alpha^\mu \frac{\partial}{\partial x^\mu}\Big|_{\bar{y}} + \beta^A \frac{\partial}{\partial y^A}\Big|_{\bar{y}} + \lambda_\mu^A \frac{\partial}{\partial v_\mu^A}\Big|_{\bar{y}}\right) = \alpha^\mu \frac{\partial}{\partial x^\mu}\Big|_{y} + v_\mu^A(\bar{y})\alpha^\mu \frac{\partial}{\partial y^A}\Big|_{y}$$

This projection is extended in a natural way to $\Lambda^m\mathrm{T}J^1E$, and so we have the following diagram

$$\begin{tikzcd}
\mathrm{T}J^1E \arrow[r, "\tau_{J^1E}"] \arrow[d, shift right, "\kappa"'] \arrow[d, shift left, "\mathrm{T}\pi^1"] & J^1E \arrow[r, shift left, "X"'] \arrow[d, "\pi^1"] & \Lambda^m\mathrm{T}J^1E \arrow[l, shift left, "\Lambda^m\tau_{J^1E}"'] \arrow[d, shift right, "\Lambda^m\mathrm{T}\pi^1"'] \arrow[d, shift left, "\Lambda^m\kappa"] \\
\mathrm{T}E \arrow[r, "\tau_E"'] & E & \Lambda^m\mathrm{T}E \arrow[l, "\Lambda^m\tau_E"]
\end{tikzcd}$$

Definition 12 *A $\bar{\pi}^1$-transverse multivector field $X \in \mathcal{X}^m(J^1E)$ verifies the* SOPDE condition *(we will say also that it is a* SOPDE*) iff $\Lambda^m\kappa \circ X = \Lambda^m\mathrm{T}\pi^1 \circ X$.*

Proposition 5 *Let $X \in \mathcal{X}^m(J^1E)$ be $\bar{\pi}^1$-transverse and locally decomposable. Then the following conditions are equivalent:*

1. *X is a SOPDE.*
2. *$i(\theta)X = 0$ (where $\theta \in \Omega^1(J^1E, \pi^{1*}\mathrm{V}(\pi))$ is the structure canonical form in J^1E [7]).*

3. *If $(W; x^\mu, y^A, v^A_\mu)$ is a natural chart in J^1E, then the local expression of X is*

$$X \equiv \bigwedge_{\mu=1}^{m} X_\mu = \bigwedge_{\mu=1}^{m} f^\mu \left(\frac{\partial}{\partial x^\mu} + v^A_\mu \frac{\partial}{\partial y^A} + G^A_{\rho\mu} \frac{\partial}{\partial v^A_\rho} \right)$$

   where f^μ are non-vanishing functions.

The relation between integrable and SOPDE multivector fields in J^1E is:

Theorem 4 *Let $X \in \mathcal{X}^m(J^1E)$ be $\bar{\pi}^1$-transverse and integrable. X is a SOPDE if, and only if, its integral manifolds are canonical prolongations of sections $\phi \colon M \to E$, that is, sections $\psi \colon M \to J^1E$ such that $j^1(\pi^1 \circ \psi) = \psi$.*

Definition 13 *A multivector field $X \in \mathcal{X}^m(J^1E)$ is said to be* holonomic *iff:*

1. *X is integrable.*
2. *X is a SOPDE.*

Finally, we have:

Theorem 5 *Every orientable jet field $\mathcal{Y} \colon J^1E \to J^1J^1E$ defines a class of locally decomposable and $\bar{\pi}^1$-transverse multivector fields $\{X\} \subset \mathcal{X}^m(J^1E)$, and conversely. They are characterized by the fact that $\mathcal{D}(\mathcal{Y}) = \mathcal{D}(X)$. In addition:*

1. *The jet field $\mathcal{Y}$ is integrable if, and only if, so is X, for every $X \in \{X\}$.*
2. *The jet field $\mathcal{Y}$ is SOPDE if, and only if, so is X, for every $X \in \{X\}$.*
3. *The jet field $\mathcal{Y}$ is holonomic if, and only if, so is X, for every $X \in \{X\}$.*

From the local expression (4), we obtain the following local expression for a representative multivector field X of the class $\{X\}$ associated to the jet vector field $\mathcal{Y}$

$$X \equiv \bigwedge_{\mu=1}^{m} X_\mu = \bigwedge_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu} + F^A_\mu \frac{\partial}{\partial y^A} + G^A_{\rho\mu} \frac{\partial}{\partial v^A_\rho} \right)$$

and, if $\mathcal{Y}$ is a SOPDE, then a representative multivector field of $\{X\}$ can be chosen such that $F^A_\mu = v^A_\mu$ (see Proposition 5).

So we have the following summarizing scheme:

$$\begin{array}{c}\text{Holonomic m.v.f.}\\ (class)\end{array}
\left\{\begin{array}{l}
\left\{\begin{array}{c}\bar{\pi}^1\text{-Transverse m.v.f.}\\ \text{Involutive m.v.f.}\end{array}\right\}(class) \quad \Leftrightarrow \quad \left\{\begin{array}{c}\text{Orientable j.f.}\\ \text{Integrable j.f.}\end{array}\right. \\
\left\{\begin{array}{c}\text{Loc. decom. m.v.f.}\\ \text{SOPDE m.v.f.}\end{array}\right\}(class) \quad \Leftrightarrow \quad \left\{\begin{array}{c}\text{SOPDE j.f.}\\ \text{Orientable j.f.}\end{array}\right.
\end{array}\right.
\left.\begin{array}{c}\\ \\ \\ \end{array}\right\} \begin{array}{c}\text{Holonomic}\\ \text{j.f.}\end{array}$$

5. The field equations for classical fields theories

At this point, our main goal is to show that the Lagrangian and Hamiltonian formalisms for field theories can also be established using jet fields, their associated connection forms or the equivalent multivector fields. In order to achive some of these results we must define an action of jet fields on differential forms [5].

Let $\mathcal{Y}\colon J^1E \to J^1J^1E$ be a jet field. A map $\bar{\mathcal{Y}}\colon \mathcal{X}(M) \to \mathcal{X}(J^1E)$ can be defined in the following way: let $Z \in \mathcal{X}(M)$, then $\bar{\mathcal{Y}}(Z) \in \mathcal{X}(J^1E)$ is the vector field defined as

$$\bar{\mathcal{Y}}(Z)(\bar{y}) := (\mathrm{T}_{\bar{\pi}^1(\bar{y})}\psi)(Z_{\bar{\pi}^1(\bar{y})})$$

for every $\bar{y} \in J^1E$ and $\psi \in \mathcal{Y}(\bar{y})$. Its local expression is

$$\bar{\mathcal{Y}}\left(f^\mu \frac{\partial}{\partial x^\mu}\right) = f^\mu \left(\frac{\partial}{\partial x^\mu} + F^A_\mu \frac{\partial}{\partial y^A} + G^A_{\rho\mu}\frac{\partial}{\partial v^A_\rho}\right)$$

This map induces an action of $\mathcal{Y}$ on $\Omega(J^1E)$. In fact, let $\xi \in \Omega^{m+j}(J^1E)$, with $j \geq 0$, we define $i(\mathcal{Y})\xi\colon \mathcal{X}(M)\times \overset{(m)}{\dots} \times\mathcal{X}(M) \longrightarrow \Omega^j(J^1E)$ given by

$$((i(\mathcal{Y})\xi)(Z_1,\dots,Z_m))(\bar{y};X_1,\dots,X_j) := \xi(\bar{y};\bar{\mathcal{Y}}(Z_1),\dots,\bar{\mathcal{Y}}(Z_m),X_1,\dots,X_j)$$

for $Z_1,\dots,Z_m \in \mathcal{X}(M)$ and $X_1,\dots,X_j \in \mathcal{X}(J^1E)$. It is a $\mathrm{C}^\infty(M)$-linear and alternate map on the vector fields $Z_1,\dots,Z_m$.

The $\mathrm{C}^\infty(J^1E)$-linear map $i(\mathcal{Y})\xi$, extended by zero to forms of degree $p < m$, is called the *inner contraction* of the jet field $\mathcal{Y}$ and the differential form ξ.

5.1. Lagrangian formalism

We assume that M is a m-dimensional oriented manifold and $\omega \in \Omega^m(M)$ is the volume m-form on M. A *Lagrangian density* is a $\bar{\pi}^1$-semibasic m-form on J^1E with respect to the projection $\bar{\pi}^1$. A Lagrangian density is usually written as $\mathcal{L} = \pounds(\bar{\pi}^{1^*}\omega)$, where $\pounds \in \mathrm{C}^\infty(J^1E)$ is the *Lagrangian function* associated with $\mathcal{L}$ and ω. In a natural system of coordinates this expression is $\mathcal{L} = \pounds(x^\mu, y^A, v^A_\mu)\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m$. Then, using the canonical structures of the bundle J^1E, we can construct the *Poincaré-Cartan* $(m+1)$*-form*, $\Omega_\mathcal{L} \in \Omega^{m+1}(J^1E)$ associated with the Lagrangian density $\mathcal{L}$ [5], [7]; whose local expression is:

$$\Omega_\mathcal{L} = -\mathrm{d}\left(\frac{\partial\pounds}{\partial v^A_\mu}\right) \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}\left(\frac{\partial\pounds}{\partial v^A_\mu}v^A_\mu - \pounds\right) \wedge \mathrm{d}^m x$$

(where $\mathrm{d}^{m-1}x_\mu \equiv i\left(\dfrac{\partial}{\partial x^\mu}\right)\mathrm{d}^m x$) In addition, a variational problem (called the *Hamilton principle*) can be posed from the Lagrangian density $\mathcal{L}$, and the (compact-supported) critical sections of this variational problem can be characterized as follows:

Theorem 6 *The critical sections of the Lagrangian variational problem are holonomic sections $j^1\phi\colon M \to J^1E$ which satisfy the following equivalent conditions*

1. $(j^1\phi)^*\, i(X)\Omega_\mathcal{L} = 0$ *, for every $X \in \mathcal{X}(J^1E)$.*
2. *The coordinates of ϕ satisfy the* Euler-Lagrange equations:

$$\left.\frac{\partial\pounds}{\partial y^A}\right|_{j^1\phi} - \frac{\partial}{\partial x^\mu}\left.\frac{\partial\pounds}{\partial v^A_\mu}\right|_{j^1\phi} = 0$$

3. *They are the integral sections of a holonomic jet field $\mathcal{Y}_\mathcal{L}\colon J^1E \to J^1J^1E$, such that $i(\mathcal{Y}_\mathcal{L})\Omega_\mathcal{L} = 0$.*

4. *They are the integral sections of a holonomic jet field* $\mathcal{Y}_\mathcal{L}\colon J^1E \to J^1J^1E$, *such that* $i(\nabla_\mathcal{L})\Omega_\mathcal{L} = (m-1)\Omega_\mathcal{L}$; *where* $\nabla_\mathcal{L}$ *is the associated connection form.*

5. *They are the integral sections of a class of locally decomposable non-vanishing, holonomic multivector fields* $\{X_\mathcal{L}\} \subset \mathcal{X}^m(J^1E)$, *such that* $i(X_\mathcal{L})\Omega_\mathcal{L} = 0$.

The conditions in the last item are the version of the Euler-Lagrange equations in terms of multivector fields.

5.2. Hamiltonian formalism

In order to establish the Hamiltonian formalism for field theories, first we need to define a *dual bundle* of J^1E. The choice of such a dual bundle is not unique in the existing bibliography. We will take the definition given in [3] for this bundle, and we refer also to this work for the details on the construction of the Hamiltonian form.

In this way, the so-called *multimomentum dual bundle* is

$$J^{1*}E := \Lambda_1^m \mathrm{T}^*E/\Lambda_0^m \mathrm{T}^*E$$

where $\Lambda_1^m \mathrm{T}^*E$ is the bundle of m-forms on E, vanishing by the action of two π-vertical vector fields, and $\Lambda_0^m \mathrm{T}^*E$ is the bundle of π-semibasic m-forms in E. $J^{1*}E \xrightarrow{\tau^1} E \xrightarrow{\pi} M$ is an affine bundle over E, and $\bar{\tau}^1 = \pi \circ \tau^1 \colon J^{1*}E \longrightarrow M$. The natural systems of coordinates on $J^{1*}E$ are denoted (x^μ, y^A, p_A^μ).

Now we can construct the *Hamilton-Cartan* $(m+1)$*-form*, $\Omega_\mathcal{H} \in \mathit{\Omega}^{m+1}(J^{1*}E)$, (which is the Hamiltonian counterpart of the Poincaré-Cartan form), whose local expression is:

$$\Omega_\mathcal{H} = -\mathrm{d}p_A^\mu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{dh} \wedge \mathrm{d}^m x$$

where h is the *Hamiltonian function*. The corresponding variational problem in this formalism is called the *Hamilton-Jacobi principle*, and the (compact-supported) critical sections of this variational problem can be characterized as follows:

Theorem 7 *The critical sections of the Hamiltonian variational problem are sections* $\psi\colon M \to J^{1*}E$ *which satisfy the following equivalent conditions*

1. $(\psi)^*\, i(X)\Omega_\mathcal{H} = 0$, *for every* $X \in \mathcal{X}(J^{1*}E)$.

2. *The coordinates of* ψ *satisfy the* Hamilton-De Donder-Weyl equations:

$$\left.\frac{\partial y^A}{\partial x^\mu}\right|_\psi = \left.\frac{\partial \mathrm{h}}{\partial p_A^\mu}\right|_\psi \quad ; \quad \left.\frac{\partial p_A^\mu}{\partial x^\mu}\right|_\psi = -\left.\frac{\partial \mathrm{h}}{\partial y^A}\right|_\psi$$

3. *They are the integral sections of an integrable jet field* $\mathcal{Y}_\mathcal{H}\colon J^{1*}E \to J^1J^*1E$, *such that* $i(\mathcal{Y}_\mathcal{H})\Omega_\mathcal{H} = 0$.

4. *They are the integral sections of an integrable jet field* $\mathcal{Y}_\mathcal{H}\colon J^{1*}E \to J^1J^{1*}E$, *such that* $i(\nabla_\mathcal{H})\Omega_\mathcal{H} = (m-1)\Omega_\mathcal{H}$; *where* $\nabla_\mathcal{H}$ *is the associated connection form.*

5. *They are the integral sections of a class of integrable and* $\bar{\tau}^1$*-transverse multivector fields* $\{X_\mathcal{H}\} \subset \mathcal{X}^m(J^{1*}E)$, *such that* $i(X_\mathcal{H})\Omega_\mathcal{H} = 0$.

The conditions in the last item are the version of the Hamilton-De Donder-Weyl equations in terms of multivector fields.

6. Conclusions and discusion

We have studied the integrability of multivector fields in a differentiable manifold and the relation between integrable jet fields and multivector fields in jet bundles, using all of these to give alternative (equivalent) geometric formulations of the Lagrangian and Hamiltonian formalisms of classical field theories (of first order).

Concerning to this last aspect of the article, each one of these formulations can be used to analize different features of these theories. For instance, the formulation of jet fields has been used in [5] for proving a version of the *Noether theorem* in field theory. Using the formulation of Ehresmann connections, in [11] is proved that, for regular Lagrangians, the critical sections solution of the Euler-Lagrange equations are necessarily holonomic and that this solution is not unique (it depends on $N(m^2-1)$ arbitrary functions, at most). This formulation is also used in [12] in order to set a *constraint algorithm* for singular (*almost-regular*) field theories. Finally, we hope that the formulation using multivector fields can serve to recover all of these results and, in addition, will allow to study some other aspects concerning singular field theories (for example, the geometric characterization of *gauge freedom*). In any case, multivector fields in the realm of multisymplectic manifolds in general, has been used for exploring some geometrical properties of these manifolds [9].

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