

Deformation on Phase Space

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Abstract. We review several procedures of quantization formulated in the framework of (classical) phase space M . These quantization methods consider quantum mechanics as a “deformation” of classical mechanics by means of the “transformation” of the commutative algebra $C^\infty(M)$ in a new non-commutative algebra $C^\infty(M)_\hbar$. These ideas lead in a natural way to quantum groups as deformation (or quantization, in a broad sense) of Poisson–Lie groups, which is also analysed here.

Deformación en el espacio de las fases

Resumen. El trabajo que presentamos constituye una revisión de varios procedimientos de cuantización basados en un espacio de fases clásico M . Estos métodos consideran a la mecánica cuántica como una “deformación” de la mecánica clásica por medio de la “transformación” del álgebra conmutativa $C^\infty(M)$ en una nueva álgebra no conmutativa $C^\infty(M)_\hbar$. Todas estas ideas conducen de modo natural a los grupos cuánticos como deformación (o cuantización en un sentido amplio) de los grupos de Poisson–Lie, lo cual también será analizado aquí.

1. Introduction

Knowledge and understanding of Nature is the object of Physics. The approach to the *real world* is usually made in successive steps, in such a way that the old theory is recovered from the new one by dropping the new effects. In practice this process is carried out by a kind of limit procedure as is illustrated in the following concrete situations.

From the beginning of Einstein relativity theory, it is well known that non-relativistic classical mechanics can be seen as the limit of relativistic mechanics when the speed of light goes to infinity. On the other hand, the Galilei group is a contraction, in the sense of Inönü and Wigner [20], of the Poincaré group when the contraction parameter $\varepsilon = 1/c$ goes to zero (Galilei and Poincaré groups are the kinematical groups of the non-relativistic classical mechanics and the relativistic mechanics, respectively).

Another interesting example of this process relating two physical theories is given by quantum and classical mechanics, the latter can be considered as the limit of the first one when the Planck constant, \hbar , goes to zero. In both examples, relativistic mechanics and quantum mechanics depend on a parameter whose limit ($c \rightarrow \infty$, $\hbar \rightarrow 0$) leads to a different physical theory.

Deformation can be considered as a kind of inverse procedure of the contraction of Lie groups as well as of the above limits for physical theories. From this viewpoint, the Poincaré group is a deformation of

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the Galilei group, relativity theory is a deformation of non-relativistic classical mechanics and quantum mechanics is a deformation of classical mechanics.

As is well known, physicists are very interested in finding procedures that allow to obtain quantum systems from classical ones, i.e., to solve the problem of quantization of classical systems, from the first days of quantum mechanics onward. Nevertheless, the deep differences between both (classical and quantum) theories at the level of mathematical formalism as well as physical interpretation have made impossible up to now to solve this problem in a complete and satisfactory way.

Among all the quantization procedures, stand out those of canonical quantization, geometric quantization [22]–[36] and group quantization [1]–[2], which are related with the usual formalism of quantum theory, as well as Moyal quantization [25], Berezin quantization [9], \ast -product formalism [7] and Fedosov quantization [14] associated with the phase space framework.

The idea that quantization is deeply related with deformation was introduced by Bayen *et al.* in [7]. For these authors, quantum mechanics can be replaced by a deformation of classical mechanics describing quantum systems in terms of functions defined on their phase spaces. This can be achieved introducing a non-commutative product (\ast -product) of these functions that replaces the usual commutative product of functions. The mathematical tool for this quantization procedure is the deformation of Lie algebras à la Gerstenhaber [18]. A particular case of this kind of deformation of classical mechanics is Moyal theory [25].

It is worthy to note that quantization in terms of \ast -products (or deformation) plays, with respect to the formalism of quantum mechanics on phase space framework, a similar role to that geometric quantization plays with respect to the standard formalism of quantum mechanics, i.e, in terms of Hilbert spaces, operators, etc.

On the other hand, this formalism is closely related with quantum groups, which are deformed (Hopf) algebras, in the sense of Gerstenhaber, of universal enveloping Lie algebras for quantum algebras, or deformation of Poisson–Lie structures for quantum groups.

In this work we review different procedures of quantization of classical systems from the point of view of the deformation theory. Incidentally, all of them try to formulate quantum mechanics in terms of the formalism of phase space. Also this paper shows how these ideas can be used in the theory of quantum groups. Now, the objects to deform are a kind of Poisson structures over a Lie group (Poisson–Lie groups), which play in some sense the role of phase spaces, and the \ast -product procedure can be implemented in order to quantize or deform these objects giving rise to one of the few procedures known in the literature to get quantum groups.

The paper is organized as follows. Section 2 presents the Moyal quantization theory. When the physical system under study has a symmetry group one can profit this fact in order to systematize Moyal quantization by means of the Stratonovich–Weyl correspondence, and this is the subject of Section 3. Two interesting examples are showed to illustrate how the theory works. In the following section we present a short review about the \ast -product. Last section is devoted to quantum groups. We also show the procedure to obtain quantum groups starting from “classical” structures like Poisson–Lie groups by means of a \ast -product that deforms these objects. As an example we quantize the group $SL(2)$.

2. Moyal quantization

The kinematical description of classical physical systems can be modeled using a symplectic manifold (M, ω) . The closed two-form ω identifies (sections of) the tangent and the cotangent bundle on M . The dynamical behaviour of the system is then controlled by a function H defined on the manifold through the vector field X_H associated to its differential dH by ω . This is the arena of classical mechanics, and the object described by (M, ω, H) is called a Hamiltonian classical system.

The physical description of the previous system in terms of states and observables carries a certain mathematical “duality” implemented by M and the set of (smooth) functions $\mathcal{C}^\infty(M)$. In a more technical language, we can say that this duality is realized by the contravariant Gelfan’d–Naimark functor, which

shows that no information is lost if we replace the manifold M by the algebra $\mathcal{A} = C^\infty(M)$. From this point of view *every* structure defined on M has a *natural* analogous on \mathcal{A} , in particular ω is transferred to a Poisson bracket on \mathcal{A} . Therefore, we can make a good definition for our system using the triplet $(\mathcal{A}, \{\cdot, \cdot\}, H)$ and this formalism permits an immediate generalization: to consider algebras such that the commutativity assumption is relaxed.

This procedure fits nicely into the problem of quantization because it is precisely what we are looking for when we try to “quantize” a classical system. Obviously, the passage from a commutative algebra \mathcal{A} to a non-commutative one can be done in a large variety of ways. Usually, one considers a new algebra \mathcal{A}_\hbar depending on one or more parameters, and imposes that for a value of the parameter, say $\hbar = 0$, the algebra reduces to the commutative one. A far-reaching idea is to build up the new algebra over the underlying set of the algebra \mathcal{A} by means of a new product denoted by $*_\hbar$. This product allows us to define a new deformed Poisson bracket

$$\{f, g\}_\hbar = f * g - g * f, \quad f, g \in \mathcal{A},$$

which is a Lie algebra deformation of the original one. We will show this construction later.

The aim of Moyal formulation of quantum mechanics is to describe it as a statistical theory taking place on a classical phase space unlike the standard formulation, which is developed by means of Hilbert space methods. In this way Moyal obtained a theory conceptually more transparent (for more details see [16] and references therein).

In this framework observables and states of a quantum system are considered as (generalized) functions on a phase space M isomorphic to \mathbb{R}^{2n} (again the algebra $\mathcal{A} = C^\infty(M)$). The expectation value of the observable A in the state ρ is given by

$$\langle A \rangle_\rho = \frac{\int_M A \rho}{\int_M \rho},$$

just like in classical statistical mechanics.

Moyal formulation unifies in a single theory two important constructions: Weyl mappings and Wigner functions. For that reason we call this theory Moyal–Weyl–Wigner formulation.

Let us see what is the role played by Moyal work in quantization. The simplest and most usual quantization procedure is canonical quantization (or principle of correspondence). This scheme works rather well for physical systems whose phase space is isomorphic to \mathbb{R}^{2n} , and it uses the Hilbert space $L^2(\mathbb{R}^n)$ of square integrable functions on \mathbb{R}^n with respect to the Lebesgue measure. This method of quantization takes advantage of the Dirac prescription [12] in order to associate functions (classical observables) with operators (quantum observables). Thus, to the position and momentum coordinates, q_i and p_i , are associated the operators Q_i (multiplication by q_i) and $P_i = -i\hbar \frac{\partial}{\partial q_i}$, respectively. The operator linked with the function $f(q_i, p_i)$ is obtained formally replacing the classical coordinates by their corresponding operators, which yields $f(Q_i, P_i)$. However, operators Q_i and P_i do not commute and, henceforth, the expression $f(Q_i, P_i)$ is meaningless unless we fix some ordering.

The mathematical meaning of the Dirac prescription is as follows: if the Poisson bracket of two canonical coordinates is

$$\{q_i, p_j\} = \delta_{ij},$$

then the commutator for the corresponding operators is

$$[Q_i, P_j] = i\hbar \delta_{ij}.$$

So, there is a faithful representation of the Lie subalgebra of $(C^\infty(M), \{\cdot, \cdot\})$, generated by the local coordinates $(q_i, p_i, i = 1, \dots, n)$, in the Hilbert space $L^2(\mathbb{R}^n)$. In other words, we have a homomorphism between the Lie algebras of classical and quantum observables (Heisenberg Lie algebra). This last interpretation leads to the general rule of canonical quantization

$$\{\cdot, \cdot\} \longrightarrow \frac{1}{i\hbar} [\cdot, \cdot].$$

It is worthy to note that there are many possibilities to extend the Dirac prescription to general functions according to the ordering we select on monomials in Q_i and P_i . The most usual ones are the normal ordering ($q^m p^n \longrightarrow Q^m P^n$), the antinormal ordering ($q^m p^n \longrightarrow P^n Q^m$), and the Weyl ordering or Weyl correspondence rule:

$$q^m p^n \longrightarrow (Q^m P^n)_S = \frac{n!m!}{(n+m)!} \sum_i P_i^{m,n}(Q^m P^n),$$

where $P_i^{m,n}$ are permutations with repetition of m operators Q and n operators P . Last ordering is the most suitable for quantum formulation on phase space. Moreover, it exhibits invariance under the Galilei and the symplectic groups.

A variant of canonical quantization is given by the Weyl postulate [32], which associates functions with operators by

$$e^{\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{p}+\mathbf{y}\cdot\mathbf{q})} \longrightarrow e^{\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{P}+\mathbf{y}\cdot\mathbf{Q})},$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Note that canonical quantization can be interpreted as a representation of the Heisenberg algebra, and Weyl quantization corresponds to a unitary representation of the Heisenberg group.

Now, if f is a regular function on \mathbb{R}^{2n} such that its Fourier transform \hat{f} exists, then

$$f(\mathbf{p}, \mathbf{q}) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} d\mathbf{x} d\mathbf{y} \hat{f}(\mathbf{x}, \mathbf{y}) e^{\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{p}+\mathbf{y}\cdot\mathbf{q})}.$$

This expression, together with the Weyl postulate, leads to the natural definition of the Weyl correspondence, which associates the operator W_f with the function f by means of

$$W_f = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} d\mathbf{x} d\mathbf{y} \hat{f}(\mathbf{x}, \mathbf{y}) e^{\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{P}+\mathbf{y}\cdot\mathbf{Q})}.$$

The mapping W can be extended to generalized functions (i.e., distributions) on \mathbb{R}^{2n} .

On the other hand, the function $\rho(\mathbf{p}, \mathbf{q})$ linked with the state operator ρ is given by

$$\rho(\mathbf{p}, \mathbf{q}) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} d\mathbf{x} d\mathbf{y} e^{\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{p}+\mathbf{y}\cdot\mathbf{q})} \text{Tr}[\rho e^{\frac{-i}{\hbar}(\mathbf{x}\cdot\mathbf{P}+\mathbf{y}\cdot\mathbf{Q})}].$$

When ρ is a pure state, i.e. $\rho = |\psi\rangle\langle\psi|$, last expression reduces to

$$\rho(\mathbf{p}, \mathbf{q}) = \int_{\mathbb{R}^n} d\mathbf{x} e^{\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{p}} \psi^*(\mathbf{q} + \frac{1}{2}\mathbf{x}) \psi(\mathbf{q} - \frac{1}{2}\mathbf{x}),$$

which coincides with the expression given by Wigner [33]. Function $\rho(p, q)$ is called Wigner function.

In fact, mappings provided by the Weyl correspondence rule and Wigner functions are inverse of each other. An elegant proof of this fact uses the Grossmann–Royer operators [19, 28] defined by

$$[\Omega(\mathbf{p}, \mathbf{q})\psi](\mathbf{x}) := 2^n e^{\frac{2i}{\hbar}\mathbf{p}\cdot(\mathbf{x}-\mathbf{q})} \psi(2\mathbf{q} - \mathbf{x}).$$

So, the Weyl mapping can be rewritten as

$$W_f = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} d\mathbf{p} d\mathbf{q} f(\mathbf{p}, \mathbf{q}) \Omega(\mathbf{p}, \mathbf{q}).$$

Using the complete set $\{|\mathbf{x}\rangle, \mathbf{x} \in \mathbb{R}^n\}$ of kets for \mathbf{Q} we can define the trace of an operator A by

$$\text{Tr} A = \int_{\mathbb{R}^n} \langle\mathbf{x}|A|\mathbf{x}\rangle.$$

Computing the trace of the product of two Grossmann–Royer operators, which has a distributional meaning, we obtain

$$\text{Tr}[\Omega(\mathbf{p}, \mathbf{q})\Omega(\mathbf{p}', \mathbf{q}')] = (2\pi\hbar)^n \delta(\mathbf{p} - \mathbf{p}')\delta(\mathbf{q} - \mathbf{q}').$$

Finally, given an operator A acting on $L^2(\mathbb{R}^n)$, its associated function on phase space is

$$W^{-1}(A) = \text{Tr}[\Omega(\mathbf{p}, \mathbf{q})A].$$

As we said before, the contribution of Moyal consists in combining both Weyl mapping and Wigner functions to construct a new product, associative but non-commutative, for functions on phase space through the equation

$$f * g := W^{-1}(W_f W_g).$$

Moyal product is stated in such a way that the the following diagram is commutative

$$\begin{array}{ccc} (W_f, W_g) & \longrightarrow & W_f W_g \\ W^{-1} \downarrow & & \downarrow W^{-1} \\ (f, g) & \longrightarrow & f * g \end{array}$$

i.e., the quantum information encoded in the non-commutative product of operators (quantum observables) is transferred via W to the space of classical observables and stored in the $*$ -product. Note that W is a linear continuous map, $W : S'(\mathbb{R}^{2n}) \longrightarrow \mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. However, the Moyal product is not defined for all pairs of elements in $S'(\mathbb{R}^{2n})$, but there exists a maximal closed subspace $\mathcal{M}(\mathbb{R}^{2n})$ of $S'(\mathbb{R}^{2n})$ where it is well defined. This space has the structure of an algebra with respect to the sum of functions, product by scalars and Moyal product. We have the following chain of inclusions

$$S(\mathbb{R}^{2n}) \subset \mathcal{L}^2(\mathbb{R}^{2n}) \subset \mathcal{M}(\mathbb{R}^{2n}) \subset S'^2(\mathbb{R}^{2n}).$$

The $*$ -product can be expressed through the integral formula

$$f * g = \frac{1}{(2\pi\hbar)^{4n}} \int_{\mathbb{R}^{4n}} d\mathbf{v} d\mathbf{w} f(\mathbf{v})g(\mathbf{w}) e^{\frac{i}{\hbar}(\mathbf{u}^t J \mathbf{v} + \mathbf{v}^t J \mathbf{w} + \mathbf{w}^t J \mathbf{u})},$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, $\mathbf{u} = (\mathbf{q}, \mathbf{p})^t$, $\mathbf{v} = (\mathbf{q}', \mathbf{p}')^t$ and $\mathbf{w} = (\mathbf{q}'', \mathbf{p}'')^t$. As a direct consequence, $*$ is a non local product, but it reduces to a local one in the limit $\hbar \rightarrow 0$.

Using the canonical Poisson bracket on \mathbb{R}^{2n}

$$f \overset{\leftrightarrow}{P} g \equiv \{f, g\} = \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i},$$

it is also possible to write down the Moyal product in differential form by means of the exponential of $\overset{\leftrightarrow}{P}$

$$f * g = f e^{-i\frac{\hbar}{2}\overset{\leftrightarrow}{P}} g.$$

Thus, Moyal product can be characterized as a bilinear and associative mapping in the following way [7, 31]. Let us consider the bidifferential operator

$$\begin{array}{ccc} \mathcal{J}_{\{\cdot, \cdot\}} : \mathcal{C}^\infty(\mathbb{R}^{2n}) \otimes \mathcal{C}^\infty(\mathbb{R}^{2n}) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}^{2n}) \otimes \mathcal{C}^\infty(\mathbb{R}^{2n}) \\ f \otimes g & \mapsto & \{f, g\} \end{array}$$

and the product in $\mathcal{C}^\infty(\mathbb{R}^{2n})$ written as

$$\begin{array}{ccc} m : \mathcal{C}^\infty(\mathbb{R}^{2n}) \otimes \mathcal{C}^\infty(\mathbb{R}^{2n}) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}^{2n}) \\ f \otimes g & \mapsto & fg, \end{array}$$

then the Moyal product can be expressed as

$$* = m \circ e^{-i\frac{\hbar}{2}\mathcal{J}_{\{\cdot, \cdot\}}}. \quad (1)$$

A remarkable property of the Moyal product is that it is equivariant under transformations belonging to the symplectic or the Galilei groups, i.e.,

$$(f_1 * f_2)^g = f_1^g * f_2^g,$$

where g is a generic group element and $f^g(\mathbf{u}) = f(g^{-1}\mathbf{u})$.

Moyal bracket is defined antisymmetrizing the Moyal product, i. e.,

$$\{f, g\}_M = \frac{1}{-i\hbar}(f * g - g * f)$$

or

$$\{f, g\}_M = \frac{1}{-i\hbar}W^{-1}[Wf, Wg].$$

This bracket (playing the role of commutator in standard formulation) allows us to determine the evolution of the observable f by

$$\{H, f\}_M = \frac{df}{dt},$$

where H is the Hamiltonian of the system. In this dynamical sense we can translate some concepts of the standard formulation of quantum mechanics to the new one. Thus, given an evolution operator $U(t)$, it is possible to construct a Moyal propagator by

$$\Xi(\mathbf{p}, \mathbf{q}, t) = W^{-1}(U(t)).$$

If $H(\mathbf{p}, \mathbf{q})$ is a classical time-independent Hamiltonian its Moyal propagator will be

$$\Xi_H(\mathbf{p}, \mathbf{q}, t) = W^{-1}(e^{-\frac{it}{\hbar}W_H}). \quad (2)$$

The Schrödinger equation, $i\hbar\frac{\partial U(t)}{\partial t} = HU(t)$, can be rewritten in terms of the propagator (2) as

$$i\hbar\frac{\partial}{\partial t}\Xi_H(\mathbf{p}, \mathbf{q}, t) = H * \Xi_H(\mathbf{p}, \mathbf{q}, t).$$

We can also define the spectral projection by the Fourier transform of the Moyal propagator with respect to the variable t

$$\Gamma_H(\mathbf{p}, \mathbf{q}, E) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} dt \Xi_H(\mathbf{p}, \mathbf{q}, t) e^{-\frac{i}{\hbar}tE}.$$

The support on E (energy) of the projection associated to H is, in a large variety of cases [7], the spectrum of the operator W_H .

3. Stratonovich–Weyl correspondence

Symmetry principles play a central role in the analysis of physical systems. In modern physics it is customary, given a Lie group G , to define its associated classical elementary systems as G -homogeneous symplectic spaces where the group acts by symplectomorphisms. After the celebrated theorem by Kostant–Kirillov–Souriau [21] these elementary systems are diffeomorphic to some orbit in \mathfrak{g}^* (the dual of the Lie algebra \mathfrak{g} of the group G) under the coadjoint action. This fact has as an immediate consequence the complete classification of all elementary systems whose symmetry is determined by the Lie group G . In a similar way, quantum elementary systems for G are introduced as projective unitary irreducible representations (PUIR) of G [34]. Picking up some ideas from geometric quantization, the link between classical and quantum systems is provided by Kirillov theorem at least for nilpotent groups [21].

The previous definition of a quantum system fits quite well in conventional formulation of quantum mechanics but we are interested in quantum systems from Moyal point of view. Therefore, we adopt, as

definition for a Moyal quantum elementary system [10], the pair formed by a coadjoint orbit and a $*$ -product in the space of smooth functions defined on the coadjoint orbit.

Kirillov theorem can be considered as a partial answer to the problem of quantization. Information provided by this geometric quantization can be also used to quantize a physical system according to Moyal theory.

3.1. Stratonovich–Weyl kernels

The tool required for passing from geometric quantization to Moyal quantization is known as Stratonovich–Weyl kernel [30], whose definition is as follows. Given a G -coadjoint orbit \mathcal{O} and its corresponding PUIR U with support space the Hilbert space \mathcal{H} , the Stratonovich–Weyl (SW) kernel is an operator valued mapping $\Omega : \mathcal{O} \rightarrow \mathcal{L}(\mathcal{H})$ that satisfies the following axioms:

1. Ω is injective,
2. $\Omega(\mathbf{x})$ is self-adjoint $\forall \mathbf{x} \in \mathcal{O}$,
3. unit trace: $\text{Tr} \Omega(\mathbf{x}) = 1, \quad \forall \mathbf{x} \in \mathcal{O}$,
4. covariance:

$$U(g)\Omega(\mathbf{x})U(g^{-1}) = \Omega(g \cdot \mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{O}, \forall g \in G. \quad (3)$$

5. traciality:

$$\int_{\mathcal{O}} d\mu(\mathbf{x}) \text{Tr}[\Omega(\mathbf{y})\Omega(\mathbf{x})]\Omega(\mathbf{x}) = \Omega(\mathbf{y}), \quad \forall \mathbf{y} \in \mathcal{O}, \quad (4)$$

where μ is a G -invariant measure on \mathcal{O} .

Fourth axiom is a natural requirement in view of the symmetry of the system. Traciality means that the quantity $\text{Tr}[\Omega(\mathbf{y})\Omega(\mathbf{x})]$ essentially works like the Dirac distribution $\delta(\mathbf{y} - \mathbf{x})$.

The SW kernel allows us to build up a symbol calculus. Thus, the “symbol” associated with an operator A is given by

$$\mathcal{W}_A(\mathbf{x}) = \text{Tr}[A\Omega(\mathbf{x})], \quad (5)$$

and the mapping

$$\begin{array}{ccc} \mathcal{L}(\mathcal{H}) & \longrightarrow & \mathcal{C}^\infty(\mathcal{O}) \\ A & \mapsto & \mathcal{W}_A, \end{array}$$

is called the SW correspondence. Observe that now the mapping \mathcal{W} is the inverse of the mapping W defined in Section 2. It is worthy to note that if $A \mapsto \mathcal{W}_A$ is injective then expression (5) can be inverted as

$$A = \int_{\mathcal{O}} d\mu(\mathbf{x}) \mathcal{W}_A(\mathbf{x})\Omega(\mathbf{x}),$$

which shows that the same kernel implements both directions $A \leftrightarrow \mathcal{W}_A$ of the correspondence. Sometimes it is said that Ω is a quantizer and also a “dequantizer”.

The properties satisfied by the kernel Ω have immediate consequences on the SW correspondence, remarkable ones are:

- Symbols associated to selfadjoint operators are real,

$$A = A^\dagger \Rightarrow \overline{\mathcal{W}_A} = \mathcal{W}_A,$$

where the bar stands for the complex conjugate.

- The identity operator has as symbol the unit function,

$$\mathcal{W}_I = 1.$$

- Covariance condition (3) leads to

$$\mathcal{W}_{U(g)AU(g^{-1})}(g \cdot \mathbf{x}) = \mathcal{W}_A(\mathbf{x}).$$

- The trace of the product of two operators can be evaluated as an integral involving their symbols

$$\text{Tr}(AB) = \int_{\mathcal{O}} d\mu(\mathbf{x}) \mathcal{W}_A(\mathbf{x}) \mathcal{W}_B(\mathbf{x}).$$

Another crucial application of the SW kernel is the construction of a non-commutative (or twisted) product

$$(f * g)(\mathbf{x}) = \int_{\mathcal{O}} d\mu(\mathbf{y}) \int_{\mathcal{O}} d\mu(\mathbf{z}) L(\mathbf{x}, \mathbf{y}, \mathbf{z}) f(\mathbf{y}) g(\mathbf{z}),$$

where $L(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \text{Tr}[\Omega(\mathbf{x})\Omega(\mathbf{y})\Omega(\mathbf{z})]$, and is called trikernel. This construction of the $*$ -product assures that the SW correspondence is an algebra morphism

$$\mathcal{W}_{AB} = \mathcal{W}_A * \mathcal{W}_B.$$

An interesting equality involving averages is

$$\int_{\mathcal{O}} d\mu(\mathbf{x}) (f_1 * f_2)(\mathbf{x}) = \int_{\mathcal{O}} d\mu(\mathbf{x}) f_1(\mathbf{x}) f_2(\mathbf{x}).$$

The geometrical meaning of covariance is reflected in the G -equivariance of the $*$ -product

$$(f_1 * f_2)^g = f_1^g * f_2^g, \quad \forall g \in G.$$

At the level of the trikernel it means invariance, i.e., $L(g \cdot \mathbf{x}, g \cdot \mathbf{y}, g \cdot \mathbf{z}) = L(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

Up to now, no general result guaranteeing the existence of Ω is available. A practical recipe to build up a SW kernel is summarized in the three following steps:

1. Select a point $\mathbf{0}$ as “origin” of the orbit and take a section

$$\begin{aligned} s : \mathcal{O} &\longrightarrow G \\ \mathbf{x} &\longmapsto s(\mathbf{x}) \cdot \mathbf{0} = \mathbf{x}, \end{aligned}$$

(the orbit \mathcal{O} is diffeomorphic to the homogeneous space G/G_0 , where G_0 is the isotopy group of the origin).

2. Choose an operator A as Ansatz for the value of Ω at the origin ($A = \Omega(\mathbf{0})$). The kernel Ω on the whole orbit is obtained using the covariance property (3), i.e.,

$$\begin{aligned} \Omega(\mathbf{x}) &= \Omega(s(\mathbf{x}) \cdot \mathbf{0}) \\ &= U(s(\mathbf{x}))\Omega(\mathbf{0})U(s(\mathbf{x})^{-1}) = U(s(\mathbf{x}))AU(s(\mathbf{x})^{-1}). \end{aligned}$$

3. Verify that A is indeed a “good” Ansatz checking if Ω , determined in the previous step, satisfies all the axioms required.

Obviously, the first axiom to be checked is covariance, if it fails Ω can not be defined!. For this purpose it is useful the following lemma [3, 23].

Lemma 1 *Propositions (a), (b) and (c) are equivalent:*

$$(a) \quad \Omega(\mathbf{x}) = U(g)\Omega(\mathbf{x})U(g^{-1}), \quad \forall(\mathbf{x}, g) \in (\mathcal{O}, G),$$

i.e., Ω verifies the covariance axiom (3),

$$(b) \quad \Omega(\mathbf{0}) = U(g)\Omega(\mathbf{0})U(g^{-1}), \quad \forall g \in G_0,$$

$$(c) \quad [U(X), \Omega(\mathbf{0})] = 0, \quad \forall X \in \mathfrak{g}_0,$$

where \mathfrak{g}_0 is the Lie subalgebra of the isotopy subgroup.

The choice for the Ansatz A is based on parity-like operators due to the form of the Grossmann–Roger operator. Nevertheless, this choice does not successfully leads to a SW kernel in many cases. In the following we present two examples that illustrate how the theory works (for more details see [3]–[23] and references therein). Other examples of SW quantization have been studied in [17]–[6].

3.2. Example 1: Galilean systems in $(1 + 1)$ dimensions

The Galilei group is the set of transformations that relate observables measured from different inertial frames in non-relativistic mechanics. This group can also be defined from an active point of view in which galilean transformations (time and space translations, galilean boosts and space rotations) act on the space-time manifold. In $(1 + 1)$ dimensions that action is given by

$$(t', x') \equiv (b, a, v) \cdot (t, x) = (t + b, x + a + vt),$$

where b, a and v denote the parameters of time and space translations and galilean boosts, respectively. The transformations (b, a, v) form a Lie group denoted $G(1, 1)$, whose composition law is obtained from the previous action

$$(b', a', v')(b, a, v) = (b' + b, a' + a + v'b, v' + v).$$

Its associated Lie algebra $\mathfrak{g}(1, 1)$ is spanned by the infinitesimal generators of time (H) and space (P) translations and galilean boosts (K), which have the following commutation relations

$$[K, H] = P, \quad [K, P] = 0, \quad [P, H] = 0.$$

Elements of the dual space $\mathfrak{g}^*(1, 1)$ of the Lie algebra $\mathfrak{g}(1, 1)$ are linear combinations, $hH^* + pP^* + kK^*$, in terms of the dual basis of $\{H, P, K\}$. The coadjoint action of $G(1, 1)$ on \mathfrak{g}^* is expressed in that coordinates as

$$(h', p', k') \equiv (b, a, v) \cdot (h, p, k) = (h - vp, p, bp + k).$$

The space $\mathfrak{g}^*(1, 1)$ is then “foliated” into two kind of orbits:

1. 0-dimensional (0D) orbits : points of the form $hH^* + kK^*$,
2. 2-dimensional (2D) orbits \mathcal{O}_α : characterized by equation $p = \alpha$.

From a physical point of view 0D orbits cannot support a dynamics and, hence, they are not interesting. In the orbit \mathcal{O}_α a set of canonical coordinates is determined by $q = \frac{1}{\alpha}k$ and $p = h$. Taking $\mathbf{0} = \alpha P^*$ (with canonical coordinates $(0, 0)$) as representative point on \mathcal{O}_α we can find a maximal subordinate subalgebra, which induces the PUIR of $G(1, 1)$ using the Kirillov method

$$[U_\alpha(b, a, v)\psi](w) = e^{-i\alpha(a-bw)}\psi(w - v),$$

realized on the Hilbert space $L^2(\mathbb{R})$ of square integrable functions on the real line. The argument of these functions can be identified with velocity.

Let us construct a SW kernel following the three steps mentioned above:

1. We had already chosen the origin $\mathbf{0} = \alpha P^* \equiv (0, 0)$. A normalized section is given by $s(p, q) = (q, 0, -\frac{p}{\alpha}) \in G(1, 1)$.

2. As Ansatz for the kernel at the origin we take the parity-like operator

$$[Af](\omega) = 2\psi(-\omega).$$

3. It is easy matter to check all the axioms. For instance, “unit trace”:

$$\begin{aligned} \text{Tr } \Omega(p, q) &= \text{Tr}[U(s(p, q))\Omega(0, 0)U(s(p, q)^{-1})] = \text{Tr}[\Omega(0, 0)] \\ &= \int_{-\infty}^{\infty} dw \langle w|\Omega(0, 0)|w\rangle = \int_{-\infty}^{\infty} dw 2\langle w| - w\rangle \\ &= \int_{-\infty}^{\infty} dw 2\delta(2w) = 1. \end{aligned}$$

To prove covariance (3) it suffices to consider the isotopy group of $\mathbf{0}$ which is made up of space translations $G(1, 1)_{\mathbf{0}} = \{(0, a, 0)\}$.

Let us show that, in this case, there exist many different SW kernels. Firstly, note that covariance (i.e., $U_{\alpha}(0, a, 0)AU_{\alpha}(0, -a, 0) = A$) is satisfied for any operator A because the elements of the isotopy group are represented by scalars.

To analyse traciality (4) we write the action of A as

$$[A\psi](w) = \int_{-\infty}^{\infty} dw' A_{w, w'}\psi(w'),$$

this leads to the rather complicated condition

$$A_{w', w} = 2\pi \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dp A_{v+w'-w, v} \overline{A}_{v+w'-w+\frac{p}{\alpha}, v+\frac{p}{\alpha}} A_{w'+\frac{p}{\alpha}, w+\frac{p}{\alpha}}.$$

However, a particular solution can be found

$$A_{w', w} = e^{i\varphi(w)}\delta(w + w'),$$

and the associated kernel acts as

$$[\Omega(p, q)\psi](w) = e^{i\varphi(w+\frac{p}{\alpha})}e^{2i\alpha q(w+\frac{p}{\alpha})}\psi(-w - 2\frac{p}{\alpha}).$$

Finally to satisfy “hermiticity” and “unit trace” it is enough that φ verifies $\varphi(w) + \varphi(-w) \in 2\pi\mathbb{Z}$ and $\varphi(0) \in 2\pi\mathbb{Z}$, respectively.

3.3. Example 2: The Newton–Hooke group $NH(1, 1)$

The kinematical group of Newton–Hooke in $(1 + 1)$ dimensions can be defined, in analogy with the case of the Galilei group, as the set of transformations (time and space translations and boosts) acting on the space-time as

$$(t', x') \equiv (b, a, v) \cdot (t, x) = (t + b, x + a \cos \frac{t}{\tau} + v\tau \sin \frac{t}{\tau}).$$

The natural topology of this universe is that of the product $\mathbb{R} \times S^1$. The parameter τ , with dimension of time, characterizes the compact direction, and can be seen as a characteristic time of this universe. In the definition of the action we have opted for the universal covering to obtain simpler formulas (i.e., $b \in \mathbb{R}$). The group law is

$$(b', a', v')(b, a, v) = (b' + b, a' \cos \frac{b}{\tau} + v'\tau \sin \frac{b}{\tau} + a, v' \cos \frac{b}{\tau} - \frac{a'}{\tau} \sin \frac{b}{\tau} + v).$$

Its associated Lie algebra, $\mathfrak{nh}(1, 1)$, spanned by the infinitesimal generators of the above mentioned transformations, H, P, K , has only two non-vanishing commutators

$$[K, H] = P, \quad [P, H] = -\frac{1}{\tau^2}K.$$

Note that this group is isomorphic to the Euclidean group of the plane.

The coadjoint action of $NH(1, 1)$

$$(b, a, v) \cdot (h, p, k) = \left(h - vp + \frac{1}{\tau^2}ak, p \cos \frac{b}{\tau} - \frac{k}{\tau} \sin \frac{b}{\tau}, p\tau \sin \frac{b}{\tau} + k \cos \frac{b}{\tau} \right),$$

splits the dual $\mathfrak{nh}^*(1, 1)$ into 0D (points of the form hH^*) and 2D orbits (cylinders \mathcal{O}_β defined by $p^2 + \frac{k^2}{\tau^2} = \beta$).

A local chart of canonical coordinates is formed by the pair $\alpha = \arctan \frac{k}{\tau p}$, $j = \tau h$. For later use we quote here a normalized section

$$s(\alpha, j) = (\tau\alpha, 0, 0)(0, 0, -\frac{j}{\beta\tau}).$$

In order to apply the Kirillov method for induced representations we take the subalgebra $\langle P, K \rangle$ which is subordinated to the point $\beta P^* \in \mathcal{O}_\beta$. The PUIR attached to \mathcal{O}_β can then be realized on the space $L^2([-\pi, \pi])$ of square integrable functions on the circle

$$[U_\beta(b, a, v)\psi](t) = e^{i\beta(\frac{a}{\tau} \cos t + v \sin t)} \psi(t - b).$$

Note that all the orbits \mathcal{O}_β are diffeomorphic and all the PUIR's U_β have the same form, hence we will take $\beta = 1$.

The following parity-like operators

$$[A\psi](t) = 2\psi(-t), \quad [A\psi](t) = 2\psi(t + 2\pi), \quad [A\psi](t) = 2\psi(-t + \pi),$$

as Ansatz for A do not yield suitable SW kernels. The first one gives a non tracial operator and the others do not accomplish covariance (3). To solve this problem we consider a generic operator A , which has to verify consecutively all the axioms. An operator A acting on $L^2([-\pi, \pi])$ can be expressed as

$$A = \sum_{r,s \in \mathbb{Z}} A_{r,s} |r\rangle \langle s|,$$

where the ket $|r\rangle$ stands for the function $\psi_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$. To apply covariance let us note that the isotopy group of the origin $\mathbf{0} = \beta P^*$ contains only space translations. Therefore, according Lemma 3.1 the covariance condition reduces to $[U(P), A] = 0$, whose most general solution reads

$$[A\psi](t) = a(t)\psi(-t) + b(t)\psi(t),$$

with a and b arbitrary functions on $[-\pi, \pi]$. Hence, the most general mapping $\Omega : \mathcal{O} \rightarrow \mathcal{L}(L^2([-\pi, \pi]))$, verifying covariance is

$$[\Omega(j, \alpha)\psi](t) = e^{2ij \sin(t-\alpha)} a(t - \alpha)\psi(2\alpha - t) + b(t - \alpha)\psi(t).$$

The following lemma is useful to improve traciality.

Lemma 2 *A covariant mapping $\Omega : \mathcal{O} \rightarrow \mathcal{L}(L^2([-\pi, \pi]))$ verifies traciality if and only if $K(\mathbf{x}, \mathbf{y}) = \text{Tr}[\Omega(\mathbf{x})\Omega(\mathbf{y})]$ is a reproducing kernel in the space of symbols generated by Ω , i.e.,*

$$\int_{\mathcal{O}} d\mu(\mathbf{y}) K(\mathbf{x}, \mathbf{y}) \mathcal{W}(\mathbf{y}) = \mathcal{W}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{O}.$$

The proof of this lemma involves only covariance of Ω and invariance of the measure μ under $NH(1+1)$. In fact, last condition is equivalent to the following one

$$\int_{\mathcal{O}} d\mu(\mathbf{y}) K(\mathbf{0}, \mathbf{y}) \mathcal{W}(\mathbf{y}) = \mathcal{W}(\mathbf{y}),$$

apparently weaker.

Imposing traciality and the other axioms we obtain the following family of SW kernels on the cylinder

$$[\Omega(j, \alpha)\psi](t) = e^{2ij \sin(t-\alpha)} a(t-\alpha) \psi(2\alpha-t),$$

where function a is essentially arbitrary, only subject to the constraints $a(-t) = \overline{a(t)}$ and $|a(t)|^2 + |a(t+\pi)|^2 = 4|\cos t|$.

This solution solves the Moyal quantization of the cylinder. Other approaches to the cylinder quantization can be found in [26] and [27].

4. Star products

The theory of deformations of algebras of classical observables, called the theory of \ast -products, was introduced by Bayen *et al.* [7] and its mathematical foundations can be found in the works of Gerstenhaber [18] about deformation of algebraic structures. As we said before in section 2 Moyal quantization can be seen as a particular case of this theory.

Let \mathcal{A} be an algebra and $\mathcal{A}[[\hbar]]$ the algebra of formal power series in \hbar with coefficients in \mathcal{A} . The algebra $\mathcal{A}[[\hbar]]$ is said to be a deformation of \mathcal{A} with deformation parameter \hbar if

$$\mathcal{A}[[\hbar]]/\hbar\mathcal{A}[[\hbar]] \simeq \mathcal{A}.$$

Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold, and let us consider the Lie algebra $\mathcal{A} = \mathcal{C}^\infty(M)$. A quantization of \mathcal{A} is a deformation of the commutative algebra \mathcal{A} into a non-commutative algebra $\mathcal{A}_\hbar = \mathcal{A}[[\hbar]]$ with a new product, $\ast_\hbar : \mathcal{A}_\hbar \times \mathcal{A}_\hbar \rightarrow \mathcal{A}_\hbar$, defined as a deformation of the commutative product on \mathcal{A} . Since the elements of \mathcal{A}_\hbar are formal series

$$f = f(x, \hbar) = \sum_{r=1}^{\infty} f_r(x) \hbar^r, \quad f_r \in \mathcal{C}^\infty(M), \quad x \in M,$$

the \ast -product is defined as

$$f \ast_\hbar g = \sum_{r=0}^{\infty} l_r(x) \hbar^r,$$

such that l_r are polynomials on f_r, g_r and their derivatives, and $l_0(x) = f_0(x)g_0(x)$. Moreover, the \ast -product has to be associative.

A commutator is defined on \mathcal{A}_\hbar by

$$[f, g] \equiv \{f, g\}_\hbar = f \ast_\hbar g - g \ast_\hbar f = \hbar\{f_0, g_0\} + o(\hbar^2).$$

Consequently

$$f \ast_\hbar g = fg + \frac{\hbar}{2}\{f, g\} + o(\hbar^2), \quad \forall f, g \in \mathcal{A} \tag{6}$$

and

$$f \ast_\hbar a = a \ast_\hbar f = af, \quad \forall a \in \mathbb{C}, \quad f \in \mathcal{A}. \tag{7}$$

In particular $f \ast_\hbar 1 = 1 \ast_\hbar f = f$, i.e., the unit element is not quantized.

Note that the original Poisson bracket on \mathcal{A} is recovered in the semiclassical limit ($\hbar \rightarrow 0$)

$$\{f, g\} = \lim_{\hbar \rightarrow 0} \frac{1}{\hbar} \{f, g\}_\hbar.$$

Additional conditions have to be added in physical situations, for instance: $\overline{f *_\hbar g} = \overline{f} *_\hbar \overline{g}$, and $\hbar = -i\hbar$. Therefore, after quantization real-valued classical observables go over into self-adjoint quantum observables (operators).

In general a $*_\hbar$ -product is defined as

$$f *_\hbar g = fg + \sum_{r=1}^{\infty} C_r(f, g) \hbar^r,$$

where the terms C_r are Hochschild 2-cochains, i.e., bidifferential operators on \mathcal{A} without constant term in each argument (C_r vanishes over constants).

If the infinite series $\sum_{r=1}^{\infty} C_r(f, g) \hbar^r$ stops at order m verifying the associativity property up to this order, we have a deformed product up to order m . Gerstenhaber showed that the obstruction to the extension of deformed products up to order m is the third space of the Hochschild cohomology.

There are some results about the existence of $*_\hbar$ -products based on cohomological techniques involving the Hochschild or de Rham cohomologies. The most interesting of them proves the existence of a $*_\hbar$ -product for any symplectic manifold [35]. Unfortunately, all these results are formal in the sense that do not give an effective or canonical construction procedure of such $*_\hbar$ -products. As far as we know the most interesting example of $*_\hbar$ -product quantization is Moyal–Weyl–Wigner quantization.

It is worthy to note that, from a physical point of view, in this quantization method the $*_\hbar$ -products have to be invariant for the elements (distinguished observables) of a sufficiently large finite subalgebra \mathcal{I} of \mathcal{A} (i.e., $\{a, f_1 *_\hbar f_2\} = \{a, f_1\} *_\hbar f_2 + f_1 *_\hbar \{a, f_2\}$, $a \in \mathcal{I}$, $\forall f_1, f_2 \in \mathcal{A}$) such that $[a, f] = \hbar \{a, f\}$, $\forall a \in \mathcal{I}$, $\forall f \in \mathcal{A}$. These distinguished observables determine a (local) coordinate system of M in terms of a basis of this subalgebra. For instance, in the case of Moyal product for $M = \mathbb{R}^{2n}$ the polynomials of degree lesser or equal to two in the usual coordinates $(q_i, p_i, i = 1, \dots, n)$ constitute this subalgebra of distinguished observables. The fact that the quadratic Hamiltonians belong to this subalgebra makes easier the study of the temporal evolution of the quantized systems in this phase space framework.

5. Quantization of Poisson–Lie structures

Quantum groups are objects which can be seen as deformation (or quantization in a broad sense) of classical structures related with $\mathcal{C}^\infty(G)$, where G is a Lie group (see [11] for a review).

The theory of $*_\hbar$ -products is one of the different approaches to quantum groups (the others are FRT method [37], matrix T [15]). It tries to construct $*_\hbar$ -products on $\mathcal{C}^\infty(G)$ that quantize this algebra and preserve in some sense its additional algebraic structure [31, 13].

The aim of this section is to quantize (or deform) the corresponding Poisson algebra of classical observables $\mathcal{A} = \mathcal{C}^\infty(G)$ of smooth functions over a Lie group G , which has a supplementary algebraic structure of Hopf coalgebra. We will start by a brief review about these structures.

5.1. Poisson–Lie groups

The Hopf coalgebra structure on $\mathcal{A} = \mathcal{C}^\infty(G)$ (also denoted $Fun(G)$) is induced by the composition law of the Lie group G . Explicitly, there are two homomorphisms: coproduct ($\Delta : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$) and counit ($\varepsilon : \mathcal{A} \rightarrow \mathbb{C}$), and the antihomomorphism antipode ($\gamma : \mathcal{A} \rightarrow \mathcal{A}$) defined by

$$(\Delta f)(g, g') = f(gg'), \quad \varepsilon(f) = f(e), \quad [\gamma(f)](g) = f(g^{-1}), \quad (8)$$

$\forall g, g' \in G$, with e the unit element of G . So, \mathcal{A} is said to be a Hopf algebra.

A Lie group G is a Poisson–Lie (PL) group if both structures, Poisson manifold and Hopf coalgebra, are compatible in the following way:

1) Coproduct verifies the following diagram

$$\begin{array}{ccc} & \{\cdot, \cdot\} & \\ \mathcal{A} \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \\ \Delta \otimes \Delta \downarrow & & \downarrow \Delta \\ \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\{\cdot, \cdot\}_{\mathcal{A} \otimes \mathcal{A}}} & \mathcal{A} \otimes \mathcal{A}, \end{array}$$

or in other words,

$$\Delta(\{f_1, f_2\}) = \{\Delta(f_1), \Delta(f_2)\}_{\mathcal{A} \otimes \mathcal{A}},$$

where

$$\{f_1 \otimes f_2, k_1 \otimes k_2\}_{\mathcal{A} \otimes \mathcal{A}} = f_1 k_1 \otimes \{f_2, k_2\} + \{f_1, k_1\} \otimes f_2 k_2.$$

2) The group multiplication law, $m : G \times G \longrightarrow G$, is a Poisson map.

An interesting problem is to construct PL structures over a Lie group. A solution is as follows. Let us consider $r \in \Lambda^2 \mathfrak{g}$ (\mathfrak{g} is the Lie algebra of G), i.e., $r = r^{ij} X_i \otimes X_j$ with $r^{ij} = -r^{ji}$ in a basis $\{X_i\}$ of \mathfrak{g} . If we define

$$r_{12} = r^{ij} X_i \otimes X_j \otimes 1, \quad r_{13} = r^{ij} X_i \otimes 1 \otimes X_j, \quad r_{23} = r^{ij} 1 \otimes X_i \otimes X_j,$$

the Schouten bracket of r with itself can be written as

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}].$$

We say that r is a classical r -matrix verifying the classical Yang–Baxter equation (CYBE) (or a non-standard r -matrix) if it verifies $[[r, r]] = 0$. When $[[r, r]] \neq 0$ but $\text{ad}_{\mathfrak{g}}^{\otimes 3} [[r, r]] = 0$, it is said that r satisfies the modified classical Yang–Baxter equation (MCYBE) or is a standard r -matrix.

On the other hand, to every $X \in \mathfrak{g}$ there are left-invariant and right-invariant vector fields defined by

$$(X^L f)(g) = \left. \frac{d}{dt} \right|_{t=0} f(ge^{tX}), \quad (X^R f)(g) = \left. \frac{d}{dt} \right|_{t=0} f(e^{tX}g). \quad (9)$$

Using both ingredients [31, 13], a classical r -matrix and invariant vector fields, it is possible to endow G with a structure of PL group $(G, \{\cdot, \cdot\})$ with the Poisson bracket defined by (Sklyanin bracket)

$$\{f_1, f_2\} = r^{ij} (X_i^R f_1 X_j^R f_2 - X_i^L f_1 X_j^L f_2), \quad f_1, f_2 \in \mathcal{C}_\infty(G). \quad (10)$$

We have seen that a PL group is a Poisson–Hopf algebra, hence it is natural to look for a Hopf algebra structure on the deformation \mathcal{A}_h , or in other words, if it exists a coproduct Δ_h (besides a counit and an antipode) such that

$$\Delta_h(f_1 *_h f_2) = \Delta_h(f_1) *_h \Delta_h(f_2),$$

and, obviously, $\lim_{h \rightarrow 0} \Delta_h = \Delta$. The noncommutative Hopf algebra obtained in this way will be called the quantum group associated to G (usually denoted $\text{Fun}_h(G)$).

5.2. Quantization of PL groups with CYBE r -matrix

The procedure of quantization of PL groups presents some differences according to the associated r -matrix is standard or non-standard. The easier case corresponds to non-standard r -matrices and, consequently, we will present firstly this case (for more details see [31, 13]).

Although the Poisson brackets $\{f_1, f_2\}_{L,R} = r^{ij} X_i^{L,R} f_1 X_j^{L,R} f_2$ do not generate separately a PL structure on G (see expression (10)), the idea is to quantize independently both Poisson structures determined by both Poisson brackets and after to join them to get a quantization of the PL group.

For instance, let us consider the left Poisson bracket, and let us go to find a left G -equivariant associative $*$ -product verifying conditions (6) and (7), and moreover

$$\Delta(f_1 *_h f_2) = \Delta f_1 *_h \Delta f_2, \quad (11)$$

where Δ is the standard coproduct on \mathcal{A} defined by (8a). The expression for the $*$ -Moyal product (1), $*_{\text{Moyal}} = m \circ e^{\frac{\hbar}{2} \mathcal{J}\{\cdot, \cdot\}}$, suggests to define $*_h^L$ in the same way. So,

$$*_h^L = m \circ \tilde{F},$$

where \tilde{F} is a formal power series in \hbar whose coefficients \tilde{F}_n are linear differential operators on $\mathcal{A} \otimes \mathcal{A}$, i.e.,

$$\tilde{F} = 1 + \sum_{n=1}^{\infty} \hbar^n \tilde{F}_n, \quad \tilde{F}_n : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}.$$

The property of left-invariance for $*_h^L$ gives rise to

$$(L_{g_1} \otimes L_{g_2}) \circ \tilde{F}_n = \tilde{F}_n \circ (L_{g_1} \otimes L_{g_2}), \quad \forall g_1, g_2 \in G,$$

where L_g is the left-translation operator.

Let π_L be the representation of the universal enveloping algebra $U\mathfrak{g}$ by left-invariant differential operators on $\mathcal{C}^\infty(G)$ such that in terms of a basis $\{X_i\}$ of \mathfrak{g} takes the form $\pi_L(X_i) = X_i^L$, then \tilde{F}_n can be expressed as

$$\tilde{F}_n = (\pi_L \otimes \pi_L)(F_n),$$

with $F_n \in U\mathfrak{g} \otimes U\mathfrak{g}$ and $F_1 = -\frac{1}{2}r$. Hence, \tilde{F} can be written as the image by the representation π_L of a formal power series in \hbar with coefficients in $U\mathfrak{g} \otimes U\mathfrak{g}$:

$$\tilde{F} = (\pi_L \otimes \pi_L)(F), \quad F \in U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]].$$

In order to get a deformation with non quantized unit and associative it is necessary to impose the following conditions on F :

$$\begin{aligned} (\epsilon \otimes id)F &= (id \otimes \epsilon)F = 1, \\ (F \otimes id)(\Delta_0 \otimes id)F &= (id \otimes F)(id \otimes \Delta_0)F, \end{aligned} \quad (12)$$

where Δ_0 denotes the coproduct in the Hopf algebra $U\mathfrak{g}$, i.e.,

$$\Delta_0(X) = 1 \otimes X + X \otimes 1, \quad X \in \mathfrak{g}.$$

It can be shown that the $*$ -product given by

$$*_h^L = m \circ (\pi_L \otimes \pi_L)(F)$$

defines a left-invariant quantization of the Poisson bracket $\{\cdot, \cdot\}_L$.

Similarly, the right-invariant $*$ -product

$$*_h^R = m \circ (\pi_R \otimes \pi_R)(F^{-1})$$

quantizes the Poisson bracket $\{\cdot, \cdot\}_R$.

Finally, the combination of both $*_h^L$ and $*_h^R$

$$*_h = m \circ (\pi_L \otimes \pi_L)(F) \circ (\pi_R \otimes \pi_R)(F^{-1}) \quad (13)$$

yields an associative quantization of the PL group verifying (11).

The existence of an element $F \in U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$ verifying (12) for a nonstandard r -matrix of \mathfrak{g} has been proved by Drinfel'd [13]. This result assures that any PL group associated to a classical r -matrix verifying CYBE can be quantized.

Introducing the flip operator $\sigma(a \otimes b) = b \otimes a$ it is possible to construct an object,

$$\mathcal{R} = \sigma(F^{-1})F = 1 - \hbar r + o(\hbar^2), \quad (14)$$

satisfying the quantum Yang-Baxter equation (QYBE) which is said to be the universal quantum R -matrix associated to the classical r -matrix. The relevance of the previous procedure is that we can get many "concrete" solutions of the QYBE by taking different representations of the universal object \mathcal{R} . Thus, if we take a finite dimensional representation ρ of \mathfrak{g} in the algebra of $n \times n$ complex matrices $M(n, \mathbb{C})$, we have

$$R = (\rho \otimes \rho)(\mathcal{R})$$

which satisfies the QYBE

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \quad (15)$$

and the unitary condition

$$RR^\sigma = 1, \quad (16)$$

where $R^\sigma = (\rho \otimes \rho)(\sigma(\mathcal{R}))$. We see that this quantization procedure leads in a natural way to QYBE.

Finally, let us consider again the matrix representation $\rho : \mathfrak{g} \rightarrow M(n, \mathbb{C})$, consequently the group G is realized as subgroup of $GL(n, \mathbb{C})$. Let $T = (t_{ij})_{i,j=1}^n$ be the matrix of coordinate functions on G

$$t_{ij}(g) = g_{ij}, \quad g \in G.$$

Left and right actions of \mathfrak{g} on matrix coordinates on G are easily described using (9) by

$$\begin{aligned} (X_L t_{ij})(g) &= (gX)_{ij} = t_{ik}(g)X_{kj}, \\ (X_R t_{ij})(g) &= (Xg)_{ij} = X_{ik}t_{kj}(g), \end{aligned} \quad \forall X \in \mathfrak{g}.$$

On the other hand, let \hat{F} be the image of F by the representation ρ (i.e. $\hat{F} = (\rho \otimes \rho)(F)$) and defining $T_1 = T \otimes 1$ and $T_2 = 1 \otimes T$, the $*$ -product between matrix coordinates of G elements can be expressed in an elegant manner by

$$T_1 *_h T_2 = \hat{F}^{-1} T \otimes T \hat{F},$$

applying the flip operator to both sides of this expression we get

$$T_2 *_h T_1 = \sigma(\hat{F}^{-1}) T \otimes T \sigma(\hat{F}).$$

Taking into account expression (14) we obtain the relation

$$RT_1 *_h T_2 = T_2 *_h T_1 R,$$

which is the well known formula that gives the commutation relations between the matrix coordinate functions of G defining the quantum group $Fun_h(G)$.

5.3. Quantization of PL groups with MCYBE r -matrix

In this case the procedure is similar to the previous one in the sense that we again define the $*$ -product by expression (13). However, here the difference is that condition (12b) over F (where $F = 1 - (\hbar/2)r + o(\hbar^2) \in U\mathfrak{g}^{\otimes 2}[[\hbar]]$) such that $(\epsilon \otimes id)F = (id \otimes \epsilon)F = 1$, which assures associativity, is now relaxed and substituted by the more general [31]:

$$(F \otimes id)(\Delta_0 \otimes id)F = \alpha(id \otimes F)(id \otimes \Delta_0)F, \quad \alpha \in U\mathfrak{g}^{\otimes 3}[[\hbar]],$$

where a formal power series in \hbar with coefficients in $U\mathfrak{g}^{\otimes 3}$ has been introduced.

The associativity of $*_{\hbar}$ is assured if α is G -invariant, i.e.,

$$\text{ad}_{\mathfrak{g}}^{\otimes 3}\alpha = [1 \otimes 1 \otimes X + 1 \otimes X \otimes 1 + X \otimes 1 \otimes 1, \alpha] = 0, \quad \forall X \in \mathfrak{g}. \quad (17)$$

Drinfel'd has proved the existence of $\alpha \in U\mathfrak{g}^{\otimes 3}[[\hbar]]$ verifying (17) and other additional conditions that we do not display here (for more details see [31]).

Also it is possible to construct an object by

$$\mathcal{R} = \sigma(F^{-1})e^{\hbar t}F,$$

where t is an $\text{ad}_{\mathfrak{g}}^{\otimes 2}$ -invariant symmetric element of $\mathfrak{g} \otimes \mathfrak{g}$ defined by $[[r, r]] = -[t_{13}, t_{23}]$. The matrix

$$R = (\rho \otimes \rho)\mathcal{R},$$

defined in terms of a matrix representation ρ of \mathfrak{g} on $M(n, \mathbb{C})$ satisfies the QYBE (15), but it does not verify the unitary condition (16).

As in the previous case, if we consider the natural representation ρ of \mathfrak{g} on $M(n, \mathbb{C})$, and the matrix representation of the group elements, T , then the $*$ -product of the matrix coordinate functions of G is given by

$$T_1 *_{\hbar} T_2 = \hat{F}^{-1}T \otimes T\hat{F}. \quad (18)$$

On the other hand, $T_2 *_{\hbar} T_1 = \sigma(\hat{F}^{-1})T \otimes T\sigma(\hat{F})$, and from this expression and (19) one gets once more

$$RT_1 *_{\hbar} T_2 = T_2 *_{\hbar} T_1R. \quad (19)$$

5.4. Example 3: Quantization of the PL group $SL(2)$

To illustrate all the above techniques of quantization of PL groups, we present the quantization of the PL group $SL(2)$.

A basis for the Lie algebra $\mathfrak{sl}(2)$ is given by three elements X_{\pm} and H with commuting relations

$$[H, X_{\pm}] = \pm 2X_{\pm}, \quad [X_+, X_-] = H.$$

A classical r -matrix satisfying MCYBE for this algebra is

$$r = 2X_+ \wedge X_- = X_+ \otimes X_- - X_- \otimes X_+ \in \Lambda^2 \mathfrak{sl}(2).$$

The associated Sklyanin bracket is

$$\{f, g\} = X_+^R f X_-^R g - X_-^R f X_+^R g - X_+^L f X_-^L g - X_-^L f X_+^L g.$$

A 2×2 matrix representation of $\mathfrak{sl}(2)$ is given by

$$\rho(X_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(X_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho(H) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The r -matrix in this representation takes the explicit form

$$\hat{r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The elements of $SL(2)$ written in matrix coordinates are

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with the condition $\det T = ad - bc = 1$.

The Poisson brackets for the group matrix coordinates can be directly computed by means of

$$\{T \otimes T\} = [\hat{r}, T \otimes T],$$

which yields

$$\begin{aligned} \{a, b\} &= ab, & \{a, c\} &= ac, & \{a, d\} &= 2bc, \\ \{b, c\} &= 0, & \{b, d\} &= bd, & \{c, d\} &= cd. \end{aligned}$$

Note that these relations fix the Poisson brackets for all pairs of functions in $C^\infty(SL(2))$ if we consider them as polynomials in the variables a, b, c, d .

The quantization of $SL(2)$ is performed by means of

$$\hat{F} = e^{-\frac{\hbar}{2}\sigma} \begin{pmatrix} \sqrt{q} & 0 & 0 & 0 \\ 0 & u^{-1} & 0 & 0 \\ 0 & v & u & 0 \\ 0 & 0 & 0 & \sqrt{q} \end{pmatrix},$$

where $q = e^h$, $u = \sqrt{\frac{2}{q+q^{-1}}}$ and $v = \frac{q-q^{-1}}{\sqrt{2(q+q^{-1})}}$. The corresponding representation for the R -matrix is

$$R_q = \sqrt{q}\sigma(\hat{F}^{-1})e^{(\sigma-\frac{1}{2}I)\hbar}\hat{F} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

A straightforward calculation using (18) gives the $*$ -product for coordinates a, b, c, d . And from (19) one gets the usual relations defining the quantum group $SL_q(2)$:

$$\begin{aligned} a *_h b &= qb *_h a, & a *_h c &= qc *_h a, & a *_h d - d *_h a &= (q - q^{-1})b *_h c, \\ b *_h c &= c *_h b, & b *_h d &= qd *_h b, & c *_h d &= qd *_h c. \end{aligned}$$

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References

- [1] Aldaya, V. and Azcárraga, J. A. de (1982). *J. Math. Phys.*, **23**, 1297.
- [2] Aldaya, V. and Azcárraga, J. A. de (1987). *Fortschr. Phys.*, **35**, 437.
- [3] Arratia, O., Martín, M. A. and Olmo, M. A. del (1996). *Rend. Sem. Mat. Univ. Pol. Torino*, **54**, 191.
- [4] Arratia, O. and Olmo, M. A. del (1997). *Rep. Math. Phys.*, **40**, 149.
- [5] Arratia, O., Olmo, M. A. del (1997). *Int. J. Mod. Phys. A*, **12**, 125.
- [6] Arratia, O. and Olmo, M. A. del (1997). *Fortschr. Phys.*, **45**, 103.
- [7] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D. (1978). *Ann. Phys.*, **111**, 61 and 111.
- [8] Ballesteros, A., Gadella, M. and Olmo, M. A. del (1992). *J. Math. Phys.*, **33**, 3379.
- [9] Berezin, F. A. (1975). *Commun. Math. Phys.*, **40**, 153.
- [10] Cariñena, J. F., Gracia Bondía, J. M. and Várilly, J. C. (1990). *J. Phys. A: Math. Gen.*, **23**, 901.

- [11] Chari, V. and Pressly, A. (1994). *A guide to Quantum Groups*, Cambridge Univ. Press, Cambridge.
- [12] Dirac, P. A. M. (1958). *Principles of Quantum Mechanics*, Oxford Univ. Press, Oxford.
- [13] Drinfel'd, V. G. (1983). *Sov. Math. Dokl.*, **27**, 68.
- [14] Fedosov, B. V. (1994). *J. Diff. Geom.*, **40**, 213.
- [15] Fronsdal, C. and Galindo, A. (1993). "The Universal T -matrix". In *Proc. of the Joint Summer Research on Conformal Field Theory, Topological Field Theory and Quantum Groups*. Holyhoke (1992), *Lett. Math. Phys.*, **27**, 39.
- [16] Gadella, M. (1995). *Fortschr. Phys.*, **43**, 229.
- [17] Gadella, M., Martín, M. A., Nieto, L. M. and Olmo, M. A. del (1991). *J. Math. Phys.*, **32**, 1182.
- [18] Gerstenhaber, M. (1964). *Ann. Math.*, **79**, 59.
- [19] Grossmann, A. (1976). *Commun. Math. Phys.*, **48**, 191.
- [20] Inönü, E. and Wigner, E. P. (1953). *Proc. Nat. Acad. Sci. USA*, **39**, 510.
- [21] Kirillov, A. A. (1976). *Elements of the Theory of Representations*, Springer, Berlin.
- [22] Kostant, B. (1970). Quantization and unitary representations. In *Lecture Notes Math.*, **170**, 87, Springer, Berlin.
- [23] Martín, M. A. (1998). *Quantization of physical systems in (1 + 1) and (2 + 1) dimensions*. Ph.D. Thesis, Univ. de Valladolid.
- [24] Martín, M. A. and Olmo, M. A. del (1996). *J. Phys. A: Math. Gen.*, **29**, 689.
- [25] Moyal, J. E. (1949). *Proc. Camb. Phil. Soc.*, **45**, 99.
- [26] Olmo, M. A. del, and J.A. González, J. A. (1999). *Coherent states on the circle and quantization*. A. Ballesteros *et al.* eds., Univ. de Burgos, Burgos.
- [27] Plebański, J. F., Przanowski, M., Tosiek, J. and Turrubiates, F. J. (2000). *Acta Phys. Pol.*, **B31**, 561.
- [28] Royer, A. (1977). *Phys. Rev. A*, **15**, 449.
- [29] Souriau, J. M. (1970). *Structure des systèmes dynamiques*, Dunod, Paris.
- [30] Stratonovich, R. L. (1957). *Sov. Phys. JETP*, **4**, 891.
- [31] Takhtajan, L. A. (1990). "Lectures on Quantum Groups". In *Introduction to Quantum Groups and integrable massive models in Quantum Field Theory*, Mo-Lin Ge, Bao-Heng Zhao eds., World Scientific, Singapore.
- [32] Weyl, H. (1950). *The Theory of Groups and Quantum Mechanics*, Dover, New York.
- [33] Wigner, E. P. (1932). *Phys. Rev.*, **40**, 749.
- [34] Wigner, E. P. (1939). *Ann. Math.*, **40**, 149.
- [35] Wilde, M. de and Lecomte, P. B. A. (1983). *Lett. Math. Phys.*, **7**, 487.
- [36] Woodhouse, N. M. J. (1992). *Geometric quantization*, Clarendon Press, Oxford.
- [37] Yu Reshetikhin, N., Takhtajan, L. A. and Faddeev, L. D. (1990). *Leningrad Math. J.*, **1**, 193.

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