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Representation of multilinear operators on $C(K, X)$ spaces

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Abstract. We present a Riesz type representation theorem for multilinear operators defined on the product of $C(K, X)$ spaces with values in a Banach space. In order to do this we make a brief exposition of the theory of operator valued polymeasures.

Representacón de operadores multilineales sobre espacios $C(K, X)$

Resumen. Probamos un teorema de representación de tipo Riesz para operadores multilineales definidos en el producto de espacios $C(K, X)$. Para poder hacer esto exponemos unas breves nociones de la teoría de polimedidas con valores en un espacio de operadores.

1. Introduction and notation

In a series of papers, Dobrakov developed the theory of polymeasures, in order, among other things, to obtain several kinds of Riesz type representation theorems for multilinear operators defined on the product of spaces of continuous functions (scalar of vector valued) and taking values in Banach spaces. Thus, he obtained in [7] a representation theorem for multilinear operators acting on the product of $C(K)$ spaces, and in [8] a similar theorem for multilinear operators acting on the product of $C(K, X)$ spaces. However, as pointed out in [2], there is a mistake in the first of those theorems, which he carries along to the second theorem. Using the representation theorem we obtained in [2] for multilinear operators on the product of $C(K)$ spaces, we obtain here a representation theorem for multilinear operators on the product of $C(K, X)$ spaces. As in [2], the representation is done in terms of *Borel* polymeasures, whereas, using Dobrakov's techniques, it is not possible to go beyond *Baire* polymeasures. This theorem has been applied in [13] and [3] to obtain information about those multilinear operators.

In this paper, n will denote a natural number, Y, X, X_i will be Banach spaces and X^* denotes the topological dual of X. We write $\mathcal{L}^n(X_1, \dots X_n; Y)$for the space of the multilinear operators from $X_1 \times \cdots \times X_n$ into Y with the usual norm. If $n = 1$ or $Y = \mathbb{K}$, we do not write them.

$X_1 \hat{\otimes} \dots \hat{\otimes} X_n$ is the complete projective tensor product of $X_1, \dots X_n$. We suppose well known that $\mathcal{L}^n(X_1 \dots, X_n; Y)$ is isometric to $\mathcal{L}(X_1 \hat{\otimes} \dots \hat{\otimes} X_n; Y)$. We use the notation $.^{[i]}.$ to mean that the i^{th} coordinate is not involved.

K, K_i denote compact Hausdorff spaces. $C(K, X)$ is the space of the continuous functions defined on K with values in X. We write $suppf$ for the support of a function f.

If

$$\lambda : \Sigma \longrightarrow \mathcal{L}(X; Y)$$

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is a finitely additive set function, we say that λ is an *operator valued measure*, and in that case we consider its *semivariation* $|\lambda| : \Sigma \longrightarrow [0, +\infty]$ to be defined by

$$|\lambda|(A) = \sup \left\{ \left\| \sum_{j=1}^{m} \lambda(A_j)(x_j) \right\| \right\}$$

where $(A_j)_{j=1}^m$ is a Σ-partition of A, $x_j \in X$ and $\|x_j\| \leq 1$.

If $|\lambda|$ is bounded, we write $\lambda \in ba(\Sigma; \mathcal{L}(X; Y))$. If λ is a measure or an operator valued measure, we denote its variation by $v(\lambda)$; it is well known, and very easy to check, that if $\lambda : \Sigma \longrightarrow X^*$ is an operator valued measure, then $|\lambda| = v(\lambda)$.

2. Operator valued polymeasures

If F is a Banach space and $\Sigma_1, \ldots, \Sigma_n$ are σ-algebras, following [6], we say that a set function $\Gamma : \Sigma_1 \times \cdots \times \Sigma_n \longrightarrow F$ is a *polymeasure* if it is separately finitely additive. If $F = \mathcal{L}^k(X_1 \ldots, X_n; Y)$ then we say that Γ is an *operator valued polymeasure*. In this last case, we define its *semivariation*

$$|\Gamma| : \Sigma_1 \times \cdots \times \Sigma_n \longrightarrow [0, +\infty]$$

by

$$|\Gamma|(A_1, \ldots, A_n) = \sup \left\{ \left\| \sum_{j_1=1}^{n_1} \cdots \sum_{j_n=1}^{n_n} \Gamma(A_{1,j_1}, \ldots, A_{k,j_n})(x_{1,j_1}, \ldots, x_{k,j_n}) \right\| \right\}$$

where $(A_{i,j_i})_{j_i=1}^{n_i}$ is a Σ_i-partition of A_i $(1 \leq i \leq k)$, $x_{i,j_i} \in X_i$ and $\|x_{i,j_i}\| \leq 1$. It is trivial to check that this definition generalizes the above mentioned semivariation of an operator valued measure. $|\Gamma|$ is separately monotone and subadditive.

Similarly to the case of measures, we have that $\|\Gamma\| \leq |\Gamma| \leq v(\Gamma)$, where $\|\Gamma\|$ and $v(\Gamma)$ are the *scalar semivariation* and *variation* of Γ, whose definitions can be seen in [6] or [11]. If $X_i = \mathbb{K}$ $(1 \leq i \leq k)$ then $\|\Gamma\| = |\Gamma|$. Notice that, opposite to the case of measures, if $k > 1$ and $Y = \mathbb{K}$, it is not necessarily true that $|\Gamma| = v(\Gamma)$.

We denote by $pm(\Sigma_1, \ldots, \Sigma_n; X)$ the set of polymeasures from $\Sigma_1 \times \cdots \times \Sigma_n$ into X. We say that $\Gamma \in pm(\Sigma_1, \ldots, \Sigma_n; X)$ is *countably additive* (resp. *regular*) if it is separately countably additive (resp. separately regular), that is, if for every $i \in \{1, \ldots, k\}$ and every $(A_1, \overset{[i]}{\ldots}, A_n) \in \Sigma_1 \times \overset{[i]}{\cdots} \times \Sigma_n$, the measure

$$\Gamma(A_1, \ldots, A_{i-1}, \cdot, A_{i+1}, \ldots, A_n) : \Sigma_i \longrightarrow X$$

is countably additive (resp. regular). In that case we write $\Gamma \in capm(\Sigma_1, \ldots, \Sigma_n; X)$ (resp. $\Gamma \in rcapm(\Sigma_1, \ldots, \Sigma_n; X)$).

If Γ is an operator valued polymeasure defined from $\Sigma_1 \times \cdots \times \Sigma_n$ into $\mathcal{L}^n(X_1, \ldots, X_n; Y)$ then, for every $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ we define the polymeasure $\Gamma_{x_1,\ldots,x_n} \in pm(\Sigma_1, \ldots, \Sigma_n; Y)$ by

$$\Gamma_{x_1,\ldots,x_n}(A_1, \ldots, A_n) = \Gamma(A_1, \ldots, A_n)(x_1, \ldots, x_n)$$

and, for every $y^* \in Y^*$, we can also define the operator valued polymeasure

$$\Gamma_{y^*} \in pm(\Sigma_1, \ldots, \Sigma_n; \mathcal{L}^n(X_1, \ldots, X_n; \mathbb{K}))$$

by

$$\Gamma_{y^*}(A_1, \ldots, A_n)(x_1, \ldots, x_n) = \langle \Gamma(A_1, \ldots, A_n)(x_1, \ldots, x_n), y^* \rangle.$$

If $D \subset Y^*$ is a subspace norming Y, then

$$|\Gamma| = \sup_{y^* \in D, \|y^*\| \leq 1} |\Gamma_{y^*}|.$$

Let $\Gamma : \Sigma_1 \times \cdots \times \Sigma_n \longrightarrow \mathcal{L}^n(X_1, \ldots, X_n)$ be an operator valued polymeasure. If, for every $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$, the polymeasure $\Gamma_{x_1,\ldots,x_n} : \Sigma_1 \times \cdots \times \Sigma_n \longrightarrow \mathbb{K}$ is regular, then we say that Γ is *weak* regular*. We shall use analogous notation for measures. For definitions, notation and basic concepts concerning polymeasures, see [2], [6] or [12].

From now on, Σ, Σ_i will be the σ-algebras of the Borel sets of K, K_i. $S(\Sigma)$ will be the normed space of the scalar Σ-simple functions defined on K endowed with the supremum norm and $S(\Sigma, X)$ will be the normed space of the X-valued Σ-simple functions defined on K endowed also with the supremum norm. $B(\Sigma)$ and $B(\Sigma, X)$ denote the completion of $S(\Sigma)$ and $S(\Sigma, X)$ respectively.

If $s_i = \sum_{j_i=1}^{n_i} \chi_{A_{i,j_i}} x_{i,j_i} \in S(\Sigma_i, X_i)$ then, for every operator valued polymeasure

$$\Gamma \in pm(\Sigma_1, \ldots, \Sigma_n; \mathcal{L}^n(X_1, \ldots, X_n; Y)),$$

the formula

$$T_\Gamma(s_1, \ldots, s_n) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_n=1}^{n_n} \Gamma(A_{1,j_1}, \ldots, A_{k,j_n})(x_{1,j_1}, \ldots, x_{k,j_n})$$

defines a multilinear map

$$T_\Gamma : S(\Sigma_1, X_1) \times \cdots \times S(\Sigma_n, X_n) \to Y$$

such that $\|T_\Gamma\| = |\Gamma|(K_1, \ldots, K_n) (\stackrel{def}{=} |\Gamma|)$.

So, if $|\Gamma| < \infty$, i.e., if Γ has *finite semivariation*, then T_Γ can be uniquely extended (with the same norm) to $B(\Sigma_1, X_1) \times \cdots \times B(\Sigma_n, X_n)$. We still denote this extension by T_Γ and we write also

$$T_\Gamma(g_1, \ldots, g_n) \stackrel{def}{=} \int (g_1, \ldots, g_n) d\Gamma.$$

Conversely, if $T : B(\Sigma_1, X_1) \times \cdots \times B(\Sigma_n, X_n) \longrightarrow Y$ is a multilinear operator, the set function $\Gamma_T : \Sigma_1 \times \cdots \times \Sigma_n \longrightarrow \mathcal{L}^n(X_1, \ldots, X_n; Y)$ defined by

$$\Gamma_T(A_1 \times \cdots \times A_n)(x_1, \ldots, x_n) = T(x_1 \chi_{A_1}, \ldots, x_n \chi_{A_n})$$

is an operator valued polymeasure which verifies $|\Gamma_T| = \|T\|$. So, we have proved the following

Proposition 1 *The correspondence $\Gamma \leftrightarrow T_\Gamma$ is an isometric isomorphism between the space*

$$bpm(\Sigma_1, \ldots, \Sigma_n; \mathcal{L}^n(X_1, \ldots, X_n; Y))$$

of all $\mathcal{L}^n(X_1, \ldots, X_n; Y)$ valued polymeasures of finite semivariation, endowed with the semivariation norm, and the space $\mathcal{L}^n(B(\Sigma_1, X_1), \ldots, B(\Sigma_n, X_n); Y)$ endowed with the usual multilinear operator norm.

For a quite exhaustive presentation of the integral with respect to polymeasures, see [8] and the references therein. See also [9], [10] and [5] for integration with respect to certain particular classes of polymeasures.

The next lemma is a multilinear generalization of a well known linear result. We will need it later on. To simplify the proof we introduce some notation. Let us choose a multilinear operator $T : B(\Sigma_1, X_1) \times \cdots \times B(\Sigma_n, X_n) \longrightarrow Y$ with representing polymeasure Γ. If we fix $(g_1, \ldots, g_k) \in B(\Sigma_1, X_1) \times \cdots \times B(\Sigma_k, X_k)$, $(1 \leq k < n)$ then $\Gamma_{g_1,\ldots,g_k}$ is the polymeasure representing the multinear operator $T_{g_1,\ldots,g_k} : B(\Sigma_{k+1}, X_{k+1}) \times \cdots \times B(\Sigma_n, X_n) \longrightarrow Y$ given by $T_{g_1,\ldots,g_k}(g_{k+1}, \ldots, g_n) = T(g_1, \ldots, g_n)$.

Lemma 1 *Let $X_1, \ldots, X_n, Y$ be Banach spaces, $K_1, \ldots, K_n$ Hausdorff compact spaces and $\Gamma : \Sigma_1 \times \cdots \times \Sigma_n \to \mathcal{L}^n(X_1, \ldots, X_n; Y)$ a polymeasure such that, for every $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ and for every $y^* \in D \subset Y^*$, where D is a norming subspace, the scalar polymeasure $\Gamma_{(x_1,\ldots,x_n),y^*}$ is regular and such that the vector valued polymeasure $\Gamma_{x_1,\ldots,x_n}$ has bounded semivariation for every $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$. Then, for every $(A_1, \ldots, A_n) \in \Sigma_1 \times \cdots \times \Sigma_n$,*

$$|\Gamma|(A_1, \ldots, A_n) = \sup \left\{ \left\| \int_{A_1,\ldots,A_n} (f_1, \ldots, f_n) d\Gamma \right\| : f_i \in C(K_i) \otimes X_i,\ \|f_i\|_{A_i} \leq 1 \right\},$$

where $\|f\|_A = \sup_{t \in A} \|f(t)\|$.

PROOF. We reason by induction on k. For $k = 1$ the result is known (see, e.g., [1, Proposición III.1.9]). Suppose it true for $k - 1$ and let Γ be as in the hypothesis. Let us fix $\epsilon > 0$. According to the definition,

$$|\Gamma|(A_1, \ldots, A_n) = \sup \left\{ \left\| \int_{A_1,\ldots,A_n} (g_1, \ldots, g_n) d\Gamma \right\| : g_i \in S(\Sigma_i, X_i),\ \|g_i\|_{A_i} \leq 1 \right\}.$$

So, for $1 \leq i \leq k$, let $g_i \in S(\Sigma_i, X_i)$, with $\|g_i\|_{A_i} \leq 1$ be such that

$$|\Gamma|(A_1, \ldots, A_n) \leq \left\| \int_{A_1 \times \cdots \times A_n} (g_1, \ldots, g_n) d\Gamma \right\| + \epsilon.$$

Now,

$$\int_{A_1 \times \cdots \times A_n} (g_1, \ldots, g_n) d\Gamma = \int_{A_1 \times \cdots \times A_{n-1}} (g_1, \ldots, g_{n-1}) d\Gamma_{g_n \chi_{A_n}},$$

and we use the induction hypothesis to find $(f_1, \ldots, f_{n-1}) \in C(K_1) \otimes X_1 \times \cdots \times C(K_{n-1}) \otimes X_{n-1}$ as in the statement of the lemma such that

$$|\Gamma_{g_n \chi_{A_n}}|(A_1, \ldots, A_{n-1}) \leq \left\| \int_{A_1 \times \cdots \times A_{n-1}} (f_1, \ldots, f_{n-1}) d\Gamma_{g_n \chi A_n} \right\| + \epsilon.$$

So,

$$|\Gamma|(A_1, \ldots, A_n) \leq \left\| \int_{A_1 \times \cdots \times A_{n-1}} (f_1, \ldots, f_{n-1}) d\Gamma_{g_n \chi A_n} \right\| + 2\epsilon$$

$$= \left\| \int_{A_1 \times \cdots \times A_n} (f_1, \ldots, f_{n-1}, g_n) d\Gamma \right\| + 2\epsilon = \left\| \int_{A_n} g_n d\Gamma_{f_1 \chi_{A_1}, \ldots, f_{n-1} \chi_{A_{n-1}}} \right\| + 2\epsilon.$$

Now we use the linear version of the result to find $f_n \in C(K_n) \otimes X_n$ as in the statement of the lemma such that

$$|\Gamma_{f_1 \chi_{A_1}, \ldots, f_{n-1} \chi_{A_n}}|(A_n) \leq \left\| \int_{A_n} f_n d\Gamma_{f_1 \chi_{A_1}, \ldots, f_{n-1} \chi_{A_n}} \right\| + \epsilon,$$

and we finally get that

$$|\Gamma|(A_1, \ldots, A_n) \leq \left\| \int_{A_n} f_n d\Gamma_{f_1 \chi_{A_1}, \ldots, f_{n-1} \chi_{A_{n-1}}} \right\| + 3\epsilon =$$

$$= \left\| \int_{A_1 \times \cdots \times A_n} (f_1, \ldots, f_n) d\Gamma \right\| + 3\epsilon. \quad \blacksquare$$

3. The representation theorems

We state next the representation theorem for multilinear forms.

Theorem 1 *Let $T \in \mathcal{L}^n(C(K_1,X_1),\dots,C(K_n,X_n))$. Then T has one only extension $\overline{T} \in \mathcal{L}^n(B(\Sigma_1,X_1),\dots,B(\Sigma_n,X_n))$ with the same norm and separately weak* continuous (the weak* topology we consider in $B(\Sigma_i,X_i)$ is the one induced by the canonic isometric inclusion $B(\Sigma_i,X_i) \hookrightarrow (C(K_i,X_i))^{**}$).*

Moreover, if we define the operator valued polymeasure

$$\Gamma : \Sigma_1 \times \cdots \times \Sigma_n \longrightarrow \mathcal{L}^n(X_1,\dots,X_n)$$

by

$$\Gamma(A_1,\dots,A_n)(x_1,\dots,x_n) = \overline{T}(x_1\chi_{A_1},\dots,x_n\chi_{A_n})$$

then we have:

i) Γ has bounded semivariation and $|\Gamma| = \|T\|$.

ii) Γ is weak regular.*

iii) For every $f_i \in C(K_i,X_i)$ $(1 \le i \le k)$,

$$T(f_1,\dots,f_n) = \int (f_1,\dots,f_n)d\Gamma.$$

Conversely, given a weak operator valued polymeasure with bounded semivariation*

$$\Gamma : \Sigma_1 \times \cdots \times \Sigma_n \longrightarrow \mathcal{L}^n(X_1,\dots,X_n)$$

the formula given in (iii) defines a multilinear form

$$T : C(K_1,X_1) \times \cdots \times C(K_n,X_n) \longrightarrow \mathbb{K}$$

for which (i) holds.

So, the correspondence between T and Γ defines an isometry between the space of the weak regular operator valued polymeasures with bounded semivariation defined on $\Sigma_1 \times \cdots \times \Sigma_n$ and with values in $\mathcal{L}^n(X_1,\dots,X_n)$, endowed with the semivariation norm, and the space $\mathcal{L}^n(C(K_1,X_1),\dots,C(K_n,X_n))$.*

PROOF. Let $T \in \mathcal{L}^n(C(K_1,X_1),\dots,C(K_n,X_n))$. Given $(x_1,\dots,x_n) \in (X_1,\dots,X_n)$ we can define the multilinear operator

$$T_{x_1,\dots,x_n} : C(K_1) \times \cdots \times C(K_n) \longrightarrow \mathbb{K}$$

by

$$T_{x_1,\dots,x_n}(\varphi_1,\dots,\varphi_n) = T(x_1\varphi_1,\dots,x_n\varphi_n).$$

Let

$$\Gamma_{x_1,\dots,x_n} \in rcapm(\Sigma_1,\dots,\Sigma_n)(\approx (C(K_1)\hat{\otimes}\dots\hat{\otimes}C(K_n))^*)$$

be its representing scalar polymeasure (see [2]). For every $(A_1,\dots,A_n) \in \Sigma_1 \times \cdots \times \Sigma_n$ we define $\Gamma(A_1,\dots,A_n) \in \mathcal{L}^n(X_1,\dots,X_n)$ by

$$\Gamma(A_1,\dots,A_n)(x_1,\dots,x_n) = \Gamma_{x_1,\dots,x_n}(A_1,\dots,A_n).$$

Γ is well defined and weak* regular. Moreover Γ verifies the hypothesis of Proposition 1, therefore

$$|\Gamma|(K_1,\dots,K_n) = \sup\left\{\left\|\int (f_1,\dots,f_n)d\Gamma\right\| ; f_i \in C(K_i)\otimes X_i, \|f_i\| \le 1\right\}$$

$$= \sup\{\|T(f_1,\dots,f_n)\| ; f_i \in C(K_i)\otimes X_i, \|f_i\| \leq 1\} = \|T\|,$$

since $C(K_i)\otimes X_i$ is dense in $C(K_i,X_i)$. Γ has bounded semivariation, so we can define the integral $\int (f_1,\dots,f_n)d\Gamma$ for every $f_i \in C(K_i,X_i)$. We know that

$$T(f_1,\dots,f_n) = \int (f_1,\dots,f_n)d\Gamma \quad \text{for every } f_i \in C(K_i)\otimes X_i,$$

so,

$$T(f_1,\dots,f_n) = \int (f_1,\dots,f_n)d\Gamma \text{ for every } f_i \in C(K_i,X_i).$$

Let now $\overline{T} \in \mathcal{L}^n(B(\Sigma_1,X_1),\dots,B(\Sigma_n,X_n))$ be the operator associated to Γ by Proposition 1. We have $\|T\| = \|\Gamma\| = \|\overline{T}\|$. To see that it is separately weak* continuous, it suffices to check in the last variable, since the others behave similarly. So, let us fix $(g_1,\dots,g_{n-1}) \in B(\Sigma_1,X_1)\times\cdots\times B(\Sigma_{n-1},X_{n-1})$ and let $(g_n^\alpha)_{\alpha\in A} \subset B(\Sigma_n,X_n)$ a net weak* converging to $g_n \in B(\Sigma_n,X_n)$. We consider the measure

$$\Gamma_{g_1,\dots,g_{n-1}} : \Sigma_n \longrightarrow X_n^* \text{ given by}$$

$$\Gamma_{g_1,\dots,g_{n-1}}(A_n)(x_n) = \overline{T}(g_1,\dots,g_{n-1},x_n\chi_{A_n}).$$

It is easy to check that $\Gamma_{g_1,\dots,g_{n-1}} \in rcabv(\Sigma_n,X_n^*) \approx (C(K_n,X_n))^*$ and that, for every $g_n \in B(\Sigma_n,X_n)$,

$$\int g_n d\Gamma_{g_1,\dots,g_{n-1}} = \int (g_1,\dots,g_n)d\Gamma = \overline{T}(g_1,\dots,g_n).$$

Therefore,

$$\int g_n^\alpha d\Gamma_{g_1,\dots,g_{n-1}} \to \int g_n d\Gamma_{g_1,\dots,g_{n-1}},$$

which means that

$$\overline{T}(g_1,\dots,g_n^\alpha) \to \overline{T}(g_1,\dots,g_n).$$

Conversely, if $\Gamma : \Sigma_1\times\cdots\times\Sigma_n \longrightarrow \mathcal{L}^n(X_1,\dots,X_n)$ is a weak* regular operator valued polymeasure with bounded semivariation, then, Proposition 1 assures that Γ gives rise to an operator $\overline{T} \in \mathcal{L}^n(B(\Sigma_1,X_1),\dots,B(\Sigma_n,X_n))$ through the formula

$$\overline{T}(g_1,\dots,g_n) = \int (g_1,\dots,g_n)d\Gamma.$$

By restriction, we get $T \in \mathcal{L}^n(C(K_1,X_1),\dots,C(K_n,X_n))$. The equality $\|T\| = \|\Gamma\|$ follows from proposition 1. ■

Note that, in general, not every operator from $C(K_i,X_i)$ into $C(K_j,X_j)^*$ is weakly compact, so a multilinear operator $T \in \mathcal{L}^n(C(K_1,X_1),\dots,C(K_n,X_n))$ need not have a separately weak* continuous extension to the product of the biduals (see [4]). The previous result tells us that T *does* have a separately weak* continuous extension to $B(\Sigma_1,X_1)\times\cdots\times B(\Sigma_n,X_n)$. Note also that the extension morphism is linear, so we can extend the result to the case of vector valued multilinear operators.

Theorem 2 *Let $T \in \mathcal{L}^n(C(K_1,X_1),\dots,C(K_n,X_n);Y)$. Then T has one only extension $\overline{T} \in \mathcal{L}^n(B(\Sigma_1,X_1),\dots,B(\Sigma_n,X_n);Y^{**})$ with the same norm and separately weak* to weak* continuous.*

Moreover, if we define the operator valued polymeasure

$$\Gamma : \Sigma_1\times\cdots\times\Sigma_n \longrightarrow \mathcal{L}^n(X_1,\dots,X_n;Y^{**})$$

by

$$\Gamma(A_1,\dots,A_n)(x_1,\dots,x_n) = \overline{T}(x_1\chi_{A_1},\dots,x_n\chi_{A_n})$$

then we have:

i) Γ has bounded semivariation and $|\Gamma|(K_1,\dots,K_n)=\|T\|$.

ii) For every $y^*\in Y^*(\subset Y^{***})$, Γ_{y^*} is weak regular.*

iii) The mapping

$$\begin{array}{ccc} Y^* & \longrightarrow & (C(K_1,X_1)\hat{\otimes}\cdots\hat{\otimes}C(K_n,X_n))^* \\ y^* & \mapsto & \Gamma_{y^*} \end{array}$$

is weak to weak* continuous.*

iv) $T(f_1,\dots,f_n)=\int(f_1,\dots,f_n)d\Gamma$, for every $f_i\in C(K_i,X_i)$.

Conversely, every polymeasure

$$\Gamma:\Sigma_1\times\cdots\times\Sigma_n\longrightarrow\mathcal{L}^n(X_1,\dots,X_n;Y^{**})$$

with bounded semivariation and verifying (ii) and (iii) defines via (iv) an n-linear operator

$$T:C(K_1,X_1)\times\cdots\times C(K_n,X_n)\longrightarrow Y$$

for which (i) holds.

PROOF. Let $T\in\mathcal{L}^n(C(K_1,X_1),\dots,C(K_n,X_n);Y)$. Let us define

$$\overline{T}:B(\Sigma_1,X_1)\times\cdots\times B(\Sigma_n,X_n)\longrightarrow Y^{**}$$

by

$$\langle\overline{T}(g_1,\dots,g_n),y^*\rangle=\overline{y^*\circ T}(g_1,\dots,g_n)$$

where $\overline{y^*\circ T}$ is the extension of $y^*\circ T$ given by Theorem 1. It follows that $\overline{T}$ is separately weak* to weak* continuous and that $\|\overline{T}\|=\|T\|$.

Let now $\Gamma:\Sigma_1\times\cdots\times\Sigma_n\longrightarrow\mathcal{L}^n(X_1,\dots,X_n;Y^{**})$ be the operator valued polymeasure associated to $\overline{T}$ by Proposition 1. It follows from the definitions that, for every $y^*\in Y^*$, the operator valued polymeasure Γ_{y^*} defined by $\Gamma_{y^*}(A_1,\dots,A_n)(x_1,\dots,x_n)=\langle\Gamma(A_1,\dots,A_n)(x_1,\dots,x_n),y^*\rangle$ is precisely the operator valued polymeasure associated to the multilinear operator $y^*\circ T$ by Theorem 1. Since

$$|\Gamma|(K_1,\dots,K_n)=\sup_{\|y^*\|\le1}|\Gamma_{y^*}|(K_1,\dots,K_n)=\sup_{\|y^*\|\le1}\|y^*\circ T\|=\|T\|,$$

we get that (i) holds. (ii) and (iv) follow immediately from the definitions.

To see (iii), let

$$\hat{T}:C(K_1,X_1)\hat{\otimes}\cdots\hat{\otimes}C(K_n,X_n)\longrightarrow Y$$

be the linear operator associated to T. Then

$$\hat{T}^*:Y^*\longrightarrow(C(K_1,X_1)\hat{\otimes}\cdots\hat{\otimes}C(K_n,X_n))^*$$

is weak* to weak* continuous. We just need to check that $\hat{T}^*(y^*)=\Gamma_{y^*}$. To see this, let us consider $f_1\otimes\cdots\otimes f_n\in C(K_1,X_1)\otimes\cdots\otimes C(K_n,X_n)$. Then

$$\hat{T}^*(y^*)(f_1\otimes\cdots\otimes f_n)=y^*\circ\hat{T}(f_1\otimes\cdots\otimes f_n)=y^*\circ T(f_1,\dots,f_n)$$

$$=y^*\left(\int(f_1,\dots,f_n)d\Gamma\right)=\int(f_1,\dots,f_n)d\Gamma_{y^*}.$$

It follows that, for every $\sum_{j=1}^m f_1^j\otimes\cdots\otimes f_n^j\in C(K_1,X_1)\otimes\cdots\otimes C(K_n,X_n)$,

$$\hat{T}^*(y^*)(\sum_{j=1}^m f_1^j\otimes\cdots\otimes f_n^j)=\sum_{j=1}^m\int(f_1^j,\dots,f_n^j)d\Gamma_{y^*}$$

Using density, we get that (iii) holds.

The uniqueness of $\overline{T}$ is equivalent to the uniqueness of Γ. To see the uniqueness of Γ note that if, for every $(f_1,\dots,f_n)\in C(K_1,X_1)\times\cdots\times C(K_n,X_n)$,

$$\int (f_1,\dots,f_n)d\Gamma_1 = \int (f_1,\dots,f_n)d\Gamma_2$$

then, for every $y^*\in Y^*$,

$$\int (f_1,\dots,f_n)d\Gamma_{1,y^*} = \int (f_1,\dots,f_n)d\Gamma_{2,y^*},$$

so, the uniqueness of the polymeasures in the scalar case suffices to finish.

Conversely, let

$$\Gamma : \Sigma_1\times\cdots\times\Sigma_n \longrightarrow \mathcal{L}^n(X_1,\dots,X_n;Y^{**})$$

be a polymeasure of bounded semivariation for which (ii) and (iii) hold. Let us then define the multilinear operator $T\in\mathcal{L}^n(C(K_1,X_1),\dots,C(K_n,X_n);Y)$ by

$$T(f_1,\dots,f_n)=\int (f_1,\dots,f_n)d\Gamma \text{ for every } f_i\in C(K_i,X_i).$$

To see that T is well defined, let us observe first that T is the restriction to $C(K_1,X_1)\times\cdots\times C(K_n,X_n)$ of the operator $\overline{T}\in\mathcal{L}^n(B(\Sigma_1,X_1),\dots,B(\Sigma_n,X_n);Y^{**})$ associated to Γ by Proposition 1, so

$$\|T\|\le\|\overline{T}\| = |\Gamma|(K_1,\dots,K_n).$$

Moreover, since Γ verifies the hypothesis of Proposition 1 we get that

$$\|T\| = |\Gamma|(K_1,\dots,K_n).$$

We just have to see that T takes values in Y. To see this, let us take into account that, for every $y^*\in Y^*$ and for every $(f_1,\dots,f_n)\in C(K_1,X_1)\times\cdots\times C(K_n,X_n)$,

$$\langle y^*, T(f_1,\dots,f_n)\rangle = \left\langle y^*, \int (f_1,\dots,f_n)d\Gamma\right\rangle = \int (f_1,\dots,f_n)d\Gamma_{y^*}$$

so, (iii) implies that $T(f_1,\dots,f_n)\in Y^{**}$ is $\sigma(Y^*,Y)$ continuous which means that $T(f_1,\dots,f_n)\in Y$ and $\hat{T}^*(y^*)=\Gamma_{y^*}$. ■

Remark 1 *It is easy to see that, if $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$, then $\Gamma_{x_1,\dots,x_n}$ is the representing polymeasure of the operator $T_{x_1,\dots,x_n}\in\mathcal{L}^n(C(K_1),\dots,C(K_n);Y)$ given by*

$$T_{x_1,\dots,x_n}(\varphi_1,\dots,\varphi_n)=T(x_1\varphi_1,\dots,x_n\varphi_n).$$

Finally, we have

Proposition 2 *Let $T\in\mathcal{L}^n(C(K_1,X_1),\dots,C(K_n,X_n);Y)$ and let Γ and $\overline{T}$ be its representing polymeasure and extension given by Theorem 2. Then the following are equivalent:*

i) Γ is $\mathcal{L}^n(X_1,\dots,X_n;Y)$ valued.

ii) $\overline{T}$ is Y valued.

iii) For every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$, $\Gamma_{x_1,\dots,x_n}$ is countably additive.

iv) For every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$, $\Gamma_{x_1,\dots,x_n}$ is regular.

v) For every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$ and for every $y^{***}\in Y^{***}$, $\Gamma_{x_1,\dots,x_n,y^{***}}$ is countably additive.

vi) For every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$ and for every $y^{***}\in Y^{***}$, $\Gamma_{x_1,\dots,x_n,y^{***}}$ is regular.

Moreover, if

vii) For every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$, $T_{x_1,\dots,x_n}: C(K_1)\times\cdots\times C(K_n)\longrightarrow Y$ is weakly compact

then (i) through (vi) hold.

PROOF. Clearly (i) and (ii) are equivalent. If (i) holds, then, for every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$, $\Gamma_{x_1,\dots,x_n}$ is weakly countably additive and the Orlicz-Pettis theorem gives (iii). The equivalence between (iii) and (iv) and between (v) and (vi) follows from [11, Theorem 5]. Clearly (iii) implies (v). Another application of the Orlicz-Pettis theorem shows that (v) implies (iii). If (iii) holds, [11, Theorem 5] implies that, for every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$, $\Gamma_{x_1,\dots,x_n}$ is Y valued, which is equivalent to (i). If (vii) holds, another application of [11, Theorem 5] suffices to prove (i). ■

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