RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **96 (**1**),** 2002**,** pp. 23–25
Análisis Matemático / Mathematical Analysis
Comunicación Preliminar / Preliminary Communication

A note about a theorem by L. Hörmander

A. García del Amo, M. Maldonado and J. Prada

Abstract. We point out that the tempered distributions that appear in the characterizations of translations-invariant operators in L^p spaces are elements of the dual of a Sobolev space.

Una nota sobre un teorema de L. Hörmander.

Resumen. Observamos que las distribuciones temperadas que aparecen en la caracterización de los operadores invariantes por traslación en los espacios L^p, son elementos del dual de un espacio de Sobolev.

In 1960, L. Hörmander in a paper published in Acta Math. proved the following characterization of bounded translations invariant operators between L^p spaces.

Theorem 1 *If A is a bounded translation invariant operator from L^p to L^q, then there is a unique distribution $T \in \mathcal{S}'$ such that*

$$Au = T * u, u \in \mathcal{S}.$$

For the proof he needs the following lemma which is a very special case of Sobolev's lemma

Lemma 1 *If a function v in $\mathbb{R}^n$ and its derivatives of order $\leq n$ are in L^p locally, the definition of v may be changed on a set of measure 0 to make it continuous. Then we have with a C*

$$|v(x)| \leq C \sum_{|\alpha| \leq n} \left(\int_{|y-x| \leq 1} |D^\alpha v|^p \, dy \right)^{\frac{1}{p}} .$$

In the proof of Theorem 1, he claims that

$$D^\alpha(Au) = A(D^\alpha u)$$

in the distribution sense and after proving it, using the previous lemma, he finds the inequality

$$|A(u)(0)| \leq C \sum_{|\alpha| \leq n} \|D^\alpha u\|_p$$

Presentado por Jesús Ildefonso Díaz.
Recibido: 19 de Octubre de 2001. *Aceptado*: 9 de Enero de 2002.
Palabras clave / Keywords: multipliers, L^p-spaces
Mathematics Subject Classifications: 42B15, 46E30

from which he deduces that $(Au)(0)$ is a continuous linear form on $\mathcal{S}$ so that it may be written

$$(Au)(0) = T(\widetilde{u}) = (T * u)(0),$$

where $\widetilde{u}(x) = u(-x)$ and $T \in \mathcal{S}'$.

In this note we just want to point out that the above inequality implies that $(Au)(0)$ is a continuous linear form on $\mathcal{S}$ taken as a subspace of the Sobolev space $\mathcal{W}_n^p(\mathbb{R}^n)$.

Assume that $p < \infty$; as $\mathcal{S}$ is a dense subspace of $\mathcal{W}_n^p(\mathbb{R}^n)$, then $(Au)(0)$ can be extended to a continuous linear form on the mentioned Sobolev space, that is, an element of $\mathcal{W}^{-n,p'}(\mathbb{R}^n) = (\mathcal{W}_n^p(\mathbb{R}^n))'$ (p and p' are conjugate exponents).

Therefore T can be written

$$T = \sum_{|\alpha| \le n} D^\alpha f_\alpha, \quad f_\alpha \in L^{p'}(\mathbb{R}^n)$$

and so

$$Au = T * u = \sum_{|\alpha| \le n} f_\alpha * D^\alpha u, \quad u \in \mathcal{S}$$

or, taking Fourier transforms,

$$\widehat{T} = \sum_{|\alpha| \le n} (2\pi i)^{|\alpha|} x^\alpha \widehat{f}_\alpha.$$

When $p = q = 2$, Theorem 1.5 of the mentioned paper reads

Theorem 2 *With equality also of the norms, we have*

$$M_2^2 = L^\infty.$$

Note that in this particular case we have

$$L^\infty \subset \left\{ \widehat{f} \,:\, f = \sum_{|\alpha| \le n} D^\alpha f_\alpha,\ f_\alpha \in L^2(\mathbb{R}^n) \right\}.$$

If $f \in L^\infty(\mathbb{R})$, then it can be written as $f(x) = f_1(x) + 2\pi i x f_2(x)$, where f_1, $f_2 \in L^2(\mathbb{R})$ are

$$f_1(x) = f(x) \cdot \chi_{[-1,1]}(x),$$

$$f_2(x) = \frac{f(x) - f_1(x)}{2\pi i x}.$$

In general, if $f \in L^\infty(\mathbb{R}^n)$, $n > 1$, it is easy to see that

$$f(x) = f_1(x) + \left[(2\pi i)^2 \sum_{k=1}^n x_k^2 \right]^{\left[\frac{n}{4}\right]+1} f_2(x),$$

where $f_1, f_2 \in L^2(R^n)$ are

$$f_1(x) = f(x) \chi_{B_1(0)}(x)$$

$$f_2(x) = \frac{f(x) - f_1(x)}{\left[(2\pi i)^2 \sum_{k=1}^n x_k^2 \right]^{\left[\frac{n}{4}\right]+1}}$$

Acknowledgement. This paper has been partially supported by SA46/00B.

References

[1] Adams, R. (1977) *Sobolev Spaces.* Academic Press, New York.

[2] Hörmander, L. (1960) Estimates for translation invariant operators in L^p spaces. *Acta Math.*, **104**, 93–140.

[3] Maldonado, M. (2000) Multiplicadores en Espacios de Funciones. *Trabajo de Grado*. Universidad de Salamanca.

A. García del Amo
Departamento de Ciencias Experimentales y Tecnología
Universidad Rey Juan Carlos
C/ Tulipán s/n. E-28933 Móstoles (Madrid)
Spain
alejandro@escet.urjc.es

M. Maldonado
Departamento de Matemáticas
Universidad de Salamanca
Plaza de la Merced 1-4. E-37008 Salamanca.
Spain
cordero@usal.es

J. Prada
Departamento de Matemáticas
Universidad de Salamanca
Plaza de la Merced 1-4. E-37008 Salamanca.
Spain
prada@usal.es