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The error term in Nevanlinna's second fundamental theorem for holomorphic mappings on coverings

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Abstract. We study the error term in Nevanlinna's second fundamental theorem for holomorphic mappings $F : Y \to \mathbf{P}^1$, where on Y is defined an holomorphic covering map $p : Y \to \mathbb{C}$, as considered by S.Lang and W.Cherry. We also obtain the logarithmic derivative lemma for this class of functions which, in particular, contains the class of algebroid functions. We show that our estimate improves a classical result of Henrik Selberg on the logarithmic derivative for algebroid functions.

Término de error del segundo teorema fundamental de Nevanlinna sobre aplicaciones recubridoras holomorfas

Resumen. Se estudia el término de error en el segundo teorema fundamental para las aplicaciones holomorfas $F : Y \to \mathbf{P}^1$, donde en Y hay definida una aplicación recubridora holomorfa $p : Y \to \mathbb{C}$, aplicaciones consideradas por S. Lang and W. Cherry. También se obtiene el lema de la derivada logarítmica para esta clase de funciones que, en particular, contiene la clase de las funciones algebroides. Se demuestra que la estimación obtenida mejora un resultado clásico de Henrik Selberg sobre la derivada logarítmica de las funciones algebroides.

1. Introduction

In [8], Lang conjectured the following form for the fundamental inequality in Nevanlinna's second fundamental theorem for meromorphic functions $F : \mathbb{C} \to \mathbf{P}^1$

$$\sum_{j=1}^{q} m(r, a_j) - 2T(r, F) + N_{Ram,F}(r) \leq \log T(r, F) + \text{ lower order terms.} \tag{1}$$

Lang's conjecture was established by Wong [19], and Ye showed by an example, see [9], that (1) is essentially the best possible estimate that we can obtain, in the sense that the coefficient one of $\log T(r, F)$, cannot be replaced by any smaller number.

Later on, he also considered the lower order terms an gave a precise estimate for the error term in Nevanlinna's fundamental inequality, that is, the left hand side of (1) see Ye [20]. He proved

$$\sum_{j=1}^{q} m(r, a_j) - 2T(r, F) + N_{Ram,F}(r) \leq \log T(r, F) + \log \psi(T(r, F)) + \frac{1}{2} \log \psi(r), \tag{2}$$

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where $\psi : [1, \infty) \to \mathbb{R}$ is a Khintchine function, that is, ψ is positive, increasing and

$$\int_1^\infty \frac{dx}{x\psi(x)} < \infty.$$

In the same paper he showed that the estimate yielded by the right hand side of (1) is also valid and sharp in the logrithmic derivative lemma.

Here we shall consider the same questions for functions $F : Y \to \mathbf{P}^1$, where on Y is defined an holomorphic covering map $p : Y \to \mathbb{C}$. That is, Y is a connected Riemann surface and p a proper surjective holomorphic map. Lang and Cherry [9] developed a value distribution theory in this setting and makes sense to study the size of the error term in the second main theorem or the logarithmic derivative lemma.

We shall see in Section 6, that we can consider the algebroid functions as a proper subclass of the holomorphic mappings defined on coverings. Let $F(z)$ be an analytic function which has algebraic character for every $z \in \mathbb{C}$ and a constant finite number of branches $F_1(z), \ldots, F_k(z)$. Such a function will satisfy an equation of the form

$$\{F(z)\}^k + A_1(z)\{F(z)\}^{k-1} + \ldots + A_k(z) = 0, \tag{3}$$

with meromorphic functions as coefficients. According to a terminology due to Poincaré, we shall call such an analytic function, an algebroid function.

Conversely, any equation of the form (3) with meromorphic coefficients defines an algebroid function if the left hand side of the equation is irreducible, that is, cannot be expressed as a product of factors of the same type, with coefficients which are meromorphic in the plane.

Given an algebroid function F, we can consider the Riemann surface

$$Y = \{(z, F_\nu(z)),\ z \in \mathbb{C},\ \nu = 1, \ldots, k\}$$

with the canonical projection

$$\begin{array}{cccc} p: & Y & \longrightarrow & \mathbb{C} \\ & (z, F_\nu(z)) & \mapsto & p(z, F_\nu(z)) = z \end{array}$$

and on Y the holomorphic function

$$\begin{array}{cccc} F: & Y & \longrightarrow & \mathbf{P}^1 \\ & (z, F_\nu(z)) & \mapsto & F(z, F_\nu(z)) = F_\nu(z) \end{array}$$

which we shall denote by F, in the same way as the original algebroid function.

H.L.Selberg [14] developed a value distribution theory for algebroid functions. We shall see that the basic concepts introduced by H.Selberg for algebroid functions, proximity function, counting function, characteristic function, are essentially the same as those introduced by Lang and Cherry for holomorphic mappings on coverings. Finally we shall compare the logarithmic derivative lemma which we obtain for holomorphic mappings on coverings in the case of functions of finite order, with that obtained by H.L.Selberg [13] and check that we improve the estimate.

2. Notation and basic facts

For a proper covering $p : Y \to \mathbb{C}$, we follow the notation of [9].

$$[Y : \mathbb{C}] = \text{ the degree of the covering,}$$

$$Y(r) = \{y \in Y \mid\mid p(y) \mid < r\},$$

$$Y[r] = \{y \in Y \mid\mid p(y) \mid\leq r\},$$
$$Y < r >= \{y \in Y \mid\mid p(y) \mid= r\},$$
$$\sigma_Y = d^c \log \mid p \mid^2 = p^* \left(\frac{d\theta}{2\pi}\right),$$

where $d\theta$ is the usual form on $\mathbb{C}$.

In a local coordinate w

$$\sigma_Y(w) = \frac{i}{4\pi}\left(\frac{\overline{p'(w)}}{\overline{p(w)}}d\bar{w} - \frac{p'(w)}{p(w)}dw\right),$$

outside $Y < 0 >$.

Next let $F : Y \to \mathbf{P}^1$ be a nonconstant holomorphic map such that $F(y) \neq 0, \infty$, and $F'(y) \neq 0$, for all $y \in Y < 0 >$. Then for $a \in \mathbf{P}^1$, we let

$$n_F(a, r) = \text{number of roots of } F(z) = a \text{ in } Y[r] \text{ counted with multiplicity},$$

$$N_F(a, r) = \int_0^r \frac{n_F(a,t)}{t}dt = \sum_{y \in Y[r]} ord_y(F - a)\log\left|\frac{r}{p(y)}\right|,$$

$$m_F(a, r) = \int_{Y<r>} -\log \|F, a\|\sigma_Y,$$

where $\|,\|$ is the chordal distance in $\mathbf{P}^1$, and finally

$$T_{F,a}(r) = m_F(a, r) + N_F(a, r) + \sum_{y \in Y\langle 0\rangle}(ord_y p)\log\|F(y), a\|.$$

With these definitions we have

First Main Theorem. *$T_{F,a}(r)$ is independent of $a \in \mathbf{P}^1$, provided $F(y) \neq a$ for $y \in Y\langle 0\rangle$.*

In the light of the first main theorem we denote $T_{F,\infty}(r)$ by $T_F(r)$ and call it the characteristic function of F.

As in the classical situation there is an Ahlfors-Shimizu expression for the characteristic function of a function F which is defined on a covering of $\mathbb{C}$. Indeed, let

$$\phi_Y = p^*\left(\frac{i}{2\pi}dz \wedge d\bar{z}\right) = dd^c\left|p^2\right| =\mid p'(w)\mid^2 \frac{i}{2\pi}dw \wedge d\bar{w} = d \mid p \mid^2 \wedge \sigma_Y,$$

be the pseudo-volume form obtained by pulling back the euclidean form on $\mathbb{C}$.

If

$$\gamma_F = \frac{\mid F' \mid^2}{(1 + |F^2|)^2 \mid p' \mid^2},$$

then

$$T_F(r) = \int_0^r \frac{dt}{t}\int_{Y(t)} \gamma_F \phi_Y = -\frac{1}{2}\int_0^r \frac{dt}{t}\int_{Y(t)} dd^c \log \gamma_F.$$

Let now $y_0 \in Y$ and let $F = F_1/F_0$ in a neighbourhood of y_0. The ramification index of F at y_0 is defined as

$$n_{F,Ram}(y_0) = ord_{y_0}(F_0 F_1' - F_0' F_1).$$

Using this, we set

$$n_{F,Ram}(t) = \sum_{y \in Y(t)} n_{F,Ram}(y)$$

and

$$N_{F,Ram}(r) = \int_0^r \frac{n_{F,Ram}(t)}{t} dt.$$

Second Main Theorem. *Let* $p : Y \to \mathbb{C}$, *and* $F : Y \to \mathbf{P}^1$ *be as above and let*

$$\delta(Y/\mathbb{C}) = \frac{1}{2}[Y : \mathbb{C}] \log [Y : \mathbb{C}] - [Y : \mathbb{C}]^{\frac{1}{2}}, \tag{4}$$

$$S(F, c, \psi, r) = \log F(r) + \log \psi(F(r)) + \log \psi(crF(r)\psi(F(r))); \tag{5}$$

Then for all $r \geq r_1$ *outside a set of finite measure and for all* $b_{1\geq} b_1(T_f)$, *one has for a finite set of distinct points* $a_1, a_2, \ldots, a_q$ *in* $\mathbf{P}^1$, *that*

$$\begin{cases} (q-2)T_F(r) - \sum_{j=1}^{q} N_F(a_j, r) + N_{F,Ram}(r) - N_{p,Ram}(r) \\ \leq \frac{1}{2}[Y : \mathbb{C}]S(BT_F^2, b_1, \psi, r) + \frac{1}{2}[Y : \mathbb{C}] \log b - \frac{1}{2} \sum_{y \in Y<0>} \log \gamma_F(y) - \delta(Y/\mathbb{C}). \end{cases} \tag{6}$$

3. Statements of the theorems

In the statement of our theorems we shall follow the notation and ideas of Ye [20]. Thus ψ, ϕ will satisfy the conditions

$$\int_1^\infty \frac{dx}{x\psi(x)} < \infty, \tag{7}$$

and

$$\int_1^\infty \frac{dx}{x\phi(x)} = \infty, \tag{8}$$

respectively.

Theorem 1 *Let* $p : Y \to \mathbb{C}$ *be a proper holomorphic covering,* $a_1, \ldots, a_q, q \geq 1$ *a finite set of distinct points in* $\mathbf{P}^1$ *and* $F : Y \to \mathbf{P}^1$ *an holomorphic function such that* $F(0) \neq 0, \infty,\ a_j,\ j = 1, \ldots, q$ *and* $F'(0) \neq 0$. *Then,*

$$(q-2)T_F(r) - \sum_{j=1}^{q} N_F(a_j, r) + N_{F,Ram}(r) - N_{p,Ram}(r)$$

$$\leq [Y : \mathbb{C}] \log T_F(r) + [Y : \mathbb{C}] \log \psi(T_F(r)) + \frac{1}{2}[Y : \mathbb{C}] \log \psi(r),$$

for all $r \geq r_1$ *outside a set of finite measure.*

Theorem 2 (Logarithmic derivative lemma for holomorphic mappings on coverings) *Let* $p : Y \to \mathbb{C},\ F \to \mathbf{P}^1$ *and* $\psi : [1, \infty) \to \mathbf{R}$ *as in Theorem 1, then we have*

$$\begin{aligned} m(r, \infty, \frac{F'}{F}) \quad &\leq \quad [Y : \mathbb{C}] \log T_F(r) + [Y : \mathbb{C}] \log \psi\,(T_F\,(r)) \\ &\quad + \frac{1}{2}\,[Y : \mathbb{C}] \log \psi(r) + \int_{Y\langle r \rangle} \log |p'|\, \sigma_Y, \end{aligned}$$

for all $r \geq r_1$ *outside a set of finite measure.*

The next two theorems show that in some sense Theorems 1,2 are the best that we can expect

Theorem 3 *Let ϕ be as above and $h : [1,\infty) \to \mathbb{R}$ any positive increasing function with $h(r) \to \infty$ as $r \to \infty$. Then for any holomorphic covering $p : Y \to \mathbb{C}$ and any finite finite set $a_1, \dots, a_q,\ q \geq 1$ of distinct points of $\mathbf{P}^1$, there exists an holomorphic function $F_Y : Y \to \mathbf{P}^1$ such that,*

$$(q-2)T_F(r) - \sum_{j=1}^{q} N_F(a_j, r) + N_{F,Ram}(r) - N_{p,Ram}(r)$$
$$\geq [Y : \mathbb{C}] \log T_f(r) + [Y : \mathbb{C}] \log \phi(T_F(r)) + h(r).$$

Theorem 4 *Let ϕ, h as in Theorem 3. Then for any holomorphic covering $p : Y \to \mathbb{C}$, there exists an holomorphic function $F_Y : Y \to \mathbf{P}^1$ such that*

$$m\left(r, \infty, \frac{F_Y'}{F_Y}\right) \geq [Y : \mathbb{C}] \log T_F(r) + [Y : \mathbb{C}] \log \phi\left(T_F(r)\right) + \int_{Y\langle r\rangle} \log |p'|\, \sigma_Y + h(r).$$

4. Proofs of Theorems 1 and 3

Proof of Theorem 1.

To prove Theorem 1, we shall follow the ideas of Ye, (cf. [20, Theorem 2])

Lemma 1 *Let $\psi : [1,\infty) \to \mathbb{R}$ be a positive increasing function satisfying (7) and let $p \geq 1$ a real number. Then there exists a positive increasing function $\psi_1(t)$, satisfying also (7) and,*

$$\lim_{t\to\infty} \frac{\psi_1(t)}{\psi(t)} = 0, \tag{9}$$

$$\psi_1(t) \leq t^{\frac{1}{p}}, \tag{10}$$

$$\psi_1(t^p) \leq \psi(t) \tag{11}$$

for all large t.

We start from the main inequality in the Second Main Theorem where, in place of ψ, we consider the associated ψ_1 of Lemma 1 for $p = 6$,

$$\begin{cases} (q-2)T_F(r) - \sum_{j=1}^{q} N_F(a_j, r) + N_{F,Ram}(r) - N_{p,Ram}(r) \\ \leq \frac{1}{2}[Y : \mathbb{C}] S(B{T_F}^2, b_1, \psi_1, r) + \frac{1}{2}[Y : \mathbb{C}] \log b - \frac{1}{2} \sum_{y \in Y<0>} \log \gamma_F(y) - \delta(Y/\mathbb{C}) \\ = [Y : \mathbb{C}] \log T_F(r) + \frac{1}{2}[Y : \mathbb{C}] \log \psi_1(B{T_F}^2(r)) + \frac{1}{2}[Y : \mathbb{C}] \log \psi_1(b_1 r B{T_F}^2(r)\psi_1(B{T_F}^2(r)) \\ + \frac{1}{2}[Y : \mathbb{C}] \log b - \frac{1}{2} \sum_{y \in Y<0>} \log \gamma_F(y) - \delta(Y/\mathbb{C}). \end{cases} \tag{12}$$

outside a set of finite measure.
Since $T_F(r) > B$ for r sufficiently large, we get from (11) that

$$\psi_1(B{T_F}^2(r) < \psi(T_F(r)), \tag{13}$$

for large r. Furthermore, when $T_F(r) < r$, (10) and (11) imply,

$$\psi_1(b_1 r B{T_F}^2(r)\psi_1(B{T_F}^2(r)) < \psi_1(b_1 B^2 r^5) < \psi(r), \tag{14}$$

on the other hand, when $T_F(r) \geq r$, we have

$$\psi_1(b_1 r B {T_F}^2(r)\psi_1(B{T_F}^2(r)) < \psi_1(b_1 B^2 {T_F}^5(r)) < \psi(T_F(r)), \tag{15}$$

thus we conclude from (14) and (15) that

$$\psi_1(b_1 r B {T_F}^2(r)\psi_1(B{T_F}^2(r)) < \psi(r)\psi(T_F(r)), \tag{16}$$

for $r > r_1$.
Finally we conclude from (12), (13) and (16) that

$$(q-2)T_F(r) - \sum_{j=1}^{q} N_F(a_j, r) + N_{F,Ram}(r) - N_{p,Ram}(r)$$
$$\leq [Y:\mathbb{C}]\log T_F(r) + [Y:\mathbb{C}]\log \psi(T_F(r)) + \frac{1}{2}[Y:\mathbb{C}]\log \psi(r),$$

for all $r \geq r_1$ outside a set of finite measure. ■

Proof of Theorem 3.

We shall use the construction of Ye [20], which yields an entire function $F : \mathbb{C} \to \mathbb{C}$ such that,

$$(q-2)T_F(r) - \sum_{j=1}^{q} N_F(a_j, r) + N_{F,Ram}(r) \geq [Y:\mathbb{C}]\lg T_F(r) + [Y:\mathbb{C}]\lg \phi(T_F(r)) + h(r), \tag{17}$$

and consider the related function $F_Y = F \circ p : Y \to \mathbb{C}$.
Now we compare the Nevanlinna magnitudes of F and F_Y

Lemma 2 *The following relations hold:*
a) $[Y:\mathbb{C}]m(r,a,F) \leq m(r,a,F_Y) \leq [Y:\mathbb{C}]\{m(r,a,F) + C(a)\}$.
b) $N(r,a,F_Y) = [Y:\mathbb{C}]N(r,a,F)$,
c) $N_{F_Y,Ram}(r) = [Y:\mathbb{C}]N_{F,Ram}(r) + N_{p,Ram}(r)$
d) $[Y:\mathbb{C}]T_F(r) \leq T_{F_Y}(r) \leq [Y:\mathbb{C}]T_F(r) + [Y:\mathbb{C}]C + \sum_{y \in Y(0)} (ord_y p) \log \|F_Y(y), \infty\|$.

PROOF.
a)

$$m(r,a,F_Y) = \int_{Y<r>} -\log \|F_Y, a\| \, \sigma_Y$$
$$= \int_{Y<r>} -\log \|F \circ p, a\| \, \sigma_Y = \int_{Y<r>} -\log \|F \circ p, a\| \, p^*\left(\frac{d\theta}{2\pi}\right)$$
$$= [Y:\mathbb{C}]\frac{1}{2\pi}\int_0^{2\pi} -\log \left\|F(re^{i\theta}), a\right\| d\theta$$
$$\geq [Y:\mathbb{C}]\frac{1}{2\pi}\int_0^{2\pi} log^+ \left|\frac{1}{F(re^{i\theta}) - a}\right| d\theta = [Y:\mathbb{C}]m(r,a,F),$$

where we used the inequality $\|w - a\| \leq |w - a|$. On the other hand

$$m(r,a,F_Y) = [Y:\mathbb{C}]\frac{1}{2\pi}\int_0^{2\pi} -\log \left\|F(re^{i\theta}), a\right\| d\theta = [Y:\mathbb{C}]\frac{1}{2\pi}\int_{|F-a|<\frac{1}{2}} + [Y:\mathbb{C}]\frac{1}{2\pi}\int_{|F-a|\geq\frac{1}{2}}$$
$$\leq [Y:\mathbb{C}]\frac{1}{2\pi}\int_{|F-a|<\frac{1}{2}} log^+ \frac{1}{|\, F(re^{i\theta}) - a\,|} d\theta + \log^+ C_1(a) + [Y:\mathbb{C}]2\pi \log \frac{1}{B_1(a)}$$
$$\leq [Y:\mathbb{C}]m(r,a,F) + C(a),$$

since

$$| w - a | \leq C_1(a) \parallel w - a \parallel, \quad \text{if } | w - a | < \tfrac{1}{2},$$

and

$$\parallel w - a \parallel \geq B_1(a), \quad \text{if } | w - a | \geq \tfrac{1}{2}.$$

b) First, we note that,

$$n(r, a, F_Y) \text{ is the number of roots of } F_Y(z) = a \text{ in } Y[r]$$
$$= \text{number of roots of } F \circ p(z) = a \text{ in } Y[r] = [Y : \mathbb{C}]n(r, a, F).$$

Thence

$$N(r, a, F_Y) = \int_0^r \frac{n(t, a, F_Y)}{t} dt = \int_0^r \frac{[Y : \mathbb{C}]n(t, a, F)}{t} dt$$
$$[Y : \mathbb{C}] \int_0^r \frac{n(t, a, F)}{t} dt = [Y : \mathbb{C}]N(r, a, F).$$

c) This follows easily from the definition of the ramification index and the relationship

$$F'_Y(z_0) = F'(p(z_0))p'(z_0),$$

for any z_0 in Y. Since $F'(p(z_0)) \neq 0$, in Ye's example, in this case (c) becomes,

$$n_{F_Y,Ram}(r) = n_{p,Ram}(r),$$

and

$$N_{F_Y,Ram}(r) = N_{p,Ram}(r).$$

d) It follows from the definition

$$T_{F_Y}(r) = T_{F_Y,\infty}(r) = m(r, \infty, F_Y) + N(r, \infty, F_Y) + \sum_{y \in Y(0)} (ord_y p) \log \|F_Y(y), \infty\|,$$

together with a) and b). ■

Proof of Theorem 3.

We obtain from Lemma 2 and (17) that

$$\begin{cases} (q-2)T_{F_Y}(r) - \sum_{j=1}^{q} N_{F_Y}(a_j, r) + N_{F_Y,Ram}(r) - N_{p,Ram}(r) \geq [Y : \mathbb{C}]\{(q-2)T_F(r) \\ \quad - \sum_{j=1}^{q} N(F, a_j, r) + N_{F,Ram}(r)\} \geq [Y : \mathbb{C}]\{\log T_F(r) + \log \phi(T_F(r)) + h(r) \\ \quad = [Y : \mathbb{C}] \log T_{F_Y}(r) + [Y : \mathbb{C}] \log \phi(T_F(r)) + [Y : \mathbb{C}]\, h(r). \end{cases} \tag{18}$$

Now we use d) again, namely,

$$T_{F_Y}(r) \leq [Y : \mathbb{C}]T_F(r) + [Y : \mathbb{C}]C, \tag{19}$$

and define $\phi_1 : [1, \infty) \to \mathbb{R}$ and $h : [1, \infty) \to \mathbb{R}$,

$$\begin{aligned} \phi_1(r) &= \phi([Y : \mathbb{C}]r + [Y : \mathbb{C}]C), \\ h_1(r) &= \frac{h(r)}{[Y : \mathbb{C}]}. \end{aligned}$$

For these functions ϕ_1 and h_1 we can find $F_1, F_{1,Y}$ such that (18) holds, $i.e.$

$$\begin{cases} (q-2)T_{F_1,Y}(r) - \sum_{j=1}^{q} N(F_{1,Y}, a_j, r) + N_{F_{1,Y},Ram}(r) - N_{p,Ram}(r) \\ \geq [Y:\mathbb{C}] \log T_{F_{1,Y}}(r) + [Y:\mathbb{C}] \log \phi_1(T_{F_1}(r)) + [Y:\mathbb{C}] h_1(r), \end{cases} \tag{20}$$

and finally making use of (19), since ϕ is increasing, we conclude from (20) that

$$\begin{aligned} & (q-2)T_{F_1,Y}(r) - \sum_{j=1}^{q} N(F_{1,Y}, a_j, r) + N_{F_{1,Y},Ram}(r) - N_{p,Ram}(r) \\ \geq \quad & [Y:\mathbb{C}] \log T_{F_{1,Y}}(r) + [Y:\mathbb{C}] \log \phi([Y:\mathbb{C}]T_{F_1}(r) + [Y:\mathbb{C}]C) + h(r) \\ & \geq [Y:\mathbb{C}] \log T_{F_{1,Y}}(r) + [Y:\mathbb{C}] \log \phi(T_{F_1,Y}(r)) + h(r). \quad \blacksquare \end{aligned}$$

5. Proofs of Theorems 2 and 4

We can derive Theorem 2 from the following Theorem 5 following exactly the same argument as the one used to obtain Theorem 1 from the Wong-Lang-Cherry version of the second fundamental theorem.

Theorem 5 *Let $F : Y \to \mathbf{P}^1$ be a holomorphic map, where on Y is defined a covering map $p : Y \to \mathbb{C}$, such that $F(0) \neq 0, \infty$ and $F'(y) \neq 0$ for every $y \in p^{-1}(0)$. Then*

$$m\left(r, \infty, \frac{F'}{F}\right) \leq \frac{1}{2} [Y:\mathbb{C}]\ S(BT_F^2, b_1(T_F), \psi, r) + \int_{Y\langle r\rangle} \log |p'|\, \sigma_Y,$$

for all $r \geq r_1$ outside a set of finite measure.

To prove Theorem 5 we shall use the Ahlfors-Wong function. Set

$$\gamma_F = \frac{|F'|^2}{\left(1+|F|^2\right)^2 |p'|^2},$$

then if $a_1, \ldots, a_q, q \geq 1$, is a finite set of distinct points in $\mathbf{P}^1$, such that $F(y) \neq a_j$ for all j and all $y \in Y\langle 0\rangle$, and Λ is a decreasing function of r with $0 < \Lambda < 1$, the Ahlfors-Wong function is defined by, (see [6]),

$$\gamma_\Lambda = \prod_{j=1}^{q} \|F, a_j\|^{-2(1-\Lambda)} \gamma_F.$$

From [9, Lemma 5.6], we have

Lemma 3 *Let*

$$\Lambda(r) = \begin{cases} \frac{1}{qT_F(r)} & for \quad r \geq r_1 \\ cons\tan t & for \quad r < r_1 \end{cases}$$

Then

$$\log \int_{Y\langle r\rangle} \gamma_\Lambda \sigma_Y \leq S\left(BT_F^2, b_1\left(T_F\right), \psi, r\right) + \log b.$$

PROOF OF THEOREM 5. We proceed as in the plane case, see [7, Theorem 6.1]. Let Λ be as in Lemma 3. It follows from the definitions that

$$\gamma_\Lambda = \left|\frac{F'}{Fp'}\right|^2 h^\Lambda,$$

where $h = \|F,0\|^2 \|F,\infty\|^2$. Let $u = \left|\frac{F'}{F}\right|^2$ and

$$v = \frac{h^\Lambda}{|p'|^2}, \quad 0 \le v \le 1.$$

Then we have

$$\begin{aligned} m\left(r,\infty,\frac{F'}{F}\right) &= \int_{Y\langle r\rangle} \log\frac{1}{\|\frac{F'}{F},\infty\|}\sigma_Y \le \int_{Y\langle r\rangle} (\log^+\left|\frac{F'}{F}\right| + C)\sigma_Y \\ &= \int_{Y\langle r\rangle} \log^+\left|\frac{F'}{F}\right|\sigma_Y + C\,[Y:\mathbb{C}] = \frac{1}{2}\int_{Y\langle r\rangle}\log^+ u\sigma_Y + C\,[Y:\mathbb{C}] \\ &= \frac{1}{2}\int_{Y\langle r\rangle}(\log^+ u + \log v)\sigma_Y - \frac{1}{2}\int_{Y\langle r\rangle}\log v\sigma_Y + C\,[Y:\mathbb{C}] \qquad (21)\\ &= \frac{1}{2}\int_{Y\langle r\rangle}\log\exp\left(\log^+ u + \log v\right)\sigma_Y - \frac{1}{2}\int_{Y\langle r\rangle}\Lambda(r)\log h\sigma_Y \\ &+ \int_{Y\langle r\rangle}\log|p'|\,\sigma_Y + C\,[Y:\mathbb{C}]\,. \end{aligned}$$

Making use of the inequality

$$\exp\left(\log^+ u + \log v\right) \le uv + 1 = \gamma_\Lambda + 1,$$

we obtain that

$$\frac{1}{2}\int_{Y\langle r\rangle}\log\exp\left(\log^+ + \log v\right)\sigma_Y \le \frac{1}{2}\int_{Y\langle r\rangle}\log\gamma_\Lambda + \frac{[Y:\mathbb{C}]}{2}. \qquad (22)$$

We estimate the right hand side of (22) as follows

$$\begin{aligned} &\int_{Y\langle r\rangle}\log\gamma_\Lambda\sigma_Y + \frac{[Y:\mathbb{C}]}{2} \\ &= [Y:\mathbb{C}]\int_{Y\langle r\rangle}\log\gamma_\Lambda\frac{\sigma_Y}{[Y:\mathbb{C}]} + \frac{[Y:\mathbf{C}]}{2} \qquad (23)\\ &= [Y:\mathbb{C}]\log\int_{Y\langle r\rangle}\gamma_\Lambda\sigma_Y - [Y:\mathbb{C}]\left(\log[Y:\mathbb{C}] + \frac{1}{2}\right), \end{aligned}$$

where we have pulled the logarithm out of the integral after a normalization of the measure.
By Lemma 2

$$[Y:\mathbb{C}]\log\int_{Y\langle r\rangle}\gamma_\Lambda\sigma_Y \le [Y:\mathbb{C}]\,S([Y:\mathbb{C}]\,BT_F^2, b_1, \psi, r) + [Y:\mathbb{C}]\log b, \qquad (24)$$

and by the First Main Theorem

$$\begin{aligned} -\frac{1}{2}\int_{Y\langle r\rangle}\Lambda(r)\log h\sigma_Y &= \Lambda(r)(m(r,0,F) + m(r,\infty,F)) \\ &\le 2\Lambda(r)T_F(r) - \Lambda(r)\sum_{y\in Y\langle 0\rangle}(ord_y p)\log\left(\|F(y),0\|\,\|F(y),\infty\|\right). \end{aligned}$$

By our definition of $\Lambda(r)$ we get

$$2\Lambda(r)T_F(r) - \Lambda(r)\sum_{y\in Y\langle 0\rangle}(ord_y p)\log\left(\|F(y),0\|\,\|F(y),\infty\|\right) \leq 2 + o(1). \tag{25}$$

Putting together (21), (22), (23), (24) and (25)we conclude

$$m\left(r,\infty,\frac{F'}{F}\right) \leq \frac{1}{2}\left[Y:\mathbb{C}\right] S(\left[Y:\mathbb{C}\right] BT_F^2, b_1, \psi, r) + \int_{Y\langle r\rangle}\log|p'|\,\sigma_Y,$$

for a certain new constant. ■

Proof of Theorem 4. We use again the results of Ye, (see [20, Theorem 2]), who constructed for any given ϕ and h as in the statement of Theorem 3, an entire function $F:\ \mathbb{C}\to\mathbf{P}^1$ such that

$$m\left(r,\infty,\frac{F'}{F}\right) \geq \log T_F(r) + \log\phi\left(T_F(r)\right) + h(r), \tag{26}$$

for large r outside a set of finite measure.
Then we consider the function $F_Y: Y\to\mathbf{P}^1$ given by $F_Y = F\circ p$. For the function F_Y defined in this way we obtain

$$\frac{F_Y'}{F_Y}(r) = \frac{\left(F'\circ p\right)p'}{F\circ p} = \left(\frac{F'}{F}\circ p\right)p',$$

or

$$\frac{F_Y'}{F_Y}\frac{1}{p'} = \left(\frac{F'}{F}\right)\circ p,$$

whence using the definition of the chordal distance, that is

$$\|a,\infty\|^2 = \frac{1}{1+|a|^2}$$

so that

$$\frac{1}{\|a,\infty\|^2} = 1+|a|^2$$

we obtain

$$\begin{aligned}\frac{1}{\left\|\frac{F_Y'}{F_Y},\infty\right\|^2|p'|^2}+1 &= \frac{1+\left|\left(\frac{F'}{F}\circ p\right)p'\right|^2}{|p'|^2}+1\\ &= 1+\left|\frac{F'}{F}\circ p\right|^2+\frac{1}{|p'|^2} = \frac{1}{\left\|\frac{F'}{F}\circ p,\infty\right\|^2}+\frac{1}{|p'|^2},\end{aligned}$$

thus

$$\frac{1}{\left\|\frac{F_Y'}{F_Y},\infty\right\|^2|p'|^2}+1 \geq \frac{1}{\left\|\frac{F'}{F}\circ p,\infty\right\|^2}.$$

Taking logarithms and integrating and recalling a) in the proof of Lemma 1, we get

$$\begin{aligned} m\left(r,\frac{F_Y'}{F_Y},\infty\right) &\geq m\left(r,\frac{F'}{F}\circ p,\infty\right)+\int_{Y\langle r\rangle}\log|p'|\,\sigma_Y + O(1)\\ &\geq \left[Y:\mathbb{C}\right] m\left(r,\frac{F'}{F}\right)+\int_{Y\langle r\rangle}\log|p'|\,\sigma_Y + O(1).\end{aligned} \tag{27}$$

Given ϕ as in the statement in Theorem 6 and an arbitrary function $h(r)$ tending to infinity, we define the new function

$$h_1(r) = \frac{h(r)}{[Y:\mathbb{C}]},$$

then for ϕ, h_1, we find F such that (26) holds

$$\begin{aligned} m\left(r, \frac{F'}{F}, \infty\right) &\geq \log T_F(r) + \log\phi(T_F(r)) + h_1(r) \\ &= \log T_F(r) + \log\phi(T_F(r)) + \frac{h(r)}{[Y:\mathbb{C}]}, \end{aligned}$$

then for the corresponding $F_Y = F \circ p$ we obtain from (27)

$$\begin{aligned} m\left(r, \frac{F'_Y}{F_Y}, \infty\right) &\geq [Y:\mathbf{C}]\, m\left(r, \frac{F'}{F}\right) + \int_{Y\langle r\rangle} \log|p'|\,\sigma_Y + O(1) \\ &\geq [Y:\mathbb{C}]\left(\log T_F(r) + \log\phi(T_F(r)) + h_1(r)\right) + \int_{Y\langle r\rangle} \log|p'|\,\sigma_Y, \end{aligned}$$

and arguing as we did in Theorem 3, we conclude

$$\begin{aligned} m\left(r, \frac{F'_Y}{F_Y}, \infty\right) &\geq [Y:\mathbf{C}]\log T_{F_Y}(r) + [Y:\mathbb{C}]\log\phi(T_{F_Y}(r)) \\ &\quad + \int_{Y\langle r\rangle} \log|p'|\,\sigma_Y + h(r), \end{aligned}$$

as it was to be proved. ■

6. Holomorphic mappings on coverings and algebroid functions

6.1. As it was observed by A. Eremenko there is a close connection between the value distribution theory for holomorphic mappings on coverings developed by Lang and Cherry and the value distribution theory of H.L. Selberg [13] for algebroid functions.

As we said before, an algebroid function $F(z)$ is an analytic function which has algebraic character for every $z \in \mathbb{C}$ and a constant number of branches $F_1(z), \ldots, F_k(z)$.

In this situation the set of pairs

$$Y_F = \{(z, F_j(z))\, | z \in \mathbb{C},\ j = 1, \ldots, k\},$$

defines a connected Riemann surface and there is a well defined analytic function on it, given by

$$\begin{array}{rccc} F: & Y_F & \longrightarrow & \mathbf{P}^1 \\ & (z, F_j(z)) & \mapsto & F_j(z)) \end{array}$$

which we have denoted in the same way as the original algebroid function F.

We can consider the natural projection

$$\begin{array}{rccc} p: & Y_F & \longrightarrow & \mathbb{C} \\ & (z, F_j(z)) & \mapsto & z \end{array}$$

so that we obtain a proper holomorphic covering of the plane.

In this sense, given an algebroid function F there is a proper covering of the plane and an holomorphic function defined on it, associated to the algebroid function.
Conversely, given a proper holomorphic covering of the plane $p : Y \to \mathbb{C}$ and an holomorphic function $F : Y \to \mathbf{P}^1$, we can consider locally the function $F \circ p^{-1}$. It is true that any branch of it can be continued without bounds in the plane, however it might happen that in this way we obtain several disjoint complete analytic functions.
If we exclude this possibility, that is, $F \circ p^{-1}$ defines a unique complete analytic function connecting all the possible branches by analytic continuation, then it will be an algebroid function which we shall denote by F^*, and the Riemann surface Y_{F^*} associated to F^* will be conformally isomorphic to Y. This isomorphism $Y \leftrightarrow Y_{F^*}$ can be defined assigning to a point $y \in Y$, the pair $(p(y), F(y)) \in Y_{F^*}$. This is a well defined analytic function between Riemann surfaces and by our restriction is also an isomorphism.

6.2. Now we shall relate the value distribution functions introduced in both settings and check that they are essentially the same, except that Selberg divides them by the degree of the algebroid function which is the same as the degree of the associated holomorphic covering.
We shall consider an holomorphic mapping F on a covering Y which gives rise to an algebroid function in the above sense, that is, we exclude the possibility of obtaining different complete analytic functions by analytic continuation of different branches of $F \circ p^{-1}$.
The notion of neighbourhood $Y[r]$ of radius r in Y and the corresponding boundary $Y\langle r\rangle$, introduced in [9] by Lang and Cherry, correspond to the subset $Y_{F^*}[r]$ of the Riemann surface Y_{F^*} associated to an algebroid function F^*, which projects over the disc $D_r = \{z \,|\, |z| \leq r\}$ and its boundary $\Gamma(r)$ in Selberg [14]. Also the angular measure

$$\sigma_Y = d^c \log |p|^2 = p^* \left(\frac{d\theta}{2\pi} \right),$$

in $Y\langle r\rangle$ on a covering surface Y, is the same as that considered in the different sheets of the Riemann surface of an algebroid function, namely the pullback of the normalized angular measure in the plane.
Given $p : Y \to \mathbb{C}, F : Y \to \mathbf{P}^1$ and the associated algebroid function $F^*(z) = F \circ p^{-1}$, the corresponding counting functions are related as follows

$$n(F^*, a, r) = n(F, a, r)$$

$$\begin{cases} N(F^*, a, r) = \dfrac{1}{k} \displaystyle\int_0^r \dfrac{n(F, a, t)}{t} dt \\ \qquad\qquad\quad = \dfrac{1}{[Y : \mathbb{C}]} N(F, a, r), \end{cases} \tag{28}$$

where k is the degree $[Y : \mathbb{C}]$ of the covering p.
As for the proximity functions we have

$$\begin{aligned} m(F^*, \infty, r) &= \frac{1}{2\pi k} \int_0^{2\pi} \left[\sum_{\nu=1}^{k} \log^+ \left| F_\nu \left(re^{i\phi} \right) \right| \right] d\phi \\ &= \frac{1}{2\pi [Y : \mathbb{C}]} \int_0^{2\pi} \left[\sum_{\nu=1}^{k} \log^+ \left| F_\nu \left(re^{i\phi} \right) \right| \right] \end{aligned}$$

whereas according to the theory of Lang-Cherry

$$m(F, \infty, r) = \int_{Y\langle r\rangle} \log \frac{1}{\|F, \infty\|} \sigma_Y = \int_{Y\langle r\rangle} \log \left(1 + |F|^2 \right)^{\frac{1}{2}} \sigma_Y.$$

Using the relationship

$$\log^+ x \le \frac{1}{2}\log\left(1+x^2\right),$$

and recalling that σ_Y restricted to each sheet is the same as $\frac{d\phi}{2\pi}$, we obtain

$$m\left(F^*,\infty,r\right) \le \frac{1}{[Y:\mathbb{C}]} m\left(F,\infty,r\right). \tag{29}$$

Conversely, using the relationship

$$\log\left(1+|x|^2\right)^{\frac{1}{2}} \le \log^+|x| + \frac{1}{2}\log 2,$$

when we integrate in each sheet, we conclude

$$m\left(F,\infty,r\right) \le [Y:\mathbb{C}]\, m\left(F^*,\infty,r\right) + \frac{[Y:\mathbb{C}]}{2}\log 2. \tag{30}$$

Finally from (28), (29) and (30) we get

$$T\left(F^*,r\right) \le \frac{1}{[Y:\mathbb{C}]} T\left(F,r\right) \le T\left(F^*,r\right) + \frac{1}{2}\log 2. \tag{31}$$

7. The logarithmic derivative lemma for algebroid functions

In [13], H.L.Selberg proved the following logarithmic derivative lemma for algebroid functions of finite order

Theorem 6 (Logarithmic derivative lemma for algebroid functions) *For every non identically zero algebroid function $G(z)$ of finite order λ, it holds*

$$\limsup_{r\to\infty} \frac{m\left(r,\frac{G'}{G}\right)}{\log r} \le \frac{|\lambda-1|+\lambda-1}{2} + 2\lambda. \tag{32}$$

In this section we shall compare Theorem 2 with the previous Selberg's logarithmic derivative lemma and check that for finite order algebroid functions Theorem 2 yields a better estimate than (32).
By the considerations in section (32), every algebroid function G can be obtained as an F^* associated to an holomorphic mapping F on a covering Y, where

$$Y = \left\{\left(z, G_j(z)\right) | z \in \mathbb{C}\right\},$$

the G_j's are the different branches of G,

$$\begin{array}{cccc} p: & Y & \longrightarrow & \mathbf{P}^1 \\ & (z, G_j(z)) & \mapsto & G_j(z) \end{array}$$

and

$$\begin{array}{cccc} p: & Y & \longrightarrow & \mathbb{C} \\ & (z, G_j(z)) & \mapsto & z \end{array}$$

Recalling Theorem 2, we have the estimate

$$\begin{aligned} m\left(\frac{F'}{F},\infty,r\right) &\le [Y:\mathbb{C}]\log T_F(r) + [Y:\mathbb{C}]\log\psi\left(T_F(r)\right) + \frac{1}{2}[Y:\mathbf{C}]\log\psi(r) \\ &\quad + [Y:\mathbb{C}]\int_{Y\langle r\rangle} \log|p'|\,\sigma_Y. \end{aligned} \tag{33}$$

In this case $p' \equiv 1$, that is $\log|p'| \equiv 0$ and the last term vanishes identically.
The terms

$$[Y : \mathbb{C}] \log \psi\left(T_F(r)\right) + \frac{1}{2}[Y : \mathbb{C}] \log \psi(r),$$

are lower order terms compared with $[Y : \mathbb{C}] \log T_F(r)$ if G is not algebraic, see Selberg [14, page 12]. Thus making use of (31), we obtain from (33)

$$m\left(\frac{F'}{F}, \infty, r\right) \leq [Y : \mathbf{C}] \log T\left(F^*, r\right) + \text{lower order terms}, \tag{34}$$

as $r \to \infty$. On the other hand we have locally

$$F^* = F \circ p^{-1}$$

so that we obtain

$$\frac{(F^*)'}{F^*} = \frac{\left(F \circ p^{-1}\right)'}{F \circ p^{-1}} = \frac{\left(F' \circ p^{-1}\right)\left(p^{-1}\right)'}{F \circ p^{-1}},$$

that is

$$\frac{(F^*)'}{F^*} = \left(\frac{F'}{F}\right)^*, \tag{35}$$

since $\left(p^{-1}\right)' \equiv 1$.

Thus we conclude from (29), (34) and (35)

$$m\left(\frac{(F^*)'}{F^*}, \infty, r\right) \leq \log T\left(F^*, r\right) + \text{lower order terms},$$

what for functions F^* of finite order λ yields

$$m\left(\frac{(F^*)'}{F^*}, \infty, r\right) \leq (\lambda + \varepsilon) \log r,$$

for any $\varepsilon > 0$ and r sufficiently large, or

$$\limsup_{r \to \infty} \frac{m\left(\frac{(F^*)'}{F^*}, \infty, r\right)}{\log r} \leq \lambda, \tag{36}$$

which improves Selberg's estimate (32).
We leave open the question whether the estimate (36) is sharp for algebroid functions. Also our estimates in Theorem 1 for the error term in the second main theorem for holomorphic mappings on coverings are valid for algebroid functions and remains the question whether they are sharp for this more restricted class of functions. We note that the example exhibited to show that such estimates are sharp for holomorphic mappings on coverings, does not give rise to a proper algebroid function.

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References

[1] Ashline, G. (1999). The defect relation of meromorphic maps on parabolic manifolds. *Memoirs AMS.* **139.** n^o 665.

[2] Cherry, W. (1992). The Nevanlinna error term for coverings, generically surjective case. In *Proceedings of the Symposium on Value Distribution Theory in Several Complex Variables*, W.Stoll, ed., Notre Dame Mathematical Lectures, **12**, University of Notre Dame Press.

[3] Fernández Arias, A. (1986). On the size of the exceptional set in Nevanlinna theory. *J. London Math. Soc.* **34** (2), 449–456.

[4] Fernández Arias, A. (1988). On the second fundamental theorem of Nevanlinna. *Trans. Amer. Math. Soc.* **306**, 141–163.

[5] Goldberg, A. A. and Grinshtein, V. A. (1976). The logarithmic derivative of a meromorphic function. *Math. Notes* **19** , 320-323. Translated from Mat. Zametki **19**, 525–530.

[6] Hayman, W. K. (1974) *Meromorphic functions.* Oxford Clarendon Press, Oxford.

[7] Hinkannen, A. (1992).A sharp form of Nevanlinna's second fundamental theorem. *Invent. Math.* **108**, 549–574.

[8] Lang. S. (1988). The error term in Nevanlinna theory. *Duke Math. J.* **56**, 193–218.

[9] Lang, S. and Cherry, W. (1990). *Topics in Nevanlinna theory.* Lectures Notes in Math. **1433**, Springer, Berlin.

[10] Nevanlinna, R. (1970). *Analytic functions.* Grundlehren der Mathematischen Wissenschaften. Springer Verlag.

[11] Noguchi, J. (1976). Meromorphic mappings of a covering space over $\mathbb{C}^m$ into a projective variety and defect relations. *Hiroshima Math. J.*, **6**, 265–280.

[12] Noguchi, J. (1985). On the value distribution of meromorphic mappings of covering spaces over $\mathbb{C}^m$ into algebraic varieties. *J. Math. Soc. Japan*, **37** , 295–313.

[13] Selberg, H. L. (1929). Über eine Eigenschaft der logarithmischen Ableitung einer meromorphen oder algebroiden Funktion endlicher Ordnung. *Avh.Norske Vid.-Akad.Oslo, Mat-naturvid. Kl.*, n ^{o}**14**.

[14] Selberg, H. L. (1934). Algebroide Funktionen und Umkehrfunktionen Abelscher Integrale. *Avh.Norske Vid.-Akad.Oslo, Mat-naturvid. Kl.*, n^o**8**.

[15] Stoll, W. (1987). *The Ahlfors-Weyl theory of meromorphic maps on parabolic manifolds.* Lectures Notes in Mathematics **1239**. Springer Verlag.

[16] Ullrich, E. (1931). Über den Einfluss der Verzweigtheit einer Algebroide auf ihre Wertverteilung. *Journal für die reine und angewandte Mathematik.* **B.167**.

[17] Valiron, G. (1931). Sur la derivée des fonctions algebroïdes. *Bulletin de Soc. Math. France*, **59**.

[18] Wang, Y. (1977). Sharp forms of Nevanlinna's error terms. *Journal d'Analyse Mathématique*. **71**.

[19] Wong, P. M. (1989). On the second main theorem in Nevanlinna theory. *Amer. J. Math.*, **111**, 549–583.

[20] Ye, Z. (1991). On Nevanlinna's error terms. *Duke Math. J.*, **64**, 243–260.

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