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Big groups of automorphisms of some Klein surfaces

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Abstract. Let X_p be a compact bordered Klein surface of algebraic genus $p \geq 2$. It is known that if G is a group of automorphisms of X_p then $|G| \leq 12(p-1)$. We call the group G a *big group of genus* p if $|G| > 4(p-1)$. In this paper we find a family of integers p such that the only big groups of genus p are dihedral groups. In terms of the real genus introduced by C. L. May this means that for such p there is no big group of real genus p.

Grupos grandes de automorfismos de ciertas superficies de Klein

Resumen. Sea X_p una superficie de Klein compacta con borde de *gen* algebraico $p \geq 2$. Se sabe que si G es un grupo de automorfismos de X_p entonces $|G| \leq 12(p-1)$. Se dice que G es un *grupo grande de gen* p si $|G| > 4(p-1)$. En el presente artículo se halla una familia de enteros p para los que el único grupo grande de *gen* p son los grupos diédricos. Esto significa que, en términos del *gen* real introducido por C. L. May, para tales valores de p no existen grupos grandes de *gen* real p.

1. Introduction

Let X_p be a compact bordered Klein surface [1] of algebraic genus $p \geq 2$. If G is a group of automorphisms of X_p then $|G| \leq 12(p-1)$ [9]. A group G is called a *big group of genus* p if $|G| > 4(p-1)$. An example of big groups are M^*-groups, groups of order $12(p-1)$ acting on bordered Klein surfaces of algebraic genus p. These groups were very extensively investigated, see [6], [7], [10] for example. In this paper we prove that if p is an integer lying between twin primes $p-1, p+1$, then $|G| \leq 4(p-1)$ or G is a dihedral group D_{2p} or $D_{2(p+1)}$. The *real genus* $\rho(G)$ of a finite group G is the smallest algebraic genus of any compact bordered Klein surface on which G acts. Its study was initiated by Coy L. May [11]. He showed that there are no groups of real genus $p = 2$ and posed the problem whether 2 is the unique value of p with this property [11]. Since dihedral groups have real genus 0, our result means that there is no big group of real genus p for the p mentioned before.

2. Preliminaries

Let $\mathcal{H}$ be the open upper half plane. A *non-euclidean crystallographic group*, *NEC group* in short, is a discrete subgroup Λ of $\Omega = \mathrm{Aut}^{\pm}(\mathcal{H})$ with compact quotient space $\mathcal{H}/\Lambda$. The algebraic structure of an

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NEC group Λ is determined by the signature, which has the form

$$(g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}) \tag{1}$$

The quotient space $\mathcal{H}/\Lambda$ is a surface of topological genus g with k holes. The surface is orientable if the plus sign is involved and nonorientable otherwise. The integers m_i are called proper or ordinary periods, n_{ij} link periods, and the brackets $(n_{i1}, \ldots, n_{is_i})$ periods cycles. A general presentation of Λ with signature (1) can be found in [5] for example. We do not give it here because it is rather complicated in general while we shall deal with rather special signatures. Instead we shall provide the presentations for considered cases. A signature of the form $(0; +; [-]; \{(n_1, \ldots, n_r)\})$ will be denoted by $(n_1, \ldots, n_r)$. The hyperbolic area of a fundamental region for Λ equals

$$\mu(\Lambda) = 2\pi \left(p - 1 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \sum_{i=1}^{k} \sum_{j=1}^{s_i} \frac{1}{2} \left(1 - \frac{1}{n_{ij}}\right) \right),$$

where p is an algebraic genus of $\mathcal{H}/\Lambda$, that is $p = \alpha g + k - 1$, where $\alpha = 1$ if the sign is $-$ and $\alpha = 2$ in the other case. If Λ' is a subgroup of Λ of finite index, then

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda) \quad (\mathrm{Riemman-Hurwitz\ formula}).$$

An NEC group Γ without orientation preserving elements of finite order is called a *surface group* and it has signature $(g; \pm; [-]; \{(-), \overset{k}{\ldots}, (-)\})$; it is said to be *bordered* if $k > 0$. A finite group G can be represented as the quotient Λ/Γ for some NEC group Λ and normal bordered surface NEC subgroup Γ. Such a quotient is said to be a *smooth* factor group of an NEC group Λ. Then the group G acts as a group of automorphisms of the compact bordered Klein surface $\mathcal{H}/\Gamma$. Conversely every compact bordered Klein surface X_p of algebraic genus $p \geq 2$ can be presented as the orbit space $X_p = \mathcal{H}/\Gamma$ for some bordered surface group Γ of algebraic genus p and if G is a group of automorphisms of X_p then $G = \Lambda/\Gamma$ for some NEC group Λ. Moreover in this case $|G| \leq 12(p-1)$ [9]. We will often use the following technical lemmata

Lemma 1 [[5]] *Let Λ be an NEC group with area $< \pi/2$ admitting a bordered surface group Γ as a normal subgroup. Then Λ has one of the following signatures:*

$Case$	$\sigma(\Lambda)$	$\mu(\Lambda)$
(a)	$(2,2,2,n)$	$\pi(n-2)/2n$, $(n \geq 3)$
(b)	$(2,2,3,3)$	$\pi/3$
(c)	$(2,2,3,4), (2,2,4,3)$	$5\pi/12$
(d)	$(2,2,3,5), (2,2,5,3)$	$7\pi/15$
(e)	$(0; +; [3]; \{(2,2)\})$	$\pi/3$
(f)	$(0; +; [2,3]; \{(-)\})$	$\pi/3.$

Lemma 2 [[5]] *A necessary and sufficient condition for a finite group G to be a smooth factor Λ/Γ, where Γ is a bordered surface group and Λ is an NEC group with signature $(2,2,m,n)$ is that G can be generated by three elements a, b and c of order 2 such that ab and ac have orders m and n respectively.*

3. Big groups of some compact bordered Klein surfaces

If G is a big group of genus p then by the Hurwitz-Riemman formula it is a factor group Λ/Γ of an NEC group Λ with area less than $\pi/2$.

Theorem 1 *If $k > 6$ is an integer lying between twin primes $k-1, k+1$ then the only big groups of genus k are D_{2k} or $D_{2(k+1)}$.*

First we shall prove some auxiliary results. The first is an easy exercise.

Lemma 3 *Let $p, p+2$ be twin primes with $p \geq 5$. Then $x \equiv 1 \mod(p)$ is the only solution of the congruence $x^3 \equiv 1 \mod(p)$. The congruence $x^3 \equiv 1 \mod(p+2)$ has two more solutions.*

Proposition 1 *Let $p > 5$ be a prime. Then there is no group G of order $6p$ generated by elements a, b, c of order 2 such that ab and bc have order 3.*

PROOF. Let G be such a group. By Sylow theorems G has a normal subgroup H of order p. Moreover $G/H = \langle \tilde{b}, \tilde{c} \rangle = D_3$. Then $\tilde{a} = (\tilde{b}\tilde{c})^\alpha \tilde{b}$ for some $\alpha = 0, 1, 2$. If $\alpha = 0$, then $ab \in H$, a contradiction. If $\alpha = 1$, then $H = \langle abcb \rangle$. Since H is normal in G, we obtain that $b(abcb)b = (abcb)^\beta$ for some $1 \leq \beta \leq p-1$. Then $abcb = (abcb)^{\beta^2}$, so $p | \beta^2 - 1$ and thus $\beta = 1$ or $\beta = p-1$. Therefore $b(abcb)b = abcb$ or $b(abcb)b = bcba$. In the first case $a = cbc$ and G has order 6, a contradiction. In the second case $abca = cb$. So $\langle bc \rangle \trianglelefteq G$, which is impossible. Finally, let $\alpha = 2$. Then $H = \langle ac \rangle$ and as before $bacb = ac$ or $bacb = ca$. In the first case $ac = 1$, which is impossible. In the second one $(ac)^3 = 1$, a contradiction like before. This completes the proof. ■

Proposition 2 *Let p be the smaller of twin primes with $p > 5$. Then there is no group G of order $6p$ generated by two elements of orders 2 and 3 respectively.*

PROOF. Let $G = \langle a, b \rangle$ have order $6p$ and let a, b have orders 2 and 3 respectively. Like in the previous proposition there is a normal subgroup H of order p in G. We have $G/H = \langle \tilde{a}, \tilde{b} \rangle$. First assume that $G/H = D_3$. Then $(\tilde{a}\tilde{b})^2 = 1$, so $H = \langle (ab)^2 \rangle$ and ab has order $2p$. Since H is normal in G, we obtain as before, $a(ab)^2a = (ab)^2$ or $a(ab)^2a = (b^2a)^2$. In both cases $(ab)^6 = 1$, a contradiction. Now let $G/H = Z_6$. Then $\tilde{a}\tilde{b}$ has order 6, so 6 divides $|ab|$. Therefore $|ab| = 6$ since otherwise G would be cyclic. Moreover $\tilde{a}\tilde{b} = \tilde{b}\tilde{a}$. So $abab^2 \in H$ and $H = \langle abab^2 \rangle$. Again, since H is normal in G, we have $b(abab^2)b^{-1} = (abab^2)^\beta$ for some $1 \leq \beta \leq p-1$. Now $abab^2 = (abab^2)^{\beta^3}$ and $p | \beta^3 - 1$. Thus $\beta^3 \equiv 1 \ mod(p)$. By lemma 3 we have $\beta = 1$ and $baba = abab$. So $K = \langle (ab)^2 \rangle$ is normal in G and $|G/K| \leq 6$. Since $|K| \leq 3$, we obtain $|G| \leq 18$. This is a contradiction, what completes the proof. ■

Proposition 3 *Let $p > 5$ be a prime. There is no group of order $8p$ generated by elements a, b, c of order 2 such that ab and bc have orders 2 and 4 respectively.*

PROOF. Let H be a normal subgroup of G of order p. Thus $G/H = D_4 = \langle \tilde{b}, \tilde{c} \rangle$ and $\tilde{a} = (\tilde{b}\tilde{c})^\alpha \tilde{b}$ for some $0 \leq \alpha \leq 3$ or $\tilde{a} = (\tilde{b}\tilde{c})^2$. Cases $\alpha = 0, 1$ and $\alpha = 3$ are easy to eliminate. If $\alpha = 2$, then $H = \langle acbc \rangle$. Moreover $c(acbc)c = (acbc)^\beta$ for some $1 \leq \beta \leq p-1$, so $\beta = 1$ or $\beta = p-1$. In both cases $\langle bc \rangle \trianglelefteq G$, which is impossible. Now let $\tilde{a} = (\tilde{b}\tilde{c})^2$. Then $H = \langle a(bc)^2 \rangle$ and in this case $\langle bc \rangle \trianglelefteq G$ again. This completes the proof. ■

Proposition 4 *Let $p > 5$ be a prime. The only group of order $4p$ generated by the elements a, b, c of order 2 such that ab and bc have orders 2 and p respectively is D_{2p}.*

PROOF. Suppose that G is such a group. By Sylow theorems $H = \langle bc \rangle$ is normal subgroup of G. Therefore $a(bc)a = (bc)^\alpha$ for some $1 \leq \alpha \leq p-1$. Since then $bc = (bc)^{\alpha^2}$, we obtain that $p | \alpha^2 - 1$ and so $\alpha = 1$ or $\alpha = p-1$. Now $a(bc)a = bc$ or $a(bc)a = cb$. In the first case $abc = bca$. Then $ac = ca$ and $(abc)^2 \in H$. Since $(abc)^2 \neq 1$ and abc can not have order p, we obtain that abc has order $2p$ and therefore $G = \langle ab, c \rangle = D_{2p}$. In the second case $(ac)^2 = (bc)^2$ and ac has order $2p$ and again $G = D_{2p}$. This completes the proof. ■

PROOF OF THE THEOREM 1.
Let $k > 6$ be such that $k-1, k+1$ are twin primes. Suppose that G is a group of automorphisms of a compact bordered Klein surface X_k. Assume that $|G| > 4(k-1)$. We will show that G is dihedral group D_{2k} or $D_{2(k+1)}$. The group G can be presented as a quotient Λ/Γ for some NEC group Λ and NEC

bordered surface group Γ. We have $\mu(\Gamma) = 2\pi(k-1)$ and by Riemann-Hurwitz formula $\mu(\Lambda) < \pi/2$, since $|G| > 4(k-1)$. Therefore we can use Lemma 1. First, let Λ has signature $(2,2,3,3)$. In this case by Lemma 2 $G = \langle a,b,c\rangle$, where a,b,c have order 2, ab and bc have order 3. Moreover $|G| = 6(k-1)$. By Proposition 1 there is no such a group. Now let Λ has signature $(0;+;[3];\{(2,2)\})$. Then Λ has the presentation $\langle x, c_0, c_1 | x^3, c_0^2, c_1^2, (c_0c_1)^2, (c_1xc_0x^{-1})^2\rangle$. Now $c_0 \in \Gamma$ or $c_1 \in \Gamma$. Suppose that $c_0 \in \Gamma$. Then $c_1 \notin \Gamma$ since otherwise c_0c_1 would be an orientation preserving element of order 2 in Γ. Similarly for $c_1 \in \Gamma$ we have $c_0 \notin \Gamma$. Thus G is generated by the images in Λ/Γ of x and c_i for $i = 0$ or $i = 1$. These generators have orders 3 and 2 respectively, since G is a smooth factor of Λ. Therefore G is a group of order $6(k-1)$ generated by two elements of order 2 and 3. By Proposition 2 there is no such a group. Let Λ has signature $(0;+;[2,3];\{(-)\})$. Then Λ is a group with the presentation $\langle x_1, x_2, c \,|\, x_1^2, x_2^3, c^2, (x_1x_2)^{-1}c(x_1x_2)c\rangle$. Clearly $c \in \Gamma$ and x_1, x_2 represent in Λ/Γ generators of order 2 and 3 respectively. Therefore again G is a group of order $6(k-1)$ generated by two elements of order 2 and 3. Like before by Proposition 2 there is no such a group. For signatures $(2,2,3,4), (2,2,4,3)$, $|G| = 24(k-1)/5$. Then order of G is not integer, which is impossible. Similarly for signatures $(2,2,3,5), (2,2,5,3)$, $|G| = 30(k-1)/7$, which is not integer again. Finally let Λ has signature $(2,2,2,n), n \geq 3$. In this case $G = \langle a,b,c\rangle, a,b,c$ and ab have order 2, bc has order n, and G has order $4n(k-1)/(n-2)$. Since $|G| \geq 2n$, we have $n \leq 2k$. For $n = 2k$ we obtain $G = D_{2k}$. Assume that $n < 2k$. Since $|G| = 4n(k-1)/(n-2)$ and $2n$ divides order of G, $n-2 = 1$ or $n-2 = 2$ or $n-2 = k-1$ or else $n-2 = 2(k-1)$. That means that $n = 3, 4$ or $k+1$. For $n = 3$ such group is an M^*- group. May proved [10] that there are no M^*-groups of genus $p+1$, if $p > 5$ is a prime. For $n = 4$ or $n = k+1$ we obtain $|G| = 8(k-1)$ or $|G| = 4(k+1)$ respectively. By Propositions 3, 4 $G = D_{2(k+1)}$. This completes the proof. ∎

Remark 1 If G is a finite group, then there is a compact bordered Klein surface X on which G acts as a group of automorphisms. The *real genus* $\rho(G)$ of G is the minimum algebraic genus of such surfaces. The real genus of a group was first studied by Coy L. May [11]. He has obtained many results related to the real genus, see for example [11], [12], [13]. There are infinitely many groups of real genus 0 and 1. Surprisingly there are no groups of real genus 2. Clearly the number of groups of real genus p for each integer $p \geq 2$ is finite. We also know that this number is a positive integer for infinitely many $p > 2$ [11]. A natural problem which was posed by May in [11] is finding integers p for which there is no group of real genus p. Since every dihedral group acts on a sphere with one hole, it has real genus 0. So Theorem 1 implies:

Theorem 2 *Let $k > 6$ be an integer lying between twin primes $k-1, k+1$. Then there is no big group of real genus k. That is if $\rho(G) = k$ then $|G| \leq 4(k-1)$.* ∎

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