RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **95** (2), 2001, pp. 297–302
Matemática Aplicada / Applied Mathematics
Comunicación Preliminar / Preliminary Communication

A new characterization of generators of differentiable semigroups

I. I. Vrabie

Abstract. Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\,;\ t \geq 0\}$. We prove that $\{S(t)\,;\ t \geq 0\}$ is differentiable if and only if, for each $\alpha \in (0,1)$, there exists $\lim_{n\to\infty} A\left(I - \frac{t}{n}A\right)^{-n}$ in the norm topology of $C([\,\alpha, 1/\alpha\,]; \mathcal{L}(X))$. A consequence concerning analytic C_0-semigroups of contractions is also included.

Una nueva caracterización de generadores infinitesimales de semigrupos diferenciables

Resumen. Sea $A : D(A) \subseteq X \to X$ el generador infinitesimal de un C_0-semigrupo de contracciones $\{S(t)\,;\ t \geq 0\}$. Se muestra que $\{S(t)\,;\ t \geq 0\}$ es diferenciable si y sólo si, para cada $\alpha \in (0,1)$, existe $\lim_{n\to\infty} A\left(I - \frac{t}{n}A\right)^{-n}$ en $C([\,\alpha, 1/\alpha\,]; \mathcal{L}(X))$ equipado con la norma supremo. Una consecuencia sobre C_0-semigrupos de contracciones analíticos es también considerada.

1. Introduction

Let X be a Banach space and let us denote by $\mathcal{L}(X)$ be the space of all linear continuous operators from X into itself. We endow $\mathcal{L}(X)$ with the operator norm $\|\cdot\|_{\mathcal{L}(X)}$, i.e. with the norm defined by $\|B\|_{\mathcal{L}(X)} = \sup\{\|Bx\|\,;\ \|x\| = 1\}$ for each B in $\mathcal{L}(X)$. Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\,;\ t \geq 0\}$. We recall that $\{S(t)\,;\ t \geq 0\}$ is called:

(i) *differentiable at* $\tau \geq 0$ if, for each $x \in X$, the mapping $t \mapsto S(t)x$ is differentiable at τ;

(ii) *differentiable* if it is differentiable at each $\tau \in (0, +\infty)$;

(iii) *uniformly differentiable at* $\tau \geq 0$ if the mapping $t \mapsto S(t)$, from $(0, +\infty)$ into $\mathcal{L}(X)$, is differentiable at τ;

(iv) *uniformly differentiable* if it is uniformly differentiable at each $\tau \in (0, +\infty)$;

It is known that a C_0-semigroup is differentiable if and only if it is uniformly differentiable. See Hille, Phillips [4] the first corollary on p. 311, or Pazy [7], Corollary 4.4, p. 53.

Presentado por Jesús Ildefonso Díaz.
Recibido: 9 de Mayo 2001. *Aceptado*: 5 de Diciembre 2001.
Palabras clave / Keywords: differentiable semigroups, analytic semigroups
Mathematics Subject Classifications: 47D03, 47B44

Our goal here is to prove two new characterizations of infinitesimal generators of differentiable (and thus uniformly differentiable) C_0-semigroups of contractions. As a consequence we obtain a new characterization of infintesimal generators of analytic C_0-semigroups of contractions. We note however that, with slight modifications, our arguments can be adapted to handle arbitrary C_0-semigroups. The complete characterization of infinitesimal generators of differentiable C_0-semigroups is due to Pazy [6]. See also Pazy [7], Theorem 4.7, p. 54. While Pazy's characterization relies on the spectral properties of the infinitesimal generator described by means of the complex values in the resolvent set, our results are expressed only in terms of the type of convergence in some exponential-like formulae, by using only real values in the resolvent set, and are of interest in approximation theory. We emphasize that while the interesting part of Pazy's Theorem 4.7 just mentioned is the suffciency, the interesting part of each of our results is the necessity. More precisely we will prove:

Theorem 1 *Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\,;\ t \geq 0\}$. Then $\{S(t)\,;\ t \geq 0\}$ is differentiable (and thus uniformly differentiable) if and only if for each $x \in X$ and each $\alpha \in (0,1)$ there exists*

$$\lim_{n\to\infty} A\left(I - \frac{t}{n}A\right)^{-n} x$$

uniformly for $t \in [\,\alpha, 1/\alpha\,]$.

Theorem 2 *Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\,;\ t \geq 0\}$. Then $\{S(t)\,;\ t \geq 0\}$ is differentiable (and thus uniformly differentiable) if and only if, for each $\alpha \in (0,1)$, there exists*

$$\lim_{n\to\infty} A\left(I - \frac{t}{n}A\right)^{-n}$$

in the norm topology of $C([\,\alpha, 1/\alpha\,]; \mathcal{L}(X))$.

Concerning analytic semigroups we have:

Theorem 3 *Let $A : D(A) \subseteq X \to X$ be a $\mathbb{C}$-linear operator which generates a C_0-semigroup of contractions $\{S(t)\,;\ t \geq 0\}$. Then, $\{S(t)\,;\ t \geq 0\}$ is analytic if and only if, for each $\alpha \in (0,1)$, there exists*

$$\lim_{n\to\infty} A\left(I - \frac{t}{n}A\right)^{-n} = T(t)$$

in the norm topology of $C([\,\alpha, 1/\alpha\,]; \mathcal{L}(X))$, and there exists $C > 0$ such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq \frac{C}{t}$$

for each $t > 0$.

We notice that there exists a very definite relationship between our Theorem 3 and the main result of Crandall-Pazy-Tartar [1], i.e. Theorem 5.5, p. 65 in Pazy [7]. We notice that the necessity part of our Theorem 3 is strictly stronger than the corresponding one in Theorem 5.5 loc.cit. Moreover, from the latter result and Theorem 3, we easily deduce:

Theorem 4 *Let $A : D(A) \subseteq X \to X$ be a $\mathbb{C}$-linear operator which generates a C_0-semigroup of contractions $\{S(t)\,;\ t \geq 0\}$. If there exists $C > 0$ such that*

$$\left\| A\left(I - \frac{t}{n}A\right)^{-n-1} \right\|_{\mathcal{L}(X)} \leq \frac{C}{t},$$

for each $t > 0$ and each $n \in \mathbb{N}$, then there exists

$$\lim_{n\to\infty} A\left(I - \frac{t}{n}A\right)^{-n}$$

in the norm topology of $C([\,\alpha, 1/\alpha\,]; \mathcal{L}(X))$.

2. Proof of Theorem 1

We assume familiarity with the basic concepts and results concerning C_0-semigroups and we refer the reader to Davies [2], Hille, Phillips [4] and Pazy [6].

The proof of the next lemma which is essentially based on the well-known Stirling's Formula is incorporated in that of Theorem 8.3 in Pazy [7], p. 33. See also Lemma 1 in Vrabie [8].

Lemma 1 *For each $a \in (0,1)$ and each $b \in (1,+\infty)$ we have*

$$\lim_{n\to\infty} \frac{n^{n+1}}{n!}\int_0^a (ve^{-v})^n dv = 0 \text{ and } \lim_{n\to\infty} \frac{n^{n+1}}{n!}\int_b^{+\infty} (ve^{-v})^n dv = 0.$$

Lemma 2 *For each $n \in \mathbb{N}$ we have*

$$\frac{n^{n+1}}{n!}\int_0^{+\infty} \left(ve^{-v}\right)^n dv = 1.$$

PROOF. Let us observe that by a simple integration by parts we get

$$\frac{n^{n+1}}{n!}\int_b^{\infty} \left(ve^{-v}\right)^n dv = e^{-nb}\sum_{k=0}^{n}\frac{(nb)^k}{k!},$$

for each $n \in \mathbb{N}$ and $b \geq 0$, and so the conclusion follows by taking $b = 0$ in the equality above. ■

The lemma below is a consequence of Lemma 2 in Dunford, Schwartz [3], p. 566 combined with Hille-Yosida's Theorem (see Pazy [7], Theorem 3.1, p. 8) and with Lemma 2 above.

Lemma 3 *Let $A : D(A) \subseteq X \to X$ be the infintesimal generator of a C_0-semigroup of contractions $\{S(t)\,;\ t \geq 0\}$, let $\lambda > 0$ and let $J_\lambda = (I - \lambda A)^{-1}$. Then, the mapping $\lambda \mapsto J_\lambda$ is of class C^∞ from $(0,+\infty)$ into $\mathcal{L}(X)$. In addition, for each $n \in \mathbb{N}^*$, each $t > 0$ and each $x \in X$, we have*

$$\left(I - \frac{t}{n}A\right)^{-n-1} x - S(t)x = \frac{n^{n+1}}{n!}\int_0^{+\infty} (ve^{-v})^n [S(tv)x - S(t)x]\, dv. \tag{1}$$

We may now proceed to the proof of Theorem 1.

PROOF. *Sufficiency*. Let $x \in X$ and $\alpha \in (0,1)$. From Hille's Exponential Formula (see [7], Theorem 8.3, p. 33) we have $\lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} x = S(t)x$ uniformly for $t \in [\,\alpha, 1/\alpha\,]$. As A is closed, from this remark and from hypothesis, it follows that $\lim_{n\to\infty} A\left(I - \frac{t}{n}A\right)^{-n} x = AS(t)x$ uniformly for $t \in [\,\alpha, 1/\alpha\,]$. But this means that, for each $t > 0$ and each $x \in X$, $S(t)x \in D(A)$, and this completes the proof of the sufficiency.

Necessity. Using once again the closedness of A, Hille's Theorem 3.7.12 in Hille, Phillips [4], p. 83, and Lemma 3, we get

$$A\left(I - \frac{t}{n}A\right)^{-n-1} x - AS(t)x = \frac{n^{n+1}}{n!}\int_0^{+\infty} (ve^{-v})^n [AS(tv)x - AS(t)x]\, dv, \tag{2}$$

for each $t > 0$ and each $x \in X$. Let $\alpha \in (0,1)$, and let us fix $\beta \in (0,\alpha)$. As $\{S(t)\,;\ t \ge 0\}$ is differentiable, it follows that, for each $x \in X$, the mapping $t \mapsto AS(t)x$ is continuous on $(0,+\infty)$. Then, for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\|AS(t)x - AS(s)x\| \le \varepsilon$ for each $t, s \in [\beta, 1/\beta]$, with $|t-s| \le \delta(\varepsilon)$. Moreover, for the same $\varepsilon > 0$, there exist $a = a(\varepsilon) \in (0,1)$ and $b = b(\varepsilon) \in (1,+\infty)$, such that, for each $t \in [\alpha, 1/\alpha]$ and $v \in [a,b]$, we have $tv \in [\beta, 1/\beta]$ and $|t - tv| \le \delta(\varepsilon)$. As a consequence, for each $t \in [\alpha, 1/\alpha]$ and $v \in [a,b]$, we conclude that

$$\|AS(tv)x - AS(t)x\| \le \varepsilon. \tag{3}$$

From (2) we deduce that

$$\left\| A\left(I - \frac{t}{n}A\right)^{-n-1} x - AS(t)x \right\| \le \sum_{k=1}^{5} \|\mathcal{I}_k^n(t)\|, \tag{4}$$

for each $n \in \mathbb{N}$ and each $t \in [\alpha, 1/\alpha]$, where

$$\mathcal{I}_1^n(t) = \frac{n^{n+1}}{n!}\int_0^a (ve^{-v})^n \frac{1}{t}\frac{d}{dv}(S(tv))x\,dv\,;$$

$$\mathcal{I}_2^n(t) = \frac{n^{n+1}}{n!}\int_0^a (ve^{-v})^n AS(t)x\,dv\,;$$

$$\mathcal{I}_3^n(t) = \frac{n^{n+1}}{n!}\int_a^b (ve^{-v})^n (AS(tv)x - AS(t)x)\,dv\,;$$

$$\mathcal{I}_4^n(t) = \frac{n^{n+1}}{n!}\int_b^{+\infty} (ve^{-v})^n AS(tv)x\,dv\,;$$

$$\mathcal{I}_5^n(t) = \frac{n^{n+1}}{n!}\int_b^{+\infty} (ve^{-v})^n AS(t)x\,dv,$$

for each $n \in \mathbb{N}$ and each $t \in [\alpha, 1/\alpha]$. Next, we shall evaluate each of the five terms on the right hand side of (4). To this aim let us observe that, for each $\rho > 0$, and for each $\tau \in [\rho, \infty)$, we have

$$\|AS(\tau)x\| \le \|AS(\rho)x\|. \tag{5}$$

Indeed, since the semigroup is differentiable, the mapping $\tau \mapsto AS(\tau)x$ is a C^1-solution of the equation $u' = Au$ on the interval $(0,+\infty)$ and then, by the dissipativity of A, we deduce (5).

We begin to evaluate $\mathcal{I}_1^n(t)$. Integrating by parts, observing that $|e^{-v} - ve^{-v}| \le 1$ for each $v \in (0,1)$, and taking into account that the mapping $v \mapsto ve^{-v}$ is nondecreasing on $(0,1)$, we deduce

$$\|\mathcal{I}_1^n(t)\| = \left\| \frac{n^{n+1}}{n!}\int_0^a (ve^{-v})^n \frac{1}{t}\frac{d}{dv}(S(tv))x\,dv \right\| =$$

$$= \frac{1}{t}\left\| \frac{n^{n+1}}{n!}(ae^{-a})^n S(ta)x - \frac{n^{n+1}}{n!}\int_0^a n(ve^{-v})^{n-1}(e^{-v} - ve^{-v})S(tv)x\,dv \right\| \le$$

$$\le \frac{\|x\|}{\alpha}\left[\frac{n^{n+1}}{n!}(ae^{-a})^n + \frac{n^{n+1}}{(n-1)!}(ae^{-a})^{n-1}\right],$$

for each $n \in \mathbb{N}$ and each $t \in [\alpha, 1/\alpha]$. Clearly we have

$$\frac{n^{n+1}}{n!}(ae^{-a})^n = \frac{n^n}{n!}e^{-n}\cdot n\left(ae^{1-a}\right)^n \quad \text{and}$$

$$\frac{n^{n+1}}{(n-1)!}(ae^{-a})^{n-1} = \frac{(n-1)^{n-1}}{(n-1)!}e^{-(n-1)}\cdot n^2\left(\frac{n}{n-1}\right)^{n-1}\left(ae^{1-a}\right)^{n-1}.$$

In view of Stirling's Formula (see Nikolsky [5], p. 393), we have $\lim_{n\to\infty} \frac{n^n}{n!} e^{-n} = 0$. In addition, since $a \in (0,1)$, we have $ae^{1-a} < 1$. Therefore, we deduce both

$$\lim_{n\to\infty} \frac{n^{n+1}}{n!}(ae^{-a})^n = 0 \;\text{ and }\; \lim_{n\to\infty} \frac{n^{n+1}}{(n-1)!}(ae^{-a})^{n-1} = 0.$$

So, we conclude that

$$\lim_{n\to\infty} \|\mathfrak{I}_1^n(t)\| = 0$$

uniformly with respect ot $t \in [\,\alpha, 1/\alpha\,]$. Next, from (5), we have

$$\|\mathfrak{I}_2^n(t)\| \le \|AS(\alpha)x\| \frac{n^{n+1}}{n!} \int_0^a (ve^{-v})^n \, dv,$$

for each $n \in \mathbb{N}$ and each $t \in [\,\alpha, 1/\alpha\,]$. From this inequality and Lemma 1, we get

$$\lim_{n\to\infty} \|\mathfrak{I}_2^n(t)\| = 0$$

uniformly for $t \in [\,\alpha, 1/\alpha\,]$. In order to evaluate $\mathfrak{I}_3^n(t)$, let us observe that, from (3) and Lemma 2, we have

$$\|\mathfrak{I}_3^n(t)\| \le \frac{n^{n+1}}{n!} \int_a^b \|(ve^{-v})^n\| \|AS(tv)x - AS(t)x\| \, dv \le \varepsilon \frac{n^{n+1}}{n!} \int_a^b (ve^{-v})^n \, dv \le$$

$$\le \varepsilon \frac{n^{n+1}}{n!} \int_0^\infty (ve^{-v})^n \, dv = \varepsilon$$

for each $n \in \mathbb{N}$ and each $t \in [\,\alpha, 1/\alpha\,]$. Consequently

$$\limsup_{n\to\infty} \|\mathfrak{I}_3^n(t)\| \le \varepsilon.$$

As concerns $\mathfrak{I}_4^n(t)$, let us remark that, in view of (5), we have

$$\|AS(tv)x\| \le \|AS(\alpha b)x\|$$

for each $t \in [\,\alpha, \frac{1}{\alpha}\,]$ and each $v \in (b, +\infty)$. Therefore

$$\|\mathfrak{I}_4^n(t)\| = \left\| \frac{n^{n+1}}{n!} \int_b^\infty (ve^{-v})^n AS(tv)x \, dv \right\| \le \|AS(\alpha b)x\| \frac{n^{n+1}}{n!} \int_b^\infty (ve^{-v})^n \, dv,$$

for each $n \in \mathbb{N}$ and each $t \in [\,\alpha, 1/\alpha\,]$. From this relation, and Lemma 1, we get

$$\lim_{n\to\infty} \|\mathfrak{I}_4^n(t)\| = 0.$$

Furthermore, from (5), we have

$$\|\mathfrak{I}_5^n(t)\| \le \|AS(\alpha)x\| \frac{n^{n+1}}{n!} \int_b^\infty (ve^{-v})^n \, dv,$$

for each $n \in \mathbb{N}$ and each $t \in [\,\alpha, 1/\alpha\,]$. Accordingly $\lim_{n\to\infty} \|\mathfrak{I}_5^n(t)\| = 0$. Summing up, we deduce

$$\limsup_{n\to\infty} \left\| A \left(I - \frac{t}{n} A \right)^{-n-1} x - AS(t)x \right\| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that, for each $\alpha \in (0,1)$, we have

$$\lim_{n\to\infty} A\left(I - \frac{t}{n}A\right)^{-n-1} x = AS(t)x \tag{6}$$

uniformly for $t \in [\,\alpha, 1/\alpha\,]$. To complete the proof, we have merely to show that

$$\lim_{n\to\infty} A\left(I - \frac{t}{n+1}A\right)^{-n-1} x = AS(t)x \tag{7}$$

uniformly for $t \in [\,\alpha, 1/\alpha\,]$. To this aim, let us observe that, from (6) and the infinite dimensional version of Arzelà-Ascoli's Theorem (see Vrabie [9], Theorem 1.3.1, p. 7), it follows that, for each sequence $(a_n)_{n\in\mathbb{N}}$ of functions from $\mathbb{R}_+^*$ în $\mathbb{R}_+^*$ satisfying $\lim_{n\to\infty} a_n(t) = t$ uniformly on every compact subset in $\mathbb{R}_+^*$, we have $\lim_{n\to\infty} A\left(I - \frac{a_n(t)}{n}A\right)^{-n} x = AS(t)x$ uniformly on every compact subset in $\mathbb{R}_+^*$. Indeed, this follows from the remark that, for each $x \in X$ and $\alpha \in (0,1)$, the family of functions $\left\{t \mapsto A\left(I - \frac{t}{n}A\right)^{-n-1} x;\ n \in \mathbb{N}^*\right\}$ is equicontinuous on $[\,\alpha, 1/\alpha\,]$ being relatively compact in $C([\,\alpha, 1/\alpha\,]; X)$. See (6) and Arzelà-Ascoli's Theorem 1.3.1, loc.cit. We complete the proof by observing that the choice $a_n(t) = \frac{nt}{n+1}$, for $n \in \mathbb{N}$, leads to (7). ■

Since the proof of Theorem 2 follows exactly the same lines as that of Theorem 1, we do not give details. Moreover, Theorem 3 is a direct consequence of Theorem 2 and the equivalence between (a) and (d) in Hille's Theorem 5.2 in Pazy [7], p. 61.

References

[1] Crandall, M. G., Pazy, A., Tartar, L. (1979). Remarks on generators of analytic semigroups, *Israel J. Math.*, **32**, 363-374.

[2] Davies, E. B. (1980). *One-Parameter Semigroups*, London Mathematical Society, Monographs, No. **15**, Academic Press London.

[3] Dunford, N., Schwartz, J. T. (1958). *Linear Operators Part I: General Theory*, Interscience Publishers, Inc. New York, Interscience Publishers Ltd. London.

[4] Hille, E., Phillips, R. S. (1981). *Functional Analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, Volume 31, Fourth Printing of Revised Edition.

[5] Nikolsky, S. M. (1981). *A Course of Mathematical Analysis*, Volume 1, Mir Publishers, Moscow.

[6] Pazy, A. (1968). On the differentiability and compactness of semi-groups of linear operators, *J. Math. and Mech.*, **17**, 1131-1141.

[7] Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, Berlin-Heidelberg-New York-Tokyo.

[8] Vrabie, I. I. (1989). A new characterization of generators of linear compact semigroups, *An Ştiinţ. Univ. Al. I. Cuza Iaşi, Secţ. I a Mat.* **35**, 145-151.

[9] Vrabie, I. I. (1995). *Compactness Methods for Nonlinear Evolutions*, Second Edition, Pitman Monographs and Surveys in Pure and Applied Mathematics **75**, Longman, Harlow/New York.

Ioan I. Vrabie
Faculty of Mathematics
"Al. I. Cuza" University of Iaşi
Iaşi 6600, ROMANIA
ivrabie@uaic.ro