RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **95 (**2), 2001, pp. 259–277
Geometría y Topología / Geometry and Topology

Geometrical aspects of the Landau-Hall problem on the hyperbolic plane

A. López Almorox and C. Tejero Prieto

Abstract. Some geometrical aspects of the classical hyperbolic Landau-Hall problem are discussed. The Lie algebra of infinitesimal symmetries of this problem is explicitly given, turning out that it is isomorphic to $\mathfrak{so}(2,1)$ and that its associated Noether invariants are the hyperbolic angular momenta. The Hamiltonian formulation is also given, allowing us to obtain the manifold of orbits of constant energy for this problem using symplectic reduction techniques.

Aspectos geométricos del problema de Landau-Hall en el plano hiperbólico

Resumen. Se discuten algunos aspectos del problema de Landau-Hall hiperbólico. El álgebra de Lie de las simetrías infinitesimales de este problema se da explícitamente, resultando ser isomorfa a $\mathfrak{so}(2,1)$ y que sus invariantes Noether asociados son los momentos angulares hiperbólicos. Asimismo se desarrolla la formulación hamiltoniana, lo que nos permitirá obtener la variedad de órbitas de energía constante de este problema mediante técnicas de reducción simpléctica.

1. Introduction

The Landau-Hall problem is the study of the motion of a charged particle in a constant and static magnetic field (in what follows we will refer to it as a magnetic Hall field) on a Riemann surface. The quantum aspects of this problem have been used to give different models of the integer Hall effect; that is, the quantization of the transverse resistivity of an electronic gas at very low temperatures and very high static magnetic fields in certain bidimensional experimental devices which was discovered by K. von Klitzing in 1980 [19].

Let us recall briefly what the classical Hall effect is on the euclidean plane. Let $\{x_1, x_2, x_3\}$ be the euclidean coordinates on $\mathbb{R}^3$ and let us suppose that X_1X_2 is the plane of the conductor. Let us consider a charged particle moving in this plane and let $\vec{B} \equiv (0,0,B)$ be a constant and static magnetic field perpendicular to X_1X_2 and let $\vec{E} \equiv (E_1, E_2, 0)$ be a constant and static electric field in this plane. If a particle of mass m and charge e is moving with velocity $\vec{v} \equiv (v_1, v_2, 0)$ in that plane then the forces acting upon it are the Lorentz force $\vec{F}_{Lorentz} = e\vec{v} \wedge \vec{B} \equiv (F_1, F_2, 0) = (eBv_2, -eBv_1, 0)$ and the electric force $\vec{F}_{electric} = e\vec{E} \equiv (eE_1, eE_2, 0)$. The motion of the particle on the plane X_1X_2 is described by a parametrized curve $\vec{x}(t) \equiv (x_1(t), x_2(t), 0)$ which is a solution of Newton equation with initial conditions $\vec{x}(0) \equiv (x_1(0), x_2(0), 0)$ and $\vec{v}(0) \equiv (v_1(0), v_2(0), 0)$ with $\vec{v}(t) \equiv (v_1(t), v_2(t), 0) = (\dot{x}_1(t), \dot{x}_2(t), 0)$ being the velocity of the particle. If we introduce complex notation $z(t) = x_1(t) + ix_2(t)$, $\dot{z}(t) = v_1(t) +$

Presentado por Pedro Luis García Pérez.
Recibido: 3 de Noviembre 1999. *Aceptado*: 20 de Abril 2000.
Palabras clave / Keywords: Landau problem, infinitesimal symmetries, hyperbolic plane, symplectic reduction, momentum map, Darboux frame
Mathematics Subject Classifications: 37J15, 37J35, 70H3, 70H06

$iv_2(t)$, $E = E_1 + iE_2$, one has that the evolution equations are $\ddot{z}(t) = -i\,\frac{eB}{m}\dot{z}(t) + \frac{e}{m}E = -i\omega_c\dot{z}(t) + \frac{e}{m}E$ where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency. Performing an integration, one has that

$$\dot{z}(t) = \frac{e}{im\omega_c}E + [\dot{z}(0) - \frac{e}{im\omega_c}E]e^{-i\omega_c t}$$

The term $\dot{z}(t)_{cyclotron} = [\dot{z}(0) - \frac{e}{im\omega_c}E]e^{-i\omega_c t}$ is called the cyclotron velocity, whereas $\dot{z}(t)_{drift} = -i\frac{eE}{m\omega_c} = -i\frac{E}{B}$ is called the drift velocity and it is constant. Let us notice that the drift velocity $\vec{v}_{drift} \equiv (\frac{eE_2}{m\omega_c}, -\frac{eE_1}{m\omega_c}, 0) = (\frac{E_2}{B}, -\frac{E_1}{B}, 0)$ is orthogonal to the applied electric field $\vec{E} \equiv (E_1, E_2, 0)$ and is independent of the mass and the charge of the particle. Performing another integration one obtains that the trajectory of the particle is given by the cycloid

$$z(t) = z(0) + \frac{e}{im\omega_c}Et - \frac{1}{i\omega_c}[\dot{z}(0) - \frac{e}{im\omega_c}E](e^{-i\omega_c t} + 1)$$

Let us consider now a gas of non-interacting charged particles moving in a bidimensional conductor contained in the X_1X_2 plane and subject to the action of a constant and static magnetic field orthogonal to that plane and to a constant and static electric field as described before. Let n be the electronic density, then the mean velocity of these particles is the drift velocity $< \dot{z}_{mean} >= \dot{z}(t)_{drift}$; therefore we will call the drift current or Hall current, and we will denote it by $j^{Hall} = j_1^{Hall} + ij_2^{Hall}$, the following expression

$$j^{Hall} = ne < \dot{z}_{mean} >= -i\frac{ne^2}{m\omega_c}E = -i\frac{ne}{B}E$$

therefore $j_1^{Hall} = -\frac{ne}{B}E_2$ and $j_2^{Hall} = \frac{ne}{B}E_1$. The Hall conductivity matrix $\sigma^{Hall} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ exhibits the proportionality between the Hall current and the applied electric field, that is $j^{Hall} = \sigma^{Hall}E$. Therefore

$$\begin{pmatrix} j_1^{Hall} \\ j_2^{Hall} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{ne}{B} \\ \frac{ne}{B} & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

In particular the transverse conductivity is $\sigma_{12} = -\frac{ne}{B}$. The inverse of the Hall conductivity matrix is called the Hall resistivity matrix $\rho^{Hall} = [\sigma^{Hall}]^{-1} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ and is given by

$$\rho^{Hall} = \begin{pmatrix} 0 & -\frac{B}{ne} \\ \frac{B}{ne} & 0 \end{pmatrix}$$

with $\rho_{12} = -\frac{B}{ne}$ being the so-called transverse Hall resistivity, which, in the approximation we have taken, depends linearly on the magnetic field B.

However, in experiments with Si-Mosfets devices at very low temperatures and very high magnetic fields this linearity is not observed [24]. Instead of it one observes a quantization of the transverse Hall conductivity, σ_{12} is an integer or a fractional number times $\frac{e^2}{h}$ with the appearance of plateaux in the graph of ρ_{12} against the applied magnetic field B. This phenomenon bears the name of quantum Hall effect (integer or fractional).

The different theoretical explanations of the quantum Hall effect have started always studying the quantum Landau problem for a particle with different boundary conditions [16, 27]. The quantum treatment of a charged particle moving on a plane and subject to the action of a constant magnetic field is well known but to arrive at a quantization of Hall conductivity one has to impose that the magnetic flux is quantized (Dirac's condition), this fact reveals the importance of the non triviality of the topology of the configuration space in the explanation of this phenomenon. The explanation of the plateaux in the integer quantum Hall effect is done through the effect that the impurities and the disorder in the semiconductor produce on the Landau levels which are split into energy bands.

Thouless *et al.* have studied in [25] the Landau-Hall problem for electrons without interaction in periodic potentials (Landau problem in the flat torus T^2), calculating the Hall conductivity for Bloch electrons in a magnetic field. Refinements of certain arguments of Laughlin and the use of the Kubo formula for the conductivity have made possible the understanding of the Hall conductivity as a topological invariant [26, 7]. The possible consideration of this problem in materials with different boundary conditions, leads to the necessity of generalizing the preceding study to other Riemann surfaces different from $\mathbb{R}^2 \simeq \mathbb{C}$ or the flat torus T^2. The quantum Landau-Hall problem in genus $g = 0$ is analogous to the study of a charged particle on a sphere and subject to the action of a magnetic monopole located at its center and can be found in [11, 5]. Some aspects related to the quantum Landau-Hall problem on Riemann surfaces of genus $g > 1$ have been studied during the last years and can be found in [9]–[3].

2. Geometrical aspects of the classical Landau-Hall problem on Riemann surfaces

For a better understanding of the quantum Hall effect on Riemann surfaces it is necessary to have a good knowledge of the classical Landau-Hall problem in configuration spaces which are Riemann surfaces. Thus, the first thing to do is to generalize the concept of a constant and static magnetic field which is orthogonal to the surface [20]; these are the so-called magnetic Hall fields in the physics literature .

Let Q be a Riemannian manifold, T_2 the metric tensor and ∇ the Levi-Civita connection associated with T_2.

Definition 1 *A magnetic Hall field on Q is a 2-form F_{Hall} on Q which verifies Maxwell equations $dF_{Hall} = 0$, $\delta F_{Hall} = 0$ and that in addition is covariantly constant, that is $D^{\nabla} F_{Hall} = 0$, for every vector field D on Q.*

Proposition 1 *If dim Q = 2 then the magnetic Hall fields are the harmonic 2-forms on Q. In particular every magnetic Hall field is of the form*

$$F_{Hall} = B\Omega_2 \qquad\qquad B \in \mathbb{R}$$

with Ω_2 being the Riemannian area element.

Remark 1 The magnetic Hall fields on a bidimensional Riemannian manifold Q can be described as the curvature 2-form Ω_{Hall} of a connection ω_{Hall} on a principal fiber bundle $\pi : P \rightarrow Q$ with structure group $U(1)$. If $\{U_\alpha\}_{\alpha \in I}$ is a trivializing covering of P and $\{\sigma_\alpha\}$ are the local sections that trivialize the bundle

$$\begin{array}{rcl} \Phi_\alpha : \pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times U(1) \\ \sigma_\alpha(x) \cdot g & \longrightarrow & (x, g) \end{array}$$

Then one defines $A^{(\alpha)}_{Hall} = \sigma^*_\alpha(\omega_{Hall})$ and $F^{(\alpha)}_{Hall} = \sigma^*_\alpha(\Omega_{Hall})$. If $\sigma_\alpha(x) = \sigma_\beta(x) \cdot g_{\alpha\beta}(x)$, with $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$ being the corresponding transition functions, on the non empty intersections one has that

$$\begin{array}{rcl} A^{(\beta)}_{Hall} & = & A^{(\alpha)}_{Hall} + g^*_{\alpha\beta}(\theta) \\ F^{(\beta)}_{Hall} & = & F^{(\alpha)}_{Hall} \end{array}$$

with θ being the Maurer-Cartan form of the structure group $U(1)$. Therefore, there exists a 2-form F_{Hall} on Q, moreover, if it satisfies the Maxwell equations we shall say that it represents a magnetic Hall field on Q. It is clear that in general there does not exist a global 1-form A on Q such that $F_{Hall} = dA$, although locally it is always verified that $F^{(\alpha)}_{Hall} = dA^{(\alpha)}_{Hall}$.

Proposition 2 *The magnetic Hall fields on a compact Riemann surface are discrete; that is,*

$$F_{Hall} = \frac{k\,h}{e\,a(Q)}\,\Omega_2 \qquad k \in \mathbb{Z}$$

where $k = c_1(P)$ is the first Chern class of the fiber bundle P and $a(Q)$ is the area of the surface.

Remark 2 If Q is compact and if we suppose that $\rho_{12}^{Hall} \sim \frac{B}{ne}$, as we deduced classically for the euclidean plane, then taking into account that $B = \frac{k\,h}{e\,a(Q)}$, it would be verified that

$$\rho_{12}^{Hall} \sim \frac{k\,h}{n\,e^2\,a(Q)} = \frac{k}{N}\,\frac{h}{e^2}$$

with $N = n\,a(Q)$ being the number of electrons in Q and $k \in \mathbb{Z}$ the Chern class of the fiber bundle P. Surprisingly one obtains a behavior similar to that predicted by quantum mechanics.

2.1. (Local) Lagrangian formulation of the motion of a particle in a magnetic Hall field

The classical description of the motion of a particle of mass m and charge e in a magnetic Hall field on a bidimensional Riemannian manifold (Q, T_2) can be formulated as a (local) variational problem on the 1-jet fiber bundle $\pi : J^1(\mathbb{R} \times Q/\mathbb{R}) = \mathbb{R} \times TQ \to \mathbb{R} \times Q$, of 1-jets of sections of the regular projection onto the first component $\pi_1 : \mathbb{R} \times Q \to Q$, with a Lagrangian density which, in general, is only locally defined.

$$\begin{array}{ccc} J^1(R \times Q/\mathbb{R}) & & P \\ \pi \downarrow & & \downarrow p \\ \mathbb{R} \times Q & \xrightarrow{\pi_2} & Q \\ \pi_1 \downarrow & & \\ \mathbb{R} & & \end{array}$$

We will suppose that the magnetic Hall field $F_{Hall} = B\,\Omega_2$ is defined by the curvature form Ω_{Hall} of a connection ω_{Hall} on a principal fiber bundle $p : P \to Q$ with structure group $U(1)$. Let $\sigma_\alpha : U_\alpha \to P$, $\forall \alpha \in I$ be the local sections of a trivializing covering $\{U_\alpha\}_{\alpha \in I}$ of P, let us denote by $A_{Hall}^{(\alpha)} = \sigma_\alpha^*(\omega_{Hall})$ the local vector potential and by $F_{Hall}^{(\alpha)} = \sigma_\alpha^*(\Omega_{Hall}) = dA_{Hall}^{(\alpha)}$ the magnetic Hall field on the open set U_α. Let us suppose moreover that $\{q_i^{(\alpha)}, 1 \leq i \leq 2\}$ are local coordinates on the open set U_α; then $T_2^{(\alpha)} = \sum_{i,j=1}^2 g_{ij}^{(\alpha)}\,dq_i^{(\alpha)} \otimes dq_j^{(\alpha)}$, $A_{Hall}^{(\alpha)} = A_1^{(\alpha)}\,dq_1^{(\alpha)} + A_2^{(\alpha)}\,dq_2^{(\alpha)}$ and $F_{Hall}^{(\alpha)} = F_{12}^{(\alpha)}\,dq_1^{(\alpha)} \wedge dq_2^{(\alpha)}$. On the open sets $\bar{U}_\alpha = \pi^{-1}(J \times U_\alpha) \subset \mathbb{R} \times TQ$ with J being an open set of $\mathbb{R}$ with coordinate $\{t\}$, one defines:

a) The Lagrangian (local in general)

$$L_{Hall}^{(\alpha)} = \frac{1}{2} m \dot{q}_i^{(\alpha)} g_{ij}^{(\alpha)} \dot{q}_j^{(\alpha)} + e\,A_i^{(\alpha)} \dot{q}_i^{(\alpha)}.$$

b) The Poincare-Cartan form (local in general)

$$\Theta_{Hall}^{(\alpha)} = p_j^{(\alpha)} dq_j^{(\alpha)} - H_{Hall}^{(\alpha)} dt = -(m\dot{q}_i^{(\alpha)} g_{ij}^{(\alpha)} + e\,A_j^{(\alpha)}) dq_j^{(\alpha)} + \frac{1}{2} m \dot{q}_i^{(\alpha)} g_{ij}^{(\alpha)} \dot{q}_j^{(\alpha)} dt$$

where

$$\begin{aligned} p_j^{(\alpha)} &= -\frac{\partial L_{Hall}^{(\alpha)}}{\partial \dot{q}_j^{(\alpha)}} = -(m\dot{q}_i^{(\alpha)} g_{ij}^{(\alpha)} + e\,A_j^{(\alpha)}) \\ H_{Hall}^{(\alpha)} &= -\left(\frac{\partial L_{Hall}^{(\alpha)}}{\partial \dot{q}_j^{(\alpha)}}\,\dot{q}_j^{(\alpha)} - L_{Hall}^{(\alpha)}\right) = -\frac{1}{2} m \dot{q}_i^{(\alpha)} g_{ij}^{(\alpha)} \dot{q}_j^{(\alpha)} \end{aligned}$$

are the generalized momentum and the Hall Hamiltonian respectively. Let us notice that the Hall Hamiltonian can also be expressed as

$$H_{Hall}^{(\alpha)} = -\frac{1}{2m}(p_i^{(\alpha)} + e\,A_i^{(\alpha)})[g^{(\alpha)}]^{ij}(p_j^{(\alpha)} + e\,A_j^{(\alpha)})$$

which is the usual Hamiltonian of a charged particle placed in an electromagnetic field. (Notice that due to our definition of the generalized momenta there is a global sign change, thus inessential, with respect to the traditional notation).

Taking into account that on the intersections $U_\alpha \cap U_\beta \neq \emptyset$, it is verified that $A_{Hall}^{(\beta)} = A_{Hall}^{(\alpha)} + d\Psi^{(\alpha\beta)}$ and $F_{Hall}^{(\beta} = F_{Hall}^{(\alpha)}$, then

$$\begin{aligned} L_{Hall}^{(\beta)} &= L_{Hall}^{(\alpha)} + e\,\frac{\partial \Psi^{(\alpha\beta)}}{\partial q_i^{(\alpha)}}\dot{q}_i^{(\alpha)} \\ \Theta_{Hall}^{(\beta)} &= \Theta_{Hall}^{(\alpha)} - e\,d\Psi^{(\alpha\beta)}. \end{aligned}$$

Therefore, since the fiber bundle P may not be trivial, the Lagrangian and the Poincare-Cartan form, in general, are not globally defined. However

$$\begin{aligned} H_{Hall}^{(\alpha)} &= H_{Hall}^{(\beta)} \\ d\Theta_{Hall}^{(\alpha)} &= d\Theta_{Hall}^{(\beta)} \end{aligned}$$

that is, there exists a global Hall Hamiltonian which we will denote by H_{Hall}. These expressions prove that there exists a global 2-form on $J^1(R \times Q/\mathbb{R}) = \mathbb{R} \times TQ$ which will be denoted by Ω_{Hall}; in general it is not exact, but on each open set $\bar{U}_\alpha$ it coincides with $d\Theta_{Hall}^{(\alpha)}$. If we denote by $\Theta_{kinetic}$ the (globally defined) Poincare-Cartan form associated with the (global) kinetic term of the Lagrangian then

$$\Omega_{Hall} = d\Theta_{kinetic} - e\,\pi^*(F_{Hall})$$

and therefore the evolution equations can be given in a global way by Cartan equations

$$(i_D\Omega_{Hall})_{|j^1\sigma} = 0 \qquad \forall D \in \Gamma(TJ^1(\mathbb{R} \times Q/\mathbb{R}))$$

The solutions of these equations are parametrized curves on Q, thus on each open set U_α, they are of the form

$$\begin{aligned} \sigma : J &\longrightarrow U_\alpha \\ t &\longrightarrow \sigma(t) = (q_1^{(\alpha)}(t), q_2^{(\alpha)}(t)) \end{aligned}$$

fulfilling the usual equations for a charged particle subject to the Lorentz force

$$m\,(\ddot{q}_k^{(\alpha)} + [\Gamma^{(\alpha)}]_{ij}^k \dot{q}_i^{(\alpha)}\dot{q}_j^{(\alpha)}) = e\,[g^{(\alpha)}]^{sk} F_{sr}^{(\alpha)}\dot{q}_r^{(\alpha)}$$

with $[\Gamma^{(\alpha)}]_{ij}^k$ being the Christoffel symbols of the Levi-Civita connection associated with T_2 on the open set U_α and $F_{12}^{(\alpha)} = B(g_{11}^{(\alpha)}g_{22}^{(\alpha)} - g_{12}^{(\alpha)}g_{21}^{(\alpha)})^{\frac{1}{2}}$ the magnetic Hall field.

2.2. Infinitesimal symmetries of the classical Landau-Hall problem

The concept of infinitesimal symmetries of a globally defined variational problem, and their associated Noether invariants can be found in [13]; however due to the fact that in the Landau-Hall problem the Lagrangian and the Poincare-Cartan are, in general, not globally defined it is necessary to generalize these concepts in the following way [20]

Definition 2 *A vector field D on $\mathbb{R}\times Q$ is said to be an infinitesimal symmetry of the Landau-Hall problem if its 1-jet extension j^1D to $J^1(\mathbb{R}\times Q/\mathbb{R})$ verifies the following conditions:*

1. *D is π_1-projectable.*
2. *$(j^1D)^L\Theta^{(\alpha)}_{Hall} = -dg^{(\alpha)}_D$ on each open set $\bar{U}_\alpha$.*
3. *On $\bar{U}_\alpha \cap \bar{U}_\beta \neq \emptyset$, it is verified that $g^{(\beta)}_D = g^{(\alpha)}_D + e\, j^1D(\Psi^{(\alpha\beta)})$.*

Notice that in the case of globally defined Landau-Hall problems, such as the euclidean or hyperbolic plane, this definition coincides with the one given in [13]. One can also compare this definition with the definition of the symmetries of a gauge field given in [18, 12]

Definition 3 *The Noether invariant f_D associated with the infinitesimal symmetry D is the global function of $J^1(\mathbb{R}\times Q/\mathbb{R})$ which on each open set $\bar{U}_\alpha$ is given by*

$$f^{(\alpha)}_D = i_{j^1D}\Theta^{(\alpha)}_{Hall} + g^{(\alpha)}_D$$

Notice that f_D is globally defined because on the intersections $\bar{U}_\alpha \cap \bar{U}_\beta \neq \emptyset$ it is verified that $f^{(\alpha)}_D = f^{(\beta)}_D$.

Corollary 1 *The set of infinitesimal symmetries for the Landau-Hall problem is a real Lie algebra.*

Let us see which are the equations that an infinitesimal symmetry of the Landau-Hall problem has to verify.

Let $\bar{U}$ be one of the open sets $\{\bar{U}_\alpha\}_{\alpha\in I}$ and let $\{t, q_1, q_2, \dot{q}_1, \dot{q}_2\}$ be fibered coordinates on $\bar{U}$; then $F_{Hall} = F_{12}(q_1,q_2)\, dq_1 \wedge dq_2 = B\sqrt{det\, G}\, dq_1 \wedge dq_2$ and $A_{Hall} = A_1(q_1,q_2)\, dq_1 + A_2(q_1,q_2)\, dq_2$ with $F_{12} = \frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}$. If D is an infinitesimal symmetry then on $U = \pi(\bar{U})$ it is expressed as

$$D = f(t)\frac{\partial}{\partial t} + f_1(t, mq_1, q_2)\frac{\partial}{\partial q_1} + f_2(t, mq_1, q_2)\frac{\partial}{\partial q_2}$$

since it is π_1-projectable. Therefore $j^1D = f\,\frac{\partial}{\partial t} + f_i\,\frac{\partial}{\partial q_i} + [\frac{\partial f_i}{\partial t} + \dot{q}_j\,\frac{\partial f_i}{\partial q_j} - \dot{q}_i\,\frac{\partial f}{\partial t}]\,\frac{\partial}{\partial \dot{q}_i}$. Since on $\bar{U}$ $(j^1D)^L\Theta_{Hall} = -dg_D$ we have [20]

Proposition 3 *If $D = f\frac{\partial}{\partial t} + f_1\frac{\partial}{\partial q_1} + f_2\frac{\partial}{\partial q_2}$ is an infinitesimal symmetry then on $\bar{U}$ the following system of partial differential equations is satisfied*

$$\begin{aligned}
&m[\frac{\partial^2 f_1}{\partial t^2}g_{11} + \frac{\partial^2 f_2}{\partial t^2}g_{21}] + e\,F_{12}\frac{\partial f_2}{\partial t} = 0\\
&m[\frac{\partial^2 f_1}{\partial t^2}g_{12} + \frac{\partial^2 f_2}{\partial t^2}g_{22}] - e\,F_{12}\frac{\partial f_1}{\partial t} = 0\\
&\frac{\partial}{\partial q_2}[m(\frac{\partial f_1}{\partial t}g_{11} + \frac{\partial f_2}{\partial t}g_{21}) + e\,F_{12}f_2] - \frac{\partial}{\partial q_1}[m(\frac{\partial f_1}{\partial t}g_{12} + \frac{\partial f_2}{\partial t}g_{22}) - e\,F_{12}f_1] = 0\\
&2(\frac{\partial f_1}{\partial q_1}g_{11} + \frac{\partial f_2}{\partial q_1}g_{21}) + (f_1\frac{\partial g_{11}}{\partial q_1} + f_2\frac{\partial g_{11}}{\partial q_2}) - g_{11}\frac{\partial f}{\partial t} = 0\\
&2(\frac{\partial f_1}{\partial q_2}g_{12} + \frac{\partial f_2}{\partial q_2}g_{22}) + (f_1\frac{\partial g_{22}}{\partial q_1} + f_2\frac{\partial g_{22}}{\partial q_2}) - g_{22}\frac{\partial f}{\partial t} = 0\\
&(\frac{\partial f_1}{\partial q_1}g_{12} + \frac{\partial f_2}{\partial q_1}g_{22}) + (\frac{\partial f_1}{\partial q_2}g_{11} + \frac{\partial f_2}{\partial q_1}g_{21}) + (f_1\frac{\partial g_{12}}{\partial q_1} + f_2\frac{\partial g_{12}}{\partial q_2}) - g_{12}\frac{\partial f}{\partial t} = 0.
\end{aligned}$$

Let us notice that in these equations the vector potential used to define the Lagrangian does not appear but, rather the magnetic Hall field, which is globally defined.

To integrate these equations it is better to use isothermal coordinates $\{x_1, x_2\}$ on the open set U, thus the complex coordinate of the underlying Riemann surface is $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$. The infinitesimal symmetries are

$$D = f(t)\,\frac{\partial}{\partial t} + h_z(t,z,\bar{z})\,\frac{\partial}{\partial z} + h_{\bar{z}}(t,z,\bar{z})\,\frac{\partial}{\partial \bar{z}}$$

with $h_z = f_1 + if_2$ and $h_{\bar{z}} = f_1 - if_2$ (notice that $\bar{h}_z = h_{\bar{z}}$ because the vector field is real) and the equations that must be verified by the infinitesimal symmetries are simplified to

$$0 = \frac{\partial^2 h_{\bar{z}}}{\partial t^2} - \frac{eF_{z\bar{z}}}{mg_{z\bar{z}}}\,\frac{\partial h_{\bar{z}}}{\partial t} \tag{1}$$

$$0 = \frac{\partial^2 h_{z}}{\partial t^2} + \frac{eF_{z\bar{z}}}{mg_{z\bar{z}}}\,\frac{\partial h_{z}}{\partial t} \tag{2}$$

$$0 = \frac{\partial^2}{\partial \bar{z}\partial t}(g_{z\bar{z}}h_{\bar{z}}) - \frac{\partial}{\partial \bar{z}}(\frac{eF_{z\bar{z}}}{m}h_{\bar{z}}) - \frac{\partial^2}{\partial z\partial t}(g_{z\bar{z}}h_{z}) - \frac{\partial}{\partial z}(\frac{eF_{z\bar{z}}}{m}h_{z}) \tag{3}$$

$$0 = \frac{\partial h_{\bar{z}}}{\partial z} \tag{4}$$

$$0 = h_z\,\frac{\partial g_{z\bar{z}}}{\partial z} + h_{\bar{z}}\,\frac{\partial g_{z\bar{z}}}{\partial \bar{z}} + g_{z\bar{z}}\,\frac{\partial h_z}{\partial z} + g_{z\bar{z}}\,\frac{\partial h_{\bar{z}}}{\partial \bar{z}} - g_{z\bar{z}}\frac{\partial f}{\partial t} \tag{5}$$

$$0 = \frac{\partial h_z}{\partial \bar{z}}. \tag{6}$$

Let us note that $\frac{eF_{z\bar{z}}}{m\,g_{z\bar{z}}} = \frac{i\,eB}{m} = i\omega_c$. These equations have been integrated by the authors for S^2 with its usual metric, the flat torus T^2, $\mathbb{C}^*$ with its usual complete flat metric, every Riemann surface of genus $g > 1$ as well as for their universal Riemannian coverings.

Corollary 2 *Every infinitesimal isometry of T_2 is an infinitesimal symmetry of the Landau-Hall problem.*

PROOF. If $X = \hat{h}_z(z,\bar{z})\,\frac{\partial}{\partial z} + \hat{h}_{\bar{z}}(z,\bar{z})\,\frac{\partial}{\partial \bar{z}}$ is an infinitesimal isometry of T_2, it is verified that $0 = X^L T_2 = 2g_{z\bar{z}}\,\frac{\partial \hat{h}_{\bar{z}}}{\partial z}\,dz\otimes dz + 2g_{z\bar{z}}\,\frac{\partial \hat{h}_z}{\partial \bar{z}}\,d\bar{z}\otimes d\bar{z} + [g_{z\bar{z}}\,(\frac{\partial \hat{h}_z}{\partial z} + \frac{\partial \hat{h}_{\bar{z}}}{\partial \bar{z}}) + \hat{h}_z\,\frac{\partial g_{z\bar{z}}}{\partial z} + \hat{h}_{\bar{z}}\,\frac{\partial g_{z\bar{z}}}{\partial \bar{z}}]\,(dz\otimes d\bar{z} + d\bar{z}\otimes dz)$ therefore equations (1),(2), (4), (5) and (6) are automatically verified. On the other hand equation (3), since $\hat{h}_z$ and $\hat{h}_{\bar{z}}$ do not depend on t, is written as

$$\frac{\partial}{\partial \bar{z}}(F_{z\bar{z}}\hat{h}_{\bar{z}}) + \frac{\partial}{\partial z}(F_{z\bar{z}}\hat{h}_{z}) = 0$$

but $F_{z\bar{z}} = iBg_{z\bar{z}}$, thus the preceding equation is written as

$$\frac{\partial}{\partial \bar{z}}(g_{z\bar{z}}\hat{h}_{\bar{z}}) + \frac{\partial}{\partial z}(g_{z\bar{z}}\hat{h}_{z}) = 0$$

which is automatically verified since $X^L T_2 = 0$. ■

Corollary 3 *It is trivially checked that $D = \frac{\partial}{\partial t}$ is an infinitesimal symmetry of the Landau-Hall problem.*

Remark 3 In the examples analyzed by the authors the Lie algebra of infinitesimal symmetries of the Landau-Hall problem is finite-dimensional. However, in the Landau-Hall problem on the euclidean plane and the torus there are infinitesimal symmetries different from the ones given in the preceding corollaries. On the sphere and on the Poincaré upper half plane it is proved, by integrating the preceding equations, that every infinitesimal symmetry of the Landau-Hall problem is a linear combination of $\frac{\partial}{\partial t}$ and the infinitesimal isometries of T_2 [20, 21]. Finally on compact Riemann surfaces of genus $g > 1$ there are no infinitesimal isometries apart from $\frac{\partial}{\partial t}$.

2.3. The evolution equations of the classical Landau-Hall problem

In complex coordinates it is easy to see that the evolution equations of the Landau-Hall problem are given by

$$\ddot{z} + \dot{z}^2 \frac{\partial ln g_{z\bar{z}}}{\partial z} = -\frac{eF_{z\bar{z}}}{mg_{z\bar{z}}}\, \dot{z} = -i\omega_c\, \dot{z} \tag{7}$$

$$\ddot{\bar{z}} + \dot{\bar{z}}^2 \frac{\partial ln g_{z\bar{z}}}{\partial \bar{z}} = \frac{eF_{z\bar{z}}}{mg_{z\bar{z}}}\, \dot{\bar{z}} = i\omega_c\, \dot{\bar{z}} \tag{8}$$

since $F_{z\bar{z}} = iBg_{z\bar{z}}$.

Using the Frenet frame, one can give an intrinsic expression for these equations in the following manner: Let $D = \sigma_*(\frac{d}{dt})$ be the velocity vector field of the particle; then the evolution equations are

$$i_{mD^\nabla D} T_2 = -e\, i_D F_{Hall}$$

Let Φ be the tensor field of type (1, 1) determined by $T_2(D_1, \Phi(D_2)) = F_{Hall}(D_1, D_2)$, $\forall D_1, D_2$ tangent vector fields to Q, then the evolution equations can be expressed as

$$mD^\nabla D = -e\, \Phi(D)$$

It is important to remark that $\frac{\Phi}{B}$ defines the complex structure of the Riemann surface underlying Q.

Lemma 1 *It is verified that*

1. $\Phi^2 = -B^2 Id_{TQ}$
2. $T_2(\Phi(D_1), \Phi(D_2)) = B^2 T_2(D_1, D_2)$

Using the bidimensional Frenet formulae, one has

Theorem 1 *The geodesic curvature χ_g and the energy of the trajectory of a charged particle placed in a magnetic Hall field are constant and $\chi_g = -\frac{eB}{mv}$.*

PROOF. Let $D_{\sigma(t)} = v(t)T_{\sigma(t)}$ with $v^2 = T_2(D, D)$ and let $< T_{\sigma(t)}, N_{\sigma(t)} >$ be an orthonormal basis of $T_{\sigma(t)}Q$. The bidimensional Frenet formulae are

$$\begin{aligned} T^\nabla T &= \chi_g N \\ T^\nabla N &= -\chi_g T \end{aligned}$$

with χ_g being the geodesic curvature. Taking into account that $D^\nabla D = D(v)T + \chi_g v^2 N$, one has that $T_2(mD^\nabla D, T) = -eF_{Hall}(D, T) = 0$, which implies $D(v) = \frac{dv(t)}{dt} = 0$; that is v is constant (which in turns implies that the energy is conserved). Bearing in mind this fact, the evolution equation can be expressed as

$$T^\nabla T = -\frac{e}{mv}\Phi(T)$$

and from this it follows that $\chi_g^2 = ||T^\nabla T|| = \frac{e^2B^2}{m^2v^2} = \frac{\omega_c^2}{v^2}$ with $\omega_c = \frac{eB}{m}$ the cyclotron frequency.
On the other hand it is easy to check that

$$\begin{aligned} T_2(\Phi(D), \Phi(D)) &= 1 \\ T_2(D, \Phi(D)) &= 0 \end{aligned}$$

thus we can take $N = \frac{\Phi(T)}{B}$, and the basis $< T, \frac{\Phi(T)}{B} >$ is positively oriented with respect to the Riemannian orientation of Q since $\Omega_2(T, \frac{\Phi(T)}{B}) = F_{Hall}(T, \Phi(T)) = T_2(T, T) = 1$. Then $T^\nabla T = \frac{\chi_g}{B}\Phi(T)$ and comparing this with the expression found before one deduces that $\chi_g = -\frac{eB}{mv}$. ∎

3. The hyperbolic Landau-Hall problem

With the aim of a better visualization of some results that we will find later, our hyperbolic model will be the hyperboloid of $\mathbb{R}^3$

$$Q = \{(x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 - z^2 = -R^2 \;;\; z > 0\,\}$$

The metric tensor $T_2^{hyp} = dx^2 + dy^2 - dz^2$ defines a hyperbolic geometry on $\mathbb{R}^3$. It is well known that the restriction of T_2^{hyp} to the hyperboloid Q defines a Riemannian geometry on Q with constant negative curvature equal to $-\frac{1}{R^2}$.

If we consider now the hyperbolic stereographic projection of Q onto the XY plane

$$x_1 = \frac{Rx}{R+z} \qquad\qquad x_2 = \frac{Ry}{R+z}$$

it is verified that $x_1^2 + x_2^2 = \frac{-R^2(R-z)}{R+z} \leq R^2$, obtaining in this way the disk model D^2, and the metric is expressed as

$$T_2 = {T_2^{hyp}}_{|Q} = \frac{4R^4}{\left(R^2 - x_1^2 - x_2^2\right)^2}\,(dx_1^2 + dx_2^2) = \frac{2R^4}{\left(R^2 - z\bar{z}\right)^2}\,(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$$

with $z = x_1 + ix_2$. Therefore the magnetic Hall fields on Q are

$$F_{Hall} = \frac{4R^4 B}{\left(R^2 - x_1^2 - x_2^2\right)^2}\,dx_1 \wedge dx_2 = \frac{2iBR^4}{\left(R^2 - z\bar{z}\right)^2}\,dz \wedge d\bar{z}$$

thus the Hall potential can be taken as

$$A_{Hall} = \frac{2R^2 B}{(R^2 - x_1^2 - x_2^2)}\,(x_1 dx_2 - x_2 dx_1) = \frac{-iBR^2}{(R^2 - z\bar{z})}\,(z d\bar{z} - \bar{z} dz)$$

It is important to point out that this Hall potential, although global, is not invariant under the isometries of the metric T_2.

One has that the Landau-Hall Lagrangian in the hyperbolic case is globally defined

$$L_{Hall} = \frac{2mR^4}{\left(R^2 - z\bar{z}\right)^2}\,\dot{z}\dot{\bar{z}} + \frac{ieBR^2}{(R^2 - z\bar{z})}\,(z\dot{\bar{z}} - \bar{z}\dot{z})$$

and the corresponding Poincaré-Cartan form, which is also global, is given by

$$\begin{aligned}
\Theta_{Hall} \quad &= \quad -[(\frac{2mR^4\dot{\bar{z}}}{\left(R^2 - z\bar{z}\right)^2} + e\,\frac{iBR^2\bar{z}}{(R^2 - z\bar{z})})\,dz + (\frac{2mR^4\dot{z}}{\left(R^2 - z\bar{z}\right)^2} - e\,\frac{iBR^2 z}{(R^2 - z\bar{z})})\,d\bar{z}] + \\
&+ \quad \frac{2mR^4\dot{z}\dot{\bar{z}}}{\left(R^2 - z\bar{z}\right)^2}\,dt.
\end{aligned}$$

Theorem 2 *The set of infinitesimal symmetries of the hyperbolic Landau-Hall problem is a real Lie algebra of dimension 4 which is generated by the following vector fields*

$$\begin{aligned}
D_0 \quad &= \quad \frac{\partial}{\partial t} \\
D_1 \quad &= \quad \frac{x_1 x_2}{R}\,\frac{\partial}{\partial x_1} - \frac{(R^2 + x_1^2 - x_2^2)}{2R}\,\frac{\partial}{\partial x_2} = -i\frac{(R^2 + z^2)}{2R}\,\frac{\partial}{\partial z} + i\frac{(R^2 + \bar{z}^2)}{2R}\,\frac{\partial}{\partial \bar{z}} \\
D_2 \quad &= \quad \frac{(R^2 - x_1^2 + x_2^2)}{2R}\,\frac{\partial}{\partial x_1} - \frac{x_1 x_2}{R}\,\frac{\partial}{\partial x_2} = \frac{(R^2 - z^2)}{2R}\,\frac{\partial}{\partial z} + \frac{(R^2 - \bar{z}^2)}{2R}\,\frac{\partial}{\partial \bar{z}} \\
D_3 \quad &= \quad x_2\,\frac{\partial}{\partial x_1} - x_1\,\frac{\partial}{\partial x_2} = -iz\,\frac{\partial}{\partial z} + i\bar{z}\,\frac{\partial}{\partial \bar{z}}.
\end{aligned}$$

The Lie brackets of the infinitesimal symmetries $< D_0, D_1, D_2, D_3 >$ are

$$\begin{aligned}
[D_0, D_i] &= 0, \qquad \forall i = 1,2,3 \\
[D_1, D_2] &= -D_3, \\
[D_1, D_3] &= -D_2, \\
[D_2, D_3] &= D_1,
\end{aligned}$$

from where it follows that the vector fields $< D_1, D_2, D_3 >$ generate the Lie algebra $\mathfrak{so}(2,1)$. These vector fields are the restriction of the infinitesimal isometries of T_2^{hyp} to the hyperboloid Q, and their expression in Cartesian coordinates $\{x, y, z\}$, on Q are

$$\begin{aligned}
D_1 &= -z\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z} \\
D_2 &= z\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial z} \\
D_3 &= y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}
\end{aligned}$$

with the condition $x^2 + y^2 - z^2 = -R^2$ and $z > 0$.

Theorem 3 *The Noether invariants associated with these infinitesimal symmetries are*

$$\begin{aligned}
f_{D_0} &= \frac{2mR^4\dot{z}\dot{\bar{z}}}{(R^2 - z\bar{z})^2} + \lambda_0 \\
f_{D_1} &= -\frac{imR^3}{(R^2 - z\bar{z})^2}\left[(R^2+\bar{z}^2)\dot{z} - (R^2+z^2)\dot{\bar{z}}\right] + e\,\frac{BR^3(z+\bar{z})}{(R^2 - z\bar{z})} + \lambda_1 \\
f_{D_2} &= -\frac{mR^3}{(R^2 - z\bar{z})^2}\left[(R^2-\bar{z}^2)\dot{z} + (R^2-z^2)\dot{\bar{z}}\right] - e\,\frac{iBR^3(z-\bar{z})}{(R^2 - z\bar{z})} + \lambda_2 \\
f_{D_3} &= -\frac{2imR^4}{(R^2 - z\bar{z})^2}\left[\bar{z}\dot{z} - z\dot{\bar{z}}\right] + e\,\frac{2BR^4}{(R^2 - z\bar{z})} + \lambda_3
\end{aligned}$$

with λ_0, λ_1, λ_2 and λ_3 being arbitrary constants.

The preceding constants can be chosen in a way such that the following additional conditions are fulfilled

$$[j^1(D_1)]f_{D_2} = -f_{D_3} \tag{9}$$

$$[j^1(D_1)]f_{D_3} = -f_{D_2} \tag{10}$$

$$[j^1(D_2)]f_{D_3} = f_{D_1}. \tag{11}$$

It is sufficient to take $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -e\,BR^2$. Taking into account that the Poisson brackets are given by $\{f_{D_r}, f_{D_s}\} \equiv [j^1(D_r)]f_{D_s}$ we shall see that these conditions guarantee the equivariance of the moment map that will be defined in section 4).

We will also take $\lambda_0 = 0$. Thus, one defines the energy E and the hyperbolic angular momenta J_1, J_2

and J_3, as the preceding Noether invariants for these concrete values of $\{\lambda_i \ , \forall i = 0,1,2,3\}$; that is

$$\begin{aligned} E &= \frac{2mR^4\dot{z}\dot{\bar{z}}}{(R^2 - z\bar{z})^2} \\ J_1 &= -\frac{imR^3}{(R^2 - z\bar{z})^2}\left[(R^2+\bar{z}^2)\dot{z} - (R^2+z^2)\dot{\bar{z}}\right] + e\,\frac{BR^3(z+\bar{z})}{(R^2 - z\bar{z})} \\ J_2 &= -\frac{mR^3}{(R^2 - z\bar{z})^2}\left[(R^2-\bar{z}^2)\dot{z} + (R^2-z^2)\dot{\bar{z}}\right] - e\,\frac{iBR^3(z-\bar{z})}{(R^2 - z\bar{z})} \\ J_3 &= -\frac{2imR^4}{(R^2 - z\bar{z})^2}\left[\bar{z}\dot{z} - z\dot{\bar{z}}\right] + e\,\frac{2BR^4}{(R^2 - z\bar{z})} - e\,BR^2. \end{aligned}$$

In Cartesian coordinates on the hyperboloid these invariants acquire a more symmetrical look

$$\begin{aligned} E &= \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 - \dot{z}^2) \\ J_1 &= m(y\dot{z} - z\dot{y}) - eBR\,x \\ J_2 &= m(z\dot{x} - x\dot{z}) - eBR\,y \\ J_3 &= m(y\dot{x} - x\dot{y}) - eBR\,z \end{aligned}$$

with the conditions $x^2 + y^2 - z^2 = -R^2$, $z > 0$ and $x\dot{x} + y\dot{y} - z\dot{z} = 0$. It is clear that E is the kinetic energy of the particle and $\vec{J} \equiv (J_1, J_2, J_3)$ represents the hyperbolic angular momenta.

Proposition 4 *The following relation is verified between the energy and the hyperbolic angular momenta*

$$J_1^2 + J_2^2 - J_3^2 = 2mR^2E - e^2B^2R^4$$

Hyperbolic notation

1. Given two tangent vectors X_1 and X_2 at a point of the hyperbolic space $(\mathbb{R}^3, T_2^{hyp})$,such that in Cartesian coordinates they are expressed as $X_1 \equiv (a_1, a_2, a_3)$ and $X_2 \equiv (b_1, b_2, b_3)$, we shall write

$$X_1 * X_2 = T_2^{hyp}(X_1, X_2) = a_1b_1 + a_2b_2 - a_3b_3$$

2. The hyperbolic cross product $X_1 \times_{hyp} X_2$ of the vectors $X_1 \equiv (a_1, a_2, a_3)$ and $X_2 \equiv (b_1, b_2, b_3)$ is defined in the usual way to be

$$i_{X_1 \times_{hyp} X_2} T_2^{hyp} = \Omega_3^{hyp}(X_1, X_2, -)$$

where $\Omega_3^{hyp} = dx \wedge dy \wedge dz$ is the volume element on $\mathbb{R}^3$ associated with the hyperbolic metric T_2^{hyp}. Due to the antisymmetry of Ω_3^{hyp}, it is verified that $T_2^{hyp}(X_1, X_1 \times_{hyp} X_2) = T_2^{hyp}(X_2, X_1 \times_{hyp} X_2) = 0$. In Cartesian coordinates

$$X_1 \times_{hyp} X_2 = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, -a_1b_2 + a_2b_1).$$

Note that with the above notation if $\vec{r} \equiv (x, y, z)$, and $\vec{p} \equiv (m\dot{x}, m\dot{y}, m\dot{z})$, then on the hyperboloid Q it is verified that

$$\begin{aligned} \vec{J} &= \vec{r} \times_{hyp} \vec{p} - eBR\,\vec{r} \\ E &= \frac{1}{2mR^2}(\vec{J} * \vec{J} + e^2B^2R^4) \end{aligned}$$

It is easy to prove the last equality using the hyperbolic notation we have just introduced; it is enough to take into account that

$$\vec{r} \times_{hyp} \vec{p} \equiv (my\dot{z} - mz\dot{y}\,,\, mz\dot{x} - mx\dot{z}\,,\, my\dot{x} - mx\dot{y})$$

and that if we call $\vec{v} \equiv (\dot{x}, \dot{y}, \dot{z})$ the velocity vector of the particle one has that

$$(\vec{r} \times_{hyp} \vec{p}) * (\vec{r} \times_{hyp} \vec{p}) = m^2[-\vec{v} * \vec{v}\,(x^2 + y^2 - z^2) + (x\dot{x} + y\dot{y} - z\dot{z})^2]$$

on the hyperboloid Q, since it is verified that $x^2 + y^2 - z^2 = -R^2$, $z > 0$ and $x\dot{x} + y\dot{y} - z\dot{z} = 0$. Then

$$(\vec{r} \times_{hyp} \vec{p}) * (\vec{r} \times_{hyp} \vec{p}) = m^2 R^2\,(\vec{v} * \vec{v}) = 2mR^2 E$$

with $E = \frac{1}{2} m\,\vec{v} * \vec{v}$ being the energy of the particle. Therefore

$$\vec{J} * \vec{J} = (\vec{r} \times_{hyp} \vec{p}) * (\vec{r} \times_{hyp} \vec{p}) + e^2 B^2 R^2\,(\vec{r} * \vec{r}) = 2mR^2 E - e^2 B^2 R^4$$

since $(\vec{r} \times_{hyp} \vec{p}) * \vec{r} = 0$ and $\vec{r} * \vec{r} = -R^2$.

One can also use the constants of motion $\{J_1, J_2, J_3\}$ to integrate the evolution equations since our system is completely integrable.

Theorem 4 *The trajectory of a particle on $Q = \{(x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 - z^2 = -R^2\ ;\ z > 0\}$ with values $\{J_1 = \alpha_1, J_2 = \alpha_2, J_3 = \alpha_3\}$ for the constants of motion is given by the intersection of this hyperboloid with the plane of equation $\alpha_1 x + \alpha_2 y - \alpha_3 z = eBR^3$.*

PROOF. Bearing in mind that the hyperbolic angular momentum is
$\vec{J} = \vec{r} \times_{hyp} \vec{p} - eBR\,\vec{r}$ one has that if $\vec{r}\,(t)$ is the trajectory of the particle on Q then it is verified

$$\vec{J} * \vec{r}\,(t) = \vec{\alpha} * \vec{r}\,(t) = -eBR\,\vec{r}\,(t) * \vec{r}\,(t) = eBR^3$$

from whence it follows that the trajectory of the particle is contained in that plane. ■

However, we shall give a geometrical method, based on the Darboux frame, to analyze the trajectory that the particle follows on the hyperboloid and to prove that it is contained on a plane.

Theorem 5 *Let $D = vT$ be the velocity vector of the curve $\sigma(t)$ that describes the motion of the particle on the hyperboloid, then the trajectory of the particle is contained in the plane $\Omega_D * \sigma(t) = \frac{eB}{mvR}$ with $\Omega_D = \frac{1}{R} N - \frac{eB}{mv} N_Q = \frac{1}{R}\frac{\Phi(T)}{B} + \frac{eB}{mv} T \times_{hyp} \frac{\Phi(T)}{B}$ being the hyperbolic angular velocity vector which is constant on the trajectory.*

PROOF. Let $(\mathbb{R}^3, T_2^{hyp})$ be the hyperbolic space and let us denote by $\hat{\nabla}$ the Levi-Civita connection associated with the hyperbolic metric. Let N_Q be a normal vector to the hyperboloid Q with respect to the metric T_2^{hyp} and such that $T_2^{hyp}(N_Q, N_Q) = -1$. Let $D = vT$ be the velocity field of the curve $\sigma(t)$ that the particle follows on Q and let N be a vector field, with support on the curve, tangent to Q and orthogonal to T such that the vectors $\{T, N, N_Q\}$ are orthonormal at each point of the curve and such that

$$\begin{aligned} T_2^{hyp}(T, T) &= 1 \\ T_2^{hyp}(N, N) &= 1 \\ T_2^{hyp}(N_Q, N_Q) &= -1. \end{aligned}$$

These vectors are the so-called Darboux frame of the curve. It is verified that

$$\begin{aligned} T^{\hat{\nabla}} T &= \chi_g N + \chi_n N_Q \\ T^{\hat{\nabla}} N &= -\chi_g T + \tau_g N_Q \\ T^{\hat{\nabla}} N_Q &= \chi_n T + \tau_g N \end{aligned}$$

with $\chi_g = -\frac{eB}{cmv}$ being the geodesic curvature, τ_g the geodesic torsion and χ_n the normal curvature. In particular one can take $\{T, N = \frac{\Phi(T)}{B}, N_Q = -T \times_{hyp} \frac{\Phi(T)}{B}\}$. as a Darboux frame. It is not difficult to check that for every vector field X tangent to the curve,

$$T^{\hat{\nabla}} X = \Omega_D \times_{hyp} X \qquad \forall X$$

where Ω_D is the vector field with support on the curve that we shall call hyperbolic angular velocity, which is given by

$$\Omega_D = -\tau_g T + \chi_n N + \chi_g N_Q$$

and it is verified that $\Omega_D * \Omega_D = \tau_g^2 + \chi_n^2 - \chi_g^2$.

Let ∇ be the Levi-Civita connection associated with the first fundamental form T_2 (Riemannian) of Q. It is easy to see that the second fundamental form Φ_2 of Q is $\Phi_2 = \frac{1}{R} T_2$. Hence if D_1 and D_2 are vector fields tangent to Q, the Gauss formula is written as

$$D_1^{\hat{\nabla}} D_2 = D_1^{\nabla} D_2 + \Phi_2(D_1, D_2) N_Q = D_1^{\nabla} D_2 + \frac{1}{R} T_2(D_1, D_2) N_Q$$

therefore

$$\begin{aligned} \Omega_D \times_{hyp} T &= T^{\hat{\nabla}} T = T^{\nabla} T + \frac{1}{R} T_2(T,T) N_Q = \chi_g N + \frac{1}{R} N_Q \\ \Omega_D \times_{hyp} N &= T^{\hat{\nabla}} N = T^{\nabla} N + \frac{1}{R} T_2(T,N) N_Q = -\chi_g T \end{aligned}$$

thus

$$\chi_g = -\frac{eB}{mv} \quad ; \quad \chi_n = \frac{1}{R} \quad ; \quad \tau_g = 0$$

and therefore

$$\Omega_D = \frac{1}{R} N - \frac{eB}{mv} N_Q = \frac{1}{R} N - \frac{\omega_c}{v} N_Q$$

with $\Omega_D * \Omega_D = \frac{1}{R^2} - \frac{\omega_c^2}{v^2}$ from which we conclude that

$$v^2 = \frac{\omega_c^2 R^2}{[1 - R^2 (\Omega_D * \Omega_D)]}.$$

If $\sigma(t_0)$ is a fixed point of the curve, then $f(t) = (\sigma(t) - \sigma(t_0)) * \Omega_D$ verifies $\dot{f}(t) = D * \Omega_D + v\,(\sigma(t) - \sigma(t_0)) * T^{\nabla} \Omega_D = 0$ since $T * \Omega_D = 0$ and $T^{\nabla} \Omega_D = 0$ because Ω_D is constant on the trajectory. Therefore $f(t) = f(0) = 0$, that is

$$(\sigma(t) - \sigma(t_0)) * \Omega_D = 0.$$

This means that the trajectory of the particle is contained in the plane that passes trough the point $\sigma(t_0)$ and is perpendicular, with respect to the hyperbolic metric, to the vector Ω_D. Since the points of the curve verify $\sigma(t) = \frac{1}{R} N_Q$, then

$$\Omega_D * \sigma(t_0) = \Omega_D * \sigma(t) = \Omega_D * \frac{1}{R} N_Q = -\frac{\omega_c}{vR} N_Q * N_Q = \frac{\omega_c}{vR}.$$

Therefore $\Omega_D * \sigma(t) = \frac{\omega_c}{vR} = \frac{eB}{cmvR}$ as was to be shown. ■

Remark 4 If we compare this result with the preceding theorem $\vec{J} * \vec{r}(t) = \frac{eBR^3}{c}$ we see that the relationship between the hyperbolic angular velocity and the angular momentum is given by $\vec{J} = mvR^4\, \Omega_D$.

4. Symplectic reduction of the hyperbolic Landau-Hall problem

One can give a Hamiltonian formulation of the hyperbolic Landau-Hall problem by considering that the configuration space is the hyperboloid $Q = \{(x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 - z^2 = -R^2 \,;\, z > 0\}$. Let $\pi : T^*Q \to Q$ be the cotangent bundle of Q, then the phase space, as is well known, is the symplectic manifold $(T^*Q, \omega_2^{Hall} = \omega_2 - e\,\pi^*(F^{Hall}))$ with ω_2 being the canonical symplectic form of T^*Q and $F^{Hall} = B\,\Omega_2$ with $B \neq 0$.

Let $SO(2,1)^o$ be the connected component of the identity of $SO(2,1)$. Then $SO(2,1)^o \simeq Sl(2,\mathbb{R})/\mathbb{Z}_2$ acts transitively on Q (the isotropy group of each point of Q is isomorphic to $SO(2,\mathbb{R})$) and it induces a symplectic action on (T^*Q, ω_2^{Hall}), without fixed points, that preserves the Hamiltonian of the system, furthermore it admits a moment map $J : T^*Q \to \mathfrak{so}(2,1)^*$ which is equivariant with respect to the coadjoint action [1, 15, 22].

The fundamental vector fields of this action are nothing else but the restriction to $T^*Q \subset T^*R^3$ of the fundamental vector fields of the natural action of $SO(2,1)^o$ on T^*R^3. If we take a basis $< A_1, A_2, A_3 >$ of the Lie algebra $\mathfrak{so}(2,1)$ with $[A_1, A_2] = -A_3$, $[A_1, A_3] = -A_2$ and $[A_2, A_3] = A_1$, one has that the fundamental vector field associated with these vector fields are, in Cartesian coordinates on Q,

$$\begin{aligned} D_{J_1} = A_1^* &= -z\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z} - \dot{z}\,\frac{\partial}{\partial \dot{y}} - \dot{y}\,\frac{\partial}{\partial \dot{z}} \\ D_{J_2} = A_2^* &= z\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial z} + \dot{z}\,\frac{\partial}{\partial \dot{x}} + \dot{x}\,\frac{\partial}{\partial \dot{z}} \\ D_{J_3} = A_3^* &= y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y} + \dot{y}\,\frac{\partial}{\partial \dot{x}} - \dot{x}\,\frac{\partial}{\partial \dot{y}} \end{aligned}$$

with $x^2 + y^2 - z^2 = -R^2$, $z > 0$ and $x\dot{x} + y\dot{y} - z\dot{z} = 0$.

Let $\{\omega_1, \omega_2, \omega_3\} \in \mathfrak{so}(2,1)^*$ be the dual basis of $\{A_1, A_2, A_3\}$ and let us denote by $(\alpha_1, \alpha_2, \alpha_3)$ the coordinates of an element $\omega \in \mathfrak{so}(2,1)^*$ with respect to this basis. Then the moment map $J : T^*Q \to \mathfrak{so}(2,1)^*$ is defined by

$$\begin{aligned} J^*(\alpha_1) &= J_1 = m(y\dot{z} - z\dot{y}) - eBR\,x \\ J^*(\alpha_2) &= J_2 = m(z\dot{x} - x\dot{z}) - eBR\,y \\ J^*(\alpha_3) &= J_3 = m(y\dot{x} - x\dot{y}) - eBR\,z \end{aligned}$$

with $x^2 + y^2 - z^2 = -R^2$, $z > 0$ and $x\dot{x} + y\dot{y} - z\dot{z} = 0$. This expression can be written as $\vec{J} = \vec{r} \times_{hyp} \vec{p} - eBR\,\vec{r}$. The energy E is a positive function on T^*Q which is not independent of the conserved hyperbolic momenta, since as we have previously indicated

$$E = \frac{1}{2mR^2}\,(J_1^2 + J_2^2 - J_3^2 + e^2B^2R^4).$$

It is easy to check, thanks to equations 9,10 and 11 of section 3, that the Poisson brackets are

$$\begin{aligned} \{J_1, J_2\} = D_{J_1}(J_2) &= -J_3 \\ \{J_1, J_3\} = D_{J_1}(J_3) &= -J_2 \\ \{J_2, J_3\} = D_{J_2}(J_3) &= J_1. \end{aligned}$$

Using the hyperbolic stereographic projection (isothermal coordinates) or hyperbolic coordinates on Q, it is proved after a long but easy computation that $dJ_1 \wedge dJ_2 \wedge dJ_3 = 0$ if and only if $E = 0$, which implies that

Proposition 5 *The moment map $J : T^*Q \to \mathfrak{so}(2,1)^*$ is a regular projection on all points except for those that verify $\dot{x}^2 + \dot{y}^2 - \dot{z}^2 = 0$ (which correspond to the points of zero energy for which $J_1^2 + J_2^2 - J_3^2 = -e^2B^2R^4$).*

Theorem 6 *Given $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathfrak{so}(2,1)^*$ and $r_\alpha = \alpha_1^2 + \alpha_2^2 - \alpha_3^2$. If we suppose that $eB < 0$ (analogously if $eB > 0$) then we have that:*

1. *If $r_\alpha < -e^2B^2R^4$, then $J^{-1}(\alpha) = \emptyset$. (Note that otherwise the particle would have a negative energy, but this is impossible since E is positively defined on T^*Q).*

2. *If $r_\alpha = -e^2B^2R^4$, then $E = 0$.*

   2a) *If $\alpha_3 < 0$ then $J^{-1}(\alpha) = \{(\frac{\alpha_1}{eBR}, \frac{\alpha_2}{eBR}, \frac{\alpha_3}{eBR}, 0, 0, 0)\} \in T^*Q \subset T^*\mathbb{R}$ is a point on the image of the zero section of the bundle $\pi : T^*Q \to Q$; that is, the velocity is zero and the particle is fixed at a point of the hyperboloid Q.*

   2b) *If $\alpha_3 > 0$ then $J^{-1}(\alpha) = \emptyset$, since in that case $(\frac{\alpha_1}{eBR}, \frac{\alpha_2}{eBR}, \frac{\alpha_3}{eBR}) \notin Q$.*

3. *If $-e^2B^2R^4 < r_\alpha < 0$, then $0 < E < \frac{e^2B^2R^2}{2m}$.*

   3a) *If $\alpha_3 < 0$ then $J^{-1}(\alpha) = \emptyset$, since in this case the intersection of the hyperboloid Q with the plane $\alpha_1 x + \alpha_2 y - \alpha_3 z = eBR^3$ is empty.*

   3b) *If $\alpha_3 > 0$ then $J^{-1}(\alpha)$ is a one dimensional submanifold of T^*Q defined by the zeroes of the functions $\{J_1 - \alpha_1, J_2 - \alpha_2, J_3 - \alpha_3\}$ and furthermore $\pi(J^{-1}(\alpha))$ is the closed trajectory on Q given by the intersection of the hyperboloid Q with the plane $\alpha_1 x + \alpha_2 y - \alpha_3 z = eBR^3$.*

4. *If $r_\alpha = 0$, the energy $E = \frac{e^2B^2R^2}{2m}$.*

   4a) *If $\alpha = (0,0,0)$ then $J^{-1}(\alpha) = \emptyset$ since $\vec{J} = \vec{r} * \vec{p} - eBR\,\vec{r} = (0,0,0)$ implies that $\vec{r} * \vec{p} = eBR\,\vec{r}$ and since $\vec{r} * \vec{p}$ is orthogonal to $\vec{r}$, it turns out that $\vec{r} = (0,0,0)$ but $\vec{r} = (0,0,0) \notin Q$.*

   4b) *If $\alpha \neq (0,0,0)$ then $J^{-1}(\alpha)$ is a onedimensional submanifold of T^*Q defined by the zeroes of the functions $\{J_1 - \alpha_1, J_2 - \alpha_2, J_3 - \alpha_3\}$ and one has that $\pi(J^{-1}(\alpha))$ is the open trajectory on Q given by the intersection of the hyperboloid Q with the plane $\alpha_1 x + \alpha_2 y - \alpha_3 z = eBR^3$.*

5. *If $r_\alpha > 0$, then $E > \frac{e^2B^2R^2}{2m}$, with $\pi(J^{-1}(\alpha))$ being the open trajectory on Q given by the intersection of the hyperboloid Q with the plane $\alpha_1 x + \alpha_2 y - \alpha_3 z = eBR^3$.*

Proposition 6 *The distinct coadjoint orbits of the group $SO(2,1)^o$ are:*

1. $\mathcal{O}_0 = \{(0,0,0)\}$
2. $\mathcal{O}^{hyp}_{k^2,+} = \{(\alpha_1, \alpha_2, \alpha_3) \ / \ \alpha_1^2 + \alpha_2^2 - \alpha_3^2 = -k^2 \ , \ \alpha_3 > 0\}$ *for each* $k^2 \neq 0 \in \mathbb{R}^+$.
3. $\mathcal{O}^{hyp}_{k^2,-} = \{(\alpha_1, \alpha_2, \alpha_3) \ / \ \alpha_1^2 + \alpha_2^2 - \alpha_3^2 = -k^2 \ , \ \alpha_3 < 0\}$ *for each* $k^2 \neq 0 \in \mathbb{R}^+$.
4. $\mathcal{O}^{cone}_{+} = \{(\alpha_1, \alpha_2, \alpha_3) \neq (0,0,0) \ / \ \alpha_1^2 + \alpha_2^2 - \alpha_3^2 = 0 \ , \ \alpha_3 > 0\}$.
5. $\mathcal{O}^{cone}_{-} = \{(\alpha_1, \alpha_2, \alpha_3) \neq (0,0,0) \ / \ \alpha_1^2 + \alpha_2^2 - \alpha_3^2 = 0 \ , \ \alpha_3 < 0\}$.
6. $\mathcal{O}^{hip\,1\,leaf}_{k^2} = \{(\alpha_1, \alpha_2, \alpha_3) \ / \ \alpha_1^2 + \alpha_2^2 - \alpha_3^2 = k^2\}$ *for each* $k^2 \neq 0 \in \mathbb{R}^+$.

Proposition 7 *If G_α is the isotropy subgroup of a point $\alpha \in \mathfrak{so}(2,1)^*$ with respect to the action of $SO(2,1)^o$, then one has that*

1. *If $\alpha \in \mathcal{O}_0$, then $G_\alpha = SO(2,1)^o$.*
2. *If $\alpha \in \mathcal{O}^{hyp}_{k^2,+}$, then G_α is isomorphic to $SO(2)$.*

3. *If* $\alpha \in \mathcal{O}^{hyp}_{k^2,-}$*, then* G_α *is isomorphic to* $SO(2)$*.*

4. *If* $\alpha \in \mathcal{O}^{cone}_{+}$*, then* G_α *is isomorphic to* $E(1)$*.*

5. *If* $\alpha \in \mathcal{O}^{cone}_{-}$*, then* G_α *is isomorphic to* $E(1)$*.*

6. *If* $\alpha \in \mathcal{O}^{hyp\,1\,leaf}_{k^2}$*, then* G_α *is isomorphic to* $SO(1,1)^o$*.*

Theorem 7 *Let us denote by* $T^*_\epsilon Q$ *the submanifold of* T^*Q *of constant energy* $E = \epsilon$*, then :*

1. $J^{-1}(\mathcal{O}_0) = \emptyset$.

2. (a) $J^{-1}(\mathcal{O}^{hyp}_{k^2,+}) = \emptyset$ *if* $-k^2 < -e^2B^2R^4$.

   (b) $J^{-1}(\mathcal{O}^{hyp}_{k^2,+}) = T^*_0 Q$ *(zero section of the cotangent bundle of* Q*) if* $-k^2 = -e^2B^2R^4$.

   (c) $J^{-1}(\mathcal{O}^{hyp}_{k^2,+}) = T^*_\epsilon Q$ *with* $\epsilon = \frac{1}{2mR^2}\left(-k^2 + e^2B^2R^4\right)$ *if* $-e^2B^2R^4 < -k^2 < 0$.

3. $J^{-1}(\mathcal{O}^{hyp}_{k^2,-}) = \emptyset$, $\forall k^2 \neq 0$.

4. $J^{-1}(\mathcal{O}^{cone}_{+}) = T^*_\epsilon Q$ *with* $\epsilon = \frac{e^2B^2R^2}{2m}$.

5. $J^{-1}(\mathcal{O}^{cone}_{-}) = \emptyset$.

6. $J^{-1}(\mathcal{O}^{hip\,1\,leaf}_{k^2}) = T^*_\epsilon Q$ *with* $\epsilon = \frac{1}{2mR^2}\left(k^2 + e^2B^2R^4\right)$.

Corollary 4 *Let* $E = \epsilon$ *be the energy of the particle, then:*
1) If $\epsilon = 0$*, then the trajectory on* Q *is a point.*
2) If $0 < \epsilon < \frac{e^2B^2R^2}{2m}$*, then the trajectory on* Q *is closed (periodic).*
3) If $\epsilon \geq \frac{e^2B^2R^2}{2m}$*, then the trajectory on* Q *is open.*

Remark 5 It may be expected that when one quantizes this problem, the states of energy $0 < \epsilon < \frac{e^2B^2R^2}{2m}$ will represent localized quantum states and therefore will contribute to the Hall resistivity, whereas for $\epsilon \geq \frac{e^2B^2R^2}{2m}$, where ergodicity phenomena appear, these states will represent extended states which will contribute to the Hall conductivity.

5. The manifold of orbits of constant energy on the hyperbolic Landau-Hall problem

As we have indicated, the submanifold of constant energy $E = 0$ is identified with the image of the zero section of the bundle T^*Q, whereas in the other cases $E = \epsilon \neq 0$, it is a tridimensional submanifold of $T^*_\epsilon Q$ of T^*Q, which is defined by the zeroes of the function $\frac{1}{2mR^2}\left(J_1^2 + J_2^2 - J_3^2 + e^2B^2R^4\right)$. Let us denote by $\mathcal{I}(\epsilon)$ the ideal of $C^\infty(T^*Q)$ generated by this function.

If $\epsilon \neq 0$, let ${w_2^{Hall}}_{|T^*_\epsilon Q}$ be the restriction to $T^*_\epsilon Q$ of the 2-form w_2^{Hall}.

Proposition 8 *If* $\epsilon \neq 0$, $T^*_\epsilon Q$ *is a coisotropic submanifold of* (T^*Q, w_2^{Hall}) *and the radical of* ${w_2^{Hall}}_{|T^*_\epsilon Q}$ *is the restriction to* $T^*_\epsilon Q$ *of the Hamiltonian vector fields associated to the functions* $\mathcal{I}(\epsilon)$*.*

Therefore it is easy to check that the vector field $D = J_1\, D_{J_1} + J_2\, D_{J_2} - J_3\, D_{J_3}$ is tangent to the submanifold of constant energy $E = \epsilon \neq 0$ and their restrictions to every $T^*_\epsilon Q$ generate the radical or null kernel $rad\, {w_2^{Hall}}_{|T^*_\epsilon Q}$ of ${w_2^{Hall}}_{|T^*_\epsilon Q}$.

Proposition 9 *The trajectory that passes through the point*

$$(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0) \in T^*Q$$

with $J(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0) = (\alpha_1, \alpha_2, \alpha_3)$ *and* $\epsilon = \frac{1}{2mR^2}\left(\alpha_1^2 + \alpha_2^2 - \alpha_3^2 + e^2B^2R^4\right) \neq 0$*, is the integral curve of the vector field* $D = J_1\, D_{J_1} + J_2\, D_{J_2} - J_3\, D_{J_3}$ *which for* $t = 0$ *passes through that point.*

Since the moment map J intersects the coadjoint orbits corresponding to non-zero energy cleanly, as a consequence of the Kazhdan-Kostant-Sternberg theorem [15], we have that

Theorem 8 *If* $\epsilon \neq 0$*, the integral curve of the radical that passes through the point* $T^*_\epsilon Q$ *is the orbit under the connected component of the isotropy group of the image by* J *of that point.*

With the above notations and results, bearing in mind that $SO(2,1)^o$ acts transitively on $T^*_\epsilon Q$, if we fix a point of this submanifold we have that

Proposition 10 *The morphisms*

1. $J : T^*_\epsilon Q \to \mathcal{O}^{hyp}_{k^2,+}$ *for* $\epsilon = \frac{1}{2mR^2}\left(-k^2 + e^2B^2R^4\right)$ *with* $-e^2B^2R^4 < -k^2 < 0$.
2. $J : T^*_\epsilon Q \to \mathcal{O}^{cone}_{+}$ *with* $\epsilon = \frac{e^2B^2R^2}{2m}$.
3. $J : T^*_\epsilon Q \to \mathcal{O}^{hyp\,1\,leaf}_{k^2}$ *for* $\epsilon = \frac{1}{2mR^2}\left(k^2 + e^2B^2R^4\right)$ *with* $e^2B^2R^4 < k^2$

are principal fiber bundles whose structure groups are $SO(2)$, $E(1)$ *and* $SO(1,1)^o$ *respectively.*

The manifold of orbits of constant energy ϵ is nothing else but the reduced phase space $\frac{T^*_\epsilon Q}{rad\, w_2^{Hall}{}_{|T^*_\epsilon Q}}$. Taking into account theorem 26.6 in [15] and that the isotropy subgroups with respect to the coadjoint action are connected for every $\alpha \in \mathfrak{so}(2,1)^*$ one has that

Theorem 9 *The moment map* J *induces diffeomorphisms, that will be denoted by* $\bar{J}$

1. $\bar{J} : \frac{T^*_\epsilon Q}{rad\, w_2^{Hall}{}_{|T^*_\epsilon Q}} \to \mathcal{O}^{hyp}_{k^2,+}$ *if* $-e^2B^2R^4 < -k^2 < 0$.
2. $\bar{J} : \frac{T^*_\epsilon Q}{rad\, w_2^{Hall}{}_{|T^*_\epsilon Q}} \to \mathcal{O}^{cone}_{+}$ *if* $k^2 = 0$.
3. $\bar{J} : \frac{T^*_\epsilon Q}{rad\, w_2^{Hall}{}_{|T^*_\epsilon Q}} \to \mathcal{O}^{hyp\,1\,leaf}_{k^2}$ *if* $k^2 > 0$).

with $\epsilon = \frac{1}{2mR^2}\left(-k^2 + e^2B^2R^4\right)$*. In this sense, every manifold of orbits of constant energy* $E = \epsilon \neq 0$*, is identified, via the moment map* J*, with the corresponding coadjoint orbit.*

Let us analyze the manifold of orbits of constant energy $\epsilon < \frac{e^2B^2R^2}{2m}$ for which the trajectories are closed

Theorem 10 *If* $\epsilon < \frac{e^2B^2R^2}{2m}$*, then there exists a unique symplectic structure* w^{Hall}_{red} *on* $\mathcal{O}^{hyp}_{k^2,+}$ *such that* $\bar{J}^* w^{Hall}_{red} = w_2^{Hall}{}_{|T^*_\epsilon Q}$.

PROOF. In hyperbolic coordinates

$$\begin{aligned} \alpha_1 &= k\ sh\,\zeta_1\ cos\,\zeta_2 \\ \alpha_2 &= k\ sh\,\zeta_1\ sin\,\zeta_2 \\ \alpha_3 &= k\ ch\,\zeta_1 \end{aligned}$$

on the coadjoint orbit $\mathcal{O}^{hyp}_{k^2,+}$, its Kirillov-Kostant symplectic 2-form is given by $w^{hyp}_{k^2,+} = k^2 \;\; sh\zeta_1 \;\; d\zeta_1 \wedge d\zeta_2$. One has then that $w^{Hall}_{red} = \lambda\, w^{hyp}_{k^2,+} = \lambda \; k^2 \;\; sh\zeta_1 \;\; d\zeta_1 \wedge d\zeta_2$. It is sufficient to calculate the value of λ such that the required condition is fulfilled. For doing this note that since on $\mathcal{O}^{hyp}_{k^2,+}$, it is verified that

$$\begin{aligned} \zeta_1 &= arc\, ch(\frac{J_3}{k}) \\ \zeta_2 &= arc\, tg(\frac{J_2}{J_1}) \end{aligned}$$

One has that

$$\begin{aligned} \bar{J}^* w^{Hall}_{red} &= \bar{J}^*(\lambda \; k^2 \;\; sh\zeta_1 \;\; d\zeta_1 \wedge d\zeta_2) = \bar{J}^*(\lambda \; k^2 \;\; d(ch\zeta_1) \wedge d\zeta_2) = \\ &= \lambda \; k^2 \;\; d(\frac{J_3}{k}) \wedge d(arctg\frac{J_2}{J_1}) = \lambda \; k \;\; d(\frac{J_3}{k}) \wedge \frac{J_1 dJ_2 - J_2 dJ_1}{J_1^2 + J_2^2} = \\ &= \lambda \; k \;\; \frac{dJ_1 \wedge dJ_2}{J_3} \end{aligned}$$

since, as $J_1^2 + J_2^2 - J_3^2 = -k^2$, is $dJ_3 = \frac{J_1 \, dJ_1 + J_2 \, dJ_2}{J_3}$.

But $w_2^{Hall}{}_{|T^*_\epsilon Q} = \frac{dJ_1 \wedge dJ_2}{J_3}$ and comparing it turns out that $\lambda = \frac{1}{k}$ and since $k^2 = 2mR^2\epsilon - \frac{e^2 B^2 R^4}{c^2}$,

$$w^{Hall}_{red} = \left[2mR^2\epsilon - e^2B^2R^4\right]^{\frac{1}{2}} \;\; sh\zeta_1 \;\; d\zeta_1 \wedge d\zeta_2$$

as was to be shown. ■

Remark 6 The application of geometric quantization to the manifold of constant energy $0 < \epsilon < \frac{e^2 B^2 R^2}{2m}$ leads to a quantization of energy that will be developed and compared to the results given in the physics literature elsewhere.

Acknowledgement. Each author was partially supported by a C.I.C.Y.T research project PB-96-1305 and by a Junta de Castilla y León research project SA27-98.

References

[1] Abraham, R. Marsden, J. (1978) *Foundations of mechanics*, second edition. Benjamin-Cummings.

[2] Alimohammadi, M., Shafei Deh Abad, A. (1996) Quantum group symmetry of the quantum Hall effect on non-flat surfaces. *J. Phys. A: Math. Gen.* **29**, 559-563.

[3] Alimohammadi, M., Mosheni Sadjadi, S. (1996) Laughlin states on the Poincaré half-plane and their quantum group symmetry. *J. Phys. A: Math. Gen.***29**, 5551-5557.

[4] Antoine, M., Comtet, A., Ouvry, S. (1990) Scattering on a hyperbolic torus in a constant magnetic field. *J. Phys. A* **23**, 3699-3710.

[5] Asorey, M. (1993) Topological phases of quantum theories. Chern-Simons theory, in *Infinite-dimensional geometry in physics (Karpacz, 1992). J. Geom. Phys.* **11**, 63–94.

[6] Avron, J. E., Klein, M., Pnueli, A., Sadun, L. (1992) Hall conductance and adiabatic charge transport of leaky tori. *Phys. Rev. Lett.* **69**, 128-131.

[7] Bellissard, J., van Elst, A., Schulz-Baldes, H. (1994) The noncommutative geometry of the quantum Hall effect. *J. Math. Phys.* **35**, 5373-5451.

[8] Bolte, J., Steiner, F. (1991) Flux quantization and quantum mechanics on Riemann surfaces in an external magnetic field. *J. Phys. A* **24**, 3817-3823.

[9] Comtet, A. (1987) On the Landau levels on the hyperbolic plane. *Ann. Physics* **173**, 185-209.

[10] Comtet, A., Georgeot, B., Ouvry, S. (1993) Trace formula for Riemann surfaces with magnetic field. *Phys. Rev. Lett.* **71**, 3786-3789.

[11] Dunne, G. V. (1992) Hilbert space for charged particles in perpendicular magnetic fields. *Ann. Physics* **215**, 233-263.

[12] Duval, C., Horvathy, P. (1982) Particles with internal structure: the geometry of classical motions and conservation laws. *Ann. Physics* **142**, 10-33.

[13] García Pérez, P. L. (1974) The Poincaré-Cartan invariant in the calculus of variations. *Proc. Symp. Mathematica* **14**, 219-246.

[14] Grosche, C. (1990) Path integration on the hyperbolic plane with a magnetic field. *Ann. Physics* **201**, 258-284.

[15] Guillemin, V., Sternberg, S. (1990) *Symplectic techniques in physics*. Cambridge University Press.

[16] Haldane, F. D. M., Rezayi, E. H. (1985) Periodic Laughlin-Jastrow wave functions for the fractional quantized Hall effect. *Phys. Rev. B* **31**, 2529-2531.

[17] Iengo, R., Li, D. (1994) Quantum mechanics and quantum Hall effect on Riemann surfaces. *Nuclear Phys. B* **413**, 735-753.

[18] Jackiw, R., Manton, N. S. (1980) Symmetries and conservation laws in gauge theories. *Ann. Physics* **127**, 257-273.

[19] Klitzing, K.V., Dorda, G., Pepper, M. (1980) New method for high accuracy determination of the fine structure constant based on the quantized Hall resistance. *Phys. Rev. Lett.* **45**, 494-497.

[20] Lopez Almorox, A., Tejero Prieto, C. (1996) Infinitesimal symmetries of Hall effect on Riemann surfaces, in *Proc. of the 6-th International Conference Differential Geometry and applications, Brno 1995, Czech Republic*, Eds: J. Janyska, I. Kolář and J. Slovak, Masaryk University, Brno, 503-512.

[21] Lopez Almorox, A., Tejero Prieto, C. (1996) BRST cohomology of the Hall effect on the sphere, in *Proc. of the 6-th International Conference Differential Geometry and applications, Brno 1995, Czech Republic*, Eds: J. Janyska, I. Kolář and J. Slovak, Masaryk University, Brno, 513-524.

[22] Marsden, J. E., Ratiu, T.S. (1994) *Introduction to Mechanics and Symmetry,* Springer-Verlag.

[23] Pnueli, A. (1995) Magnetocharge-transport from twisting arguments. *J. Phys. A* **28**, 2015-2024.

[24] Prange, R., Girvin, S. (1990), *The Quantum Hall Effect,* Springer-Verlag.

[25] Thouless, D. J., Kohmoto, M., Nightingale, M. P., den Nijs, M. (1982) Quantized Hall conductance in a two-dimensional periodic potential. *Phys. Rev. Letters* **49**, 405-408.

[26] Thouless, D.J. (1994) Topological interpretations of quantum Hall conductance. Topology and physics. *J. Math. Phys.* **35**, 5362-5372.

[27] Wu, Y-S. (1990) Topological aspects of the quantum Hall effect, in Physics, Geometry and Topology (Banff 1989), *Nato Adv. Sci. Ser. B Phys.* **238**, Plenum, 461-491.

A. López Almorox
Departamento de Matemáticas
Universidad de Salamanca
Pl. de la Merced 1-4 E-37008 Salamanca
Spain
alm@usal.es

C. Tejero Prieto
Departamento de Matemáticas
Universidad de Salamanca
Pl. de la Merced 1-4 E-37008 Salamanca.
Spain
carlost@usal.es