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Probabilistic models for vortex filaments based on fractional Brownian motion

D. Nualart, C. Rovira and S. Tindel

Abstract. We introduce a vortex structure based on a three dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. The purpose of this note is to present the following result: Under a suitable integrability condition on the measure ρ which controls the spread of the vorticity around the filaments, the kynetic energy of the configuration is well defined and it has moments of all orders.

Modelos probabilistas para filamentos de vorticidad basados en el movimiento browniano fraccionario

Resumen. Se introduce una estructura de vorticidad basada en el movimiento browniano fraccionario con parámetro de Hurst $H > \frac{1}{2}$. El objeto de esta nota es presentar el siguiente resultado: Bajo una condición de integrabilidad adecuada sobre la medida ρ que controla la concentración de la vorticidad a lo largo de los filamentos, la energía cinética de la configuración està bien definida y tiene momentos de todos los órdenes.

1. Vortex filaments based on fractional Brownian motion

The observations of three-dimensional turbulent fluids in a number of experiments indicate that the vorticity field of the fluid is concentrated along thin structures called vortex filaments. In his book Chorin [3] suggests probabilistic descriptions of vortex filaments by trajectories of self-avoiding walks on a lattice. Flandoli introduced in [5] a model of vortex filaments based on a three-dimensional Brownian motion. A basic problem in these models is the computation of the kynetic energy of a given configuration.

Denote by $u(x)$ the velocity field of the fluid at point $x \in \mathbb{R}^3$, and let $\xi = \mathrm{curl} u$ be the associated vorticity field. The kynetic energy of the fluid will be

$$\mathbb{H} = \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^2 dx = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\xi(x)\cdot\xi(y)}{|x-y|}\, dxdy. \tag{1}$$

If the vorticity field is concentrated along a curve $\gamma = \{\gamma(t), 0 \le t \le T\}$ this expression is divergent even if the curve γ is smooth. For this reason, following Flandoli [5], we will assume that the vorticity field is concentrated along a thin tube centered in a curve γ. Moreover, we will choose a random model and consider this curve as the trajectory of a stochastic process.

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Let us describe in detail our random model for vorticity filaments. Suppose that $B = \{B_t, t \in [0,T]\}$ is a three-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$. This means that B is a zero mean Gaussian stochastic process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the covariance function

$$\mathbb{E}(B_t^i B_s^j) = \frac{1}{2}\delta_{i,j}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{2}$$

In particular, this means that the components of B are independent one-dimensional fractional Brownian motions. We will assume that the vorticity field is concentrated along a a trajectory of B. This can be formally expressed as

$$\xi(x) = \Gamma \int_{\mathbb{R}^3} \left(\int_0^T \delta(x - y - B_s)\dot{B}_s ds \right) \rho(dy), \tag{3}$$

where Γ is a parameter called the circuitation, and ρ is a probability measure on $\mathbb{R}^3$ with compact support. Substituting (3) into (1) we derive the following formal expression for the kynetic energy:

$$\mathbb{H} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbb{H}_{xy}\, \rho(dx)\rho(dy), \tag{4}$$

where the so-called interaction energy $\mathbb{H}_{xy}$ is given by

$$\mathbb{H}_{xy} = \frac{\Gamma^2}{8\pi}\sum_{i=1}^{3}\int_0^T\int_0^T \frac{1}{|x + B_t - y - B_s|} dB_s^i dB_t^i. \tag{5}$$

The purpose of this paper is to give a rigorous meaning to the expressions (4) and (5) and to show that $\mathbb{H}$ is a well-defined nonnegative random variable with moments of all orders.

Notice first that (2) implies that $\mathbb{E}|B_t - B_s|^2 = 3|t-s|^{2H}$. As a consequence, by Kolmogorov's continuity criterion, the trajectories $t \to B_t(\omega)$ of the fBm are Hölder continuous of order $H - \varepsilon$, for all $\varepsilon > 0$. This means that the Hurst parameter H controls the regularity of the paths. Taking into account the results by Young [10], for any Hölder continuous function f of order strictly greater than $1 - H$, the Stieltjes integral $\int_0^T f(t) dB_t(\omega)$ exists.

For any integer $n \geq 1$ we denote by σ_n the convolution of $|x|^{-1}$ with a three-dimensional Gaussian kernel of variance $1/n$. By the above remark, the Stieltjes integral $\int_0^T \sigma_n(x + B_t - y - B_s)\, dB_s^i$ exists for any trajectory of the fBm, and as a function of t is again Hölder continuous of order $H - \varepsilon$, for all $\varepsilon > 0$. Hence, the smoothed interaction energy

$$\mathbb{H}_{xy}^n = \frac{\Gamma^2}{8\pi}\sum_{i=1}^{3}\int_0^T \left(\int_0^T \sigma_n(x + B_t - y - B_s)\, dB_s^i \right) dB_t^i, \tag{6}$$

is well defined.

The main result of this paper is the following:

Theorem 1 *Suppose that the measure ρ satisfies the integrability condition*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x-y|^{1-\frac{1}{H}} \rho(dx)\rho(dy) < \infty. \tag{7}$$

Let $\mathbb{H}_{xy}^n$ be the smoothed interaction energy defined by (6). Then

$$\mathbb{H}^n = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbb{H}_{xy}^n \rho(dx)\rho(dy).$$

converges, for all $k \geq 1$, in $L^k(\Omega)$ to a random variable $\mathbb{H} \geq 0$ that we call the energy associated with the vorticity field (3).

If $H = \frac{1}{2}$, the process B is a classical three-dimensional Brownian motion. In this case condition (7) would be $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x-y|^{-1} \rho(dx)\rho(dy) < \infty$, which is the assumption made by Flandoli [5] and Flandoli and Gubinelli [6]. In this last paper, using Fourier approach and Itô's stochastic calculus, the authors show that $\mathbb{E}e^{-\beta \mathbb{H}} < \infty$ for sufficiently small negative β.

The proof of Theorem 1 is based on the stochastic calculus of variations (Malliavin Calculus) with respect to the fBm. We present the main notation and results of this calculus in the next section.

2. Stochastic calculus for the fractional Brownian motion

Suppose that $B = \{B_t, t \in [0,T]\}$ is a one-dimensional fractional Brownian motion (fBm) of Hurst parameter $H \in (\frac{1}{2}, 1)$. We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = \mathbb{E}(B_t B_s)$. It is easy to show that

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr$$

for all φ and ψ in $\mathcal{E}$. The mapping $B : \mathbf{1}_{[0,t]} \longrightarrow B_t$ can be extended to an isometry between $\mathcal{H}$ and the closed linear span of the random variables $\{B_s, 0 \leq s \leq T\}$. The space $L^{1/H}([0,T])$ is continuously embedded in $\mathcal{H}$ (see [8]).

We can construct a stochastic calculus of variations with respect to the Gaussian process B following the general approach introduced, for instance, in Nualart [9]. The derivative operator D is defined on smooth and cylindrical random variables of the form $F = f(B(\phi_1), \ldots, B(\phi_n))$ by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (B(\phi_1), \ldots, B(\phi_n)) \phi_i,$$

where $\phi_i \in \mathcal{H}$. Notice that DF is an $\mathcal{H}$-valued random variable. The divergence operator δ is the adjoint of the derivative D, defined by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u \rangle_{\mathcal{H}}). \tag{8}$$

It holds that the set of processes u such that

$$\mathbb{E}\left(\left| \int_0^T |u_t|^{\frac{1}{H}} dt + \int_0^T \int_0^T |D_s u_t|^{\frac{1}{H}} ds dt \right|^{2H} \right) < \infty \tag{9}$$

is included in the domain of the divergence operator in L^2.

In the case of the Brownian motion, the divergence operator is an extension of the Itô stochastic integral, and it coincides with an extension of the stochastic integral for anticipating processes introduced by Skorohod (see [9]). The following result proved in [2] provides also an interpretation of the divergence operator as a stochastic integral. We will make use of the notation $\delta(u) = \int_0^T u_t \delta B_t$.

Proposition 1 *Let u be a process such that (9) holds andand $\int_0^T \int_0^T |D_s u_t| \, |t-s|^{2H-2} ds dt < \infty$ a.s. Then the pathwise integral $\int_0^T u_t dB_t$, defined as the limit in probability of $(2\varepsilon)^{-1} \int_0^T u_s (B_{s+\varepsilon} - B_{s-\varepsilon}) ds$, exists and*

$$\int_0^T u_s dB_s = \int_0^T u_s \delta B_s + \alpha_H \int_0^T \int_0^T D_s u_t \, |t-s|^{2H-2} ds dt, \tag{10}$$

where $\alpha_H = H(2H-1)$.

In particular, if we apply (10) to the process $u_t = f(B_t)$, where f is a C^1-function with polynomial growth, we obtain

$$\int_0^T f(B_t)dB_t = \int_0^T f(B_t)\delta B_t + H\int_0^T f'(B_t)t^{2H-1}dt. \tag{11}$$

As a consequence, from the change of variables formula for the Stieltjes integral

$$f(B_T) = f(0) + \int_0^T f'(B_t)dB_t$$

we deduce the following Itô's formula for the fBm in terms of the divergence operator (see also [1], [4], [7]):

$$f(B_T) = f(0) + \int_0^T f'(B_t)\delta B_t + H\int_0^T f''(B_t)t^{2H-1}dt. \tag{12}$$

The previous results have the corresponding natural generalization to the case of a multidimensional fractional Brownian motion.

3. Proof of the main result

In this section we sketch the proof of Theorem 1 for $k = 1$. The main steps of the proof of this result are:

Step 1 Using that the Fourier transform of $|z|^{-1}$ is $(2\pi)^3|\xi|^{-2}$, we get

$$\sigma_n(x) = \int_{\mathbb{R}^3} |\xi|^{-2} e^{i\langle \xi, x\rangle - |\xi|^2/2n}\, d\xi.$$

Substituting this expression in (6), we obtain the following formula for the smoothed kynetic energy

$$\mathbb{H}^n = \frac{\Gamma^2}{8\pi}\int_{\mathbb{R}^3} \|Y_\xi\|_{\mathbb{C}}^2\, |\xi|^{-2}\, |\widehat{\rho}(\xi)|^2\, e^{-|\xi|^2/2n} d\xi, \tag{13}$$

where $\widehat{\rho}(\xi)$ is the Fourier transform of the measure ρ,

$$Y_\xi = \int_0^T e^{i\langle \xi, B_t\rangle} dB_t$$

and $\|Y_\xi\|_{\mathbb{C}}^2 = \sum_{i=1}^3 Y_\xi^i \overline{Y_\xi}^i$.

Step 2 From Fourier analysis we can write

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x-y|^{1-\frac{1}{H}} \rho(dx)\rho(dy) = C_H \int_{\mathbb{R}^3} |\widehat{\rho}(\xi)|^2\, |\xi|^{\frac{1}{H}-4} d\xi < \infty. \tag{14}$$

Then, taking into account (13) and (14), in order to show the convergence in $L^1(\Omega)$ of $\mathbb{H}^n$ to a random variable $\mathbb{H} \geq 0$ it suffices to check that

$$E\,\|Y_\xi\|_{\mathbb{C}}^2 \leq C|\xi|^{\frac{1}{H}-2}, \tag{15}$$

for some constant $C > 0$.

Step 3 We will present the main arguments for the proof of the estimate (15). Relation (11) applied to the process $u_t = e^{i\langle \xi, B_t\rangle}$ allows us to decompose the pathwise integral $Y_\xi = \int_0^T e^{i\langle \xi, B_t\rangle} dB_t$ into the sum of a divergence plus a trace term:

$$Y_\xi = \int_0^T e^{i\langle \xi, B_t\rangle}\delta B_t + H\int_0^T i\xi e^{i\langle \xi, B_t\rangle} t^{2H-1}dt. \tag{16}$$

On the other hand, applying the three-dimensional version of Itô's formula (12) to the exponential function $e^{i\langle \xi, B_T}$ we obtain

$$e^{i\langle \xi, B_T\rangle} = 1 + \sum_{j=1}^{3} \int_0^T i\xi_j e^{i\langle \xi, B_t\rangle} \delta B_t^j - H \int_0^T t^{2H-1} |\xi|^2 e^{i\langle \xi, B_t\rangle} dt. \tag{17}$$

Multiplying both members of (17) by $i\xi|\xi|^{-2}$ and adding the result fo (16) yields

$$Y_\xi = p_\xi \left(\int_0^T e^{i\langle \xi, B_t\rangle} \delta B_t \right) - \frac{i\xi}{|\xi|^2} \left(e^{i\langle \xi, B_T\rangle} - 1 \right) := Y_\xi^{(1)} + Y_\xi^{(2)},$$

where $p_\xi(v) = v - \frac{\xi}{|\xi|^2} \langle \xi, v\rangle$ is the orthogonal projection of v on $\langle \xi \rangle^\perp$. It suffices to derive the estimate (15) for the term $Y_\xi^{(1)}$. Using the duality relationship (8) we can write for each $j = 1, 2, 3$

$$\mathbb{E}\left(Y_\xi^{(1),j} \overline{Y}_\xi^{(1),j} \right) = \mathbb{E} \left\langle e^{i\langle \xi, B_\cdot\rangle}, p_\xi^j D_\cdot \left(p_\xi^j \int_0^T e^{-i\langle \xi, B_t\rangle} \delta B_t \right) \right\rangle_{\mathcal{H}}. \tag{18}$$

The commutation relation $\langle D(\delta(u)), h\rangle_{\mathcal{H}} = \langle u, h\rangle_{\mathcal{H}} + \delta(\langle Du, h\rangle_{\mathcal{H}})$ implies

$$\begin{aligned} p_\xi^j D_r \left(p_\xi^j \int_0^T e^{-i\langle \xi, B_t\rangle} \delta B_t \right) &= e^{-i\langle \xi, B_r\rangle} \left(1 - \frac{(\xi^j)^2}{|\xi|^2} \right) + p_\xi^j (-i\xi) p_\xi^j \left(\int_0^T \mathbf{1}_{[0,t]}(r) e^{-i\langle \xi, B_t\rangle} \delta B_t \right) \\ &= e^{-i\langle \xi, B_r\rangle} \left(1 - \frac{(\xi^j)^2}{|\xi|^2} \right). \end{aligned}$$

This means, the term involving derivatives in the expectation (18) vanishes. This cancellation is similar to what happens in the computation of tha variance of the divergence of an adapted process, in the case of the Brownian motion.

As a consequence, we obtain

$$\begin{aligned} E\|Y_\xi\|_{\mathbb{C}}^2 &= 2\alpha_H \int_0^T \int_0^T \mathbb{E} e^{i\langle \xi, B_s - B_r\rangle} |s-r|^{2H-2} \, ds dr \\ &= 2\alpha_H \int_0^T \int_0^T e^{-\frac{|s-r|^{2H}}{2}|\xi|^2} \, |s-r|^{2H-2} \, ds dr, \end{aligned}$$

which leads to the estimate (15).

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D. Nualart and C. Rovira
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585, 08007 Barcelona
Spain
nualart@mat.ub.es, rovira@mat.ub.es

S. Tindel
Département de Mathématiques
nstitut Galilée - Université Paris 13
99 Avenue J. B. Clément, 93430 Villetaneuse
France
tindel@math.univ-paris13.fr