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Multivariate Probability Integral Transformation: Application to maximum likelihood estimation

A. Chakak and L. Imlahi

Abstract. Let (X_1, X_2) be a continuous random vector with a cdf F. The probability integral transformation (pit) is the univariate random variable $P_2 = F(X_1, X_2)$. The expression of its cdf and a simulation algorithm in terms of the quantile function given by Chakak et al [2000], when the distribution is absolutely continuous, are extended for distributions that may present singularity. Maximum likelihood estimation of the dependence parameter based on the pit is investigated by simulation. It is shown to perform well for singular families of distributions. Extension to higher dimensions is considered.

Transformación integral de distributión multidimensional: Applicación a la estimación de máxima verosimilitud

Resumen. Sea (X_1, X_2) un vector aleatorio cuya función de distribución F. La pit es la variable aleatoria unidimensional $P_2 = F(X_1, X_2)$. La expresion de su F.D., y un algoritmo de simulación en términos de la función cuantil, derda por Chakak et al [2000], cuando la distribución es absolumente continua, son extendidas a distribuciones qui pueden tener singularidades. La estimación de máxima verosimilitud del parametro de dependencia basada sobre la pit se hace por simutación. Esta estimación funciona bien con familias de distribuciones singulares. La extensión a grandes dimensiones es considerada.

1. Introduction

A strategy of analyzing bivariate data consists of estimating the dependence function and the marginals separately. This two step approach to stochastic modelling is often convenient because many tractable models are readily available for the marginal distributions. It is clearly appropriate in situations where the marginals are known, for example from previous experience, or when the marginals are not of interest. When the structure of the underlying distribution presents some singularity such as the shock model of Marshall and Olkin [1967], parametric estimation procedures of the dependence parameters commonly using the joint density may not be valid specially when the copula is singular with a null density. A nonlinear transformation of all the univariate marginals might be useful. In the present work, we consider the pit as such.

Let (X_1, X_2) be a continuous bivariate random variable with a distribution function F and C be the unique copula, which is a continuous cdf on $[0,1]^2$, associated to F through the relation $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$. The pit is the univariate random variable on $[0,1]$ defined by $P_2 = F(X_1, X_2) =$

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$C(U_1, U_2)$, which is clearly defined from copula that is invariant under strictly increasing transformation of X_1 and X_2, and hence it is free of the marginal distributions. Unlike univariate continuous random variables where the pit is uniform(0,1), the pit for a higher dimensional random vector is not uniform(0,1). For example when the marginals are independent, $K_2(p) = P[F_1(X_1)F_2(X_2) \leq p] = P[U_1U_2 \leq p] = p[1 - log(p)],\ p \in (0,1]$. The distribution of the pit sometimes characterizes completely the dependence structure of the copula, such as when it is Archimedian for example. Moreover, as noted in the examples herein the pit mitigates singularity; the distribution of the pit of a singular copula is a mixture of a dirac and a uniform(0,1), and that of a copula with a singular component is absolutely continuous.

To study the effects of the reliability of the maximum likelihood estimates based on the pit, we consider three families of copulas: (i) the family of singular copulas obtained by linear combination of boundary quantile functions, (ii) the family of copulas associated to the survival bivariate exponential function of Marshall and Olkin [1967] which has the loss of memory property, and (iii) the Clayton [1978] family of copulas which is a proportional hazards and a frailty model.

Recall that an absolutely continuous distribution has a positive density a.e., a singular distribution has a null density a.e., and a distribution with a singular component is an absolutely continuous distribution whose pdf does not integrate to 1.

In section 2 we give some useful properties of quantile functions for bivariate copulas that may present singularity. We consider the cdf of the pit and note its validity in the presence of singularity. An alternative formula to that developped by Imlahi et al [1999], for the pit's cdf of a higher dimensional continuous random vector that uncovers the effect of higher dimensionality is presented. In section 3 we give a generalized version of a simulation algorithm of Chakak *et al.* [2000] for bivariate absolutely continuous copulas and extend it to the higher dimensional setup. Section 4 deals with the performance of maximum likelihood estimation based on the pit.

2. Quantile function and the P.I.T.'s distribution

2.1. The quantile function

Let X be a univariate continuous random variable distributed as $F(x)$, the quantile function is $Q(p) = F^{-1}(p) = \inf\{x :\ F(x) \geq p\}$. For a copula C, we consider sections $S_2(p) = \{(u,v) : C(u,v) = p\},\ 0 < p \leq 1$ of constant copula C. They are also known as distribution contours (Conway, 1979). We describe $S_2(p)$ by a function $\psi(p,.)$ defined on the interval $[p,1]$ as $\psi(p,u) = \inf\{v :\ C(u,v) \geq p\}$. Since C continuous

$$\psi(p,u) = \inf\{v :\ C(u,v) = p\},\ 0 < p \leq u \leq 1, \tag{1}$$

also called quantile function. The quantile function $\psi(p,u)$ is simply the inverse of the increasing function $C(u,.)$. The graph of $\psi(p,u)$ may be distinct from $S_2(p)$. For example when $C(u,v) = \min\{u,v\},\ S_2(p) = \{\{p\} \times [p,1]\} \cup \{[p,1] \times \{p\}\}$, the quantile function $\psi(p,u) = p,\ 0 < p \leq u \leq 1$ whose graph is $\{p\} \times [p,1]$. The quantile function $Q(p)$, in the univariate case, and the quantile function $\psi(p,.)$ both satisfy $F(Q(p)) = p$ when F is continuous, and $C(u,\psi(p,u)) = p$, (Proposition 1 below). When C is absolutely continuous the quantile function is obtained by solving in v the equation $C(u,v) = p$. That is,

$$C(u,v) = p \Longleftrightarrow \psi(p,u) = v,\ (u,v) \in [p,1]^2,\ p \in (0,1].$$

The quantile function is informative by itself because it expresses the dependence between the marginals for any fixed value $p \in (0,1]$, which is important in safe-design reliability applications, besides its theretical importance as it characterizes the copula and takes place in the expression of the distribution of the pit. The quantile function for the cdf F can be obtained from that of the copula as $F_2(x_2) = \psi(p, F_1(x_1))$.

Proposition 1 *A copula C and its quantile function $\psi(.,.)$ satisfy:*
(1) $C(u,\psi(p,u)) = p$ and $\psi(C(u,v),u) \leq v$,
(2) $C(u,v) \leq p \Longleftrightarrow v \leq \psi(p,u),\ p \leq u, \forall p \in (0,1]$.

PROOF. Since C is continuous $C(u,\psi(p,u)) = C(u, inf\{v : C(u,v) = p\}) = p$, and $\psi(C(u,v),u) = inf\{w : C(u,w) = C(u,v)\} \leq v$, which completes (1). Finally $C(u,v) \leq p$, using $p = C(u,\psi(p,u))$, is equivalent to $v \leq \psi(p,u)$, as C is increasing. ■

Examples

(1) The family of symmetric singular copulas derived from the convex combination, in a rotated axes, of the upper and lower Frêchet bound copulas:

$$C_\lambda(u,v) = \max\{-\frac{1-\lambda}{\lambda+1} + u \wedge v + \frac{1-\lambda}{\lambda+1} u \vee v, 0\},\ \lambda \in [0,1], \tag{2}$$

where $u \wedge v = \min\{u,v\},\ u \vee v = \max\{u,v\}$. This family of singular copulas attains the Frêchet bound copulas (lower for $\lambda = 0$, and upper for $\lambda = 1$). Note that the copula (2) is not a convex combination of the Frêchet bound copulas. For $\lambda \in (0,1)$, let u_0 be such that $C(u_0,u_0) = p$, i.e. $u_0 = \frac{1}{2}[p+1+\lambda(p-1)]$. The quantile function is

$$\psi_\lambda(p,u) = \begin{cases} (p-u)\frac{1+\lambda}{1-\lambda} + 1, & p \leq u \leq u_0 \\ p + \frac{1-\lambda}{1+\lambda}(1-u), & u_0 \leq u \leq 1. \end{cases}$$

Recall that $C_0(u,v) = \max\{-1+u+v, 0\}$ with a quantile function $\psi(p,u) = p+1-u,\ p \leq u \leq 1$, and $C_1(u,v) = u \wedge v$, with a quantile function $\psi(p,u) = p,\ p \leq u \leq 1$.

(2) The family of copulas associated to the survival bivariate exponential distribution of Marshall & Olkin [1967] (BVE) is

$$C_\lambda(u,v) = u+v-1+\min\{(1-u)(1-v)^{1-\lambda}, (1-u)^{1-\lambda}(1-v)\}, \tag{3}$$

$\lambda \in [0,1]$. The only absolutely continuous copula in this family is $C_0(u,v) = uv$ with a quantile function $\psi_0(p,u) = \frac{p}{u}$. Except $\lambda = 0$, C_λ has a singular component. When $\lambda \in (0,1)$, let u_0 be such that $C_\lambda(u_0,u_0) = p$, i.e. u_0 is the solution of

$$2u_0 + (1-u_0)^{2-\lambda} = 1+p. \tag{4}$$

The quantile function is $\psi_\lambda(p,u) = 1 - \frac{u-p}{1-(1-u)^{1-\lambda}}$. There no closed form for the quantile function when $u \in (u_0,1]$ (Conway [1979]). The copula is symmetric, hence the quantile function has a symmetric graph about the diagonal $v = u$.

(3) Let ϕ be a decreasing convex function on $(0,1)$ satisfying $\phi(1) = 0$. It defines the Archimedian copula $C(u,v) = \phi^{-1}[\phi(u) + \phi(v)]$, whose quantile function is $\psi(p,u) = \phi^{-1}[\phi(p) - \phi(u)]$. For the purpose of illustration, we consider the Clayton (1978) family of copulas

$$C_\lambda(u,v) = [u^{-\lambda} + v^{-\lambda} - 1]^{-1/\lambda}, \tag{5}$$

generated by $\phi(u) = \frac{u^{-\lambda}-1}{\lambda},\ \lambda > 0$. Its quantile function is $\psi(p,u) = (p^{-\lambda} - u^{-\lambda} + 1)^{-1/\lambda}$.

For a vector $(u_1,u_2,u_3) \in [0,1]^3$, we define the quantile function for the trivariate copula as

$$\psi(p,u_1,u_2) = \inf\{u_3; C(u_1,u_2,u_3) = p\},\ p \in (0,1],\ (u_1,u_2) \in [p,1]^2.$$

This can be obtained directly for an absolutely continuous copula C as

$$\psi(p,u_1,u_2) = u_3 \Longleftrightarrow C(u_1,u_2,u_3) = p,\ 0 < p \leq u_1,u_2 \leq 1.$$

Similar to proposition 1, we have

$$\begin{array}{l} C(u_1,u_2,u_3) \le p \Longleftrightarrow \psi(p,u_1,u_2) \ge u_3, \\ C(u_1,u_2,\psi(p,u_1,u_2)) = p, \\ \psi(C(u_1,u_2,u_3),u_1,u_2) \le u_3. \end{array}$$

The quantile function for the multivariate case preserves the marginals. That is, for $s=3$ for example, $\psi(p,u_1,1)=\psi(p,u_1)$ and $\psi(p,1,u_2)=\psi(p,u_2)$. Other properties for multivariate quantile function are presented in Imlahi et al [1999].

Examples

(1) The upper Frêchet bound copula is $C(u_1,u_2,u_3)=u_1\wedge u_2\wedge u_3$, with a quantile function $\psi(p,u_1,u_2)=p,\ \ p\in(0,1],\ (u_1,u_2)\in[p,1]^2$.

(2) A natural and restrictive extension of a bivariate Archimedian copula generated by one convex function such that $\phi(1)=0$, is $C(u_1,u_2,u_3)=\phi^{-1}[\phi(u_1)+\phi(u_2)+\phi(u_3)]$, where $(-1)^k\frac{\partial^k\phi}{\partial p^k}(p)\ge 0, k=1,2,3.$ Its quantile function is $\psi(p,u_1,u_2)=\phi^{-1}[\phi(p)-\phi(u_1)-\phi(u_2)]$. Less retrictive extensions are generated by two convex functions ϕ_1 and ϕ_2 are:

$$\begin{array}{rcl} C_{12,3}(u_1,u_2,u_3) & = & \phi_2^{-1}[\phi_2(\phi_1^{-1}(\phi_1(u_1)+\phi_1(u_2)))+\phi_2(u_3)] \\ C_{23,1}(u_1,u_2,u_3) & = & \phi_2^{-1}[\phi_2(u_1)+\phi_2(\phi_1^{-1}(\phi_1(u_2)+\phi_1(u_3)))] \\ C_{13,2}(u_1,u_2,u_3) & = & \phi_2^{-1}[\phi_2(u_2)+\phi_2(\phi_1^{-1}(\phi_1(u_1)+\phi_1(u_3)))], \end{array} \tag{6}$$

where $C_{ij,k}, 1\le i\ne j\ne k\ne i\le 3$, exhibits the same dependence between (U_i,U_k) and between (U_j,U_k) but possibly a different dependence between (U_i,U_j). All these families of copulas are particular cases of Chakak and Koehler [1995]. Conditions under which, $C_{12,3}$ for example, is a copula can be found in Joe [1990] and Hillali [1998]. The quantile function is $\psi_{12,3}(p,u_1,u_2)=\phi_2^{-1}[\phi_2(p)-\phi_2(\phi_1^{-1}(\phi_1(u_1)+\phi_1(u_2)))],\ p\in(0,1],\ (u_1,u_2)\in[p,1]^2$.
Taking $\phi_i(u)=\frac{u^{\lambda_i}-1}{\lambda_i}, i=1,2,$

$$C_{12,3}(u_1,u_2,u_3)=[(u_1^{-\lambda_1}+u_2^{-\lambda_1}-1)^{\frac{\lambda_2}{\lambda_1}}+u_3^{-\lambda_2}-1]^{-1/\lambda_2},$$

and $\psi_{12,3}(p,u_1,u_2)=[p^{-\lambda_2}-(u_1^{-\lambda_1}+u_2^{-\lambda_1}-1)^{\frac{\lambda_2}{\lambda_1}}+1]^{-1/\lambda_2}$.

2.2. The pit's distribution

Let (X_1,X_2) be a bivariate random variable with continuous cdf F. The cdf of the pit is $K_2(p)=P[P_2\le p]=P[C(U_1,U_2)\le p]=P_C[C(u_1,u_2)\le p]=P[\psi(p,U_1)\le U_2]=P_C[(u_1,u_2):\psi(p,u_1)\le u_2]$, where $P_C(A)=\int\int_A dC(u,v)$ for a measurable set A.
Various authors have given expressions for $K_2(p)$ for specific families of copulas (see Genest *et al.* [1993], Capéràa *et al.* [1997], Ghoudi *et al.* [1998], Barbe *et al.* [1996]). A general formula is presented in terms of the quantile function by Chakak *et al.* [2000] when the copula is absolutely continuous,

$$K_2(p)=p+\int_p^1\frac{\partial C}{\partial u}(u,v)|_{(u,v)=(u,\psi(p,u))}du, 0<p\le 1. \tag{7}$$

Using (1) and (2) of Proposition 1, formula (7) holds true for copulas with singularities as well.

Sometimes an explicit expression for the quantile function is available only on $\{(u,v):u\le v\}$, such as for the BVE copula. When C is symmetric, a useful expression is

$$K_2(p)=p+2\int_p^{u_0}\frac{\partial C}{\partial u}(u,v)|_{(u,v)=(u,\psi(p,u))}du, \tag{8}$$

where u_0 is the solution of (4).

Using $p \le \psi(p,u) \le 1+p-u$, we have $2p - C(p,p) \le K_2(p) \le p + \int_p^1 \frac{\partial C}{\partial u}(u, 1+p-u)\, du$. These bounds are tighter than $K_U(p) = p$ (resp.$K_L(p) = 1$) which is the pit's cdf for the upper (resp. lower) Frêchet bound copula $C(u,v) = u \wedge v$ (resp. $C(u,v) = \max\{-1+u+v, 0\}$).

Examples

(1) For the singular family of copulas (2), the derivative in u is

$$\frac{\partial C_\lambda}{\partial u}(u,v) = \begin{cases} 1, & (u,v) \in [u < v, 0 < u - (1-v)\frac{1-\lambda}{1+\lambda}] \\ \frac{1-\lambda}{1+\lambda}, & (u,v) \in [v < u, 0 < v - (1-u)\frac{1-\lambda}{1+\lambda}]. \end{cases}$$

Application of (7) provides

$$K_2(p) = \lambda p + (1-\lambda),\ 0 \le p \le 1,$$

with density

$$k_2(p) = \begin{cases} (1-\lambda), & if\ p = 0 \\ \lambda, & if\ 0 < p \le 1 \end{cases}$$

Although the copula is singular, the pit's density is not null, making maximum likelihood estimation of λ possible.
Note that $K_2(p) = \lambda K_U(p) + (1-\lambda)K_L(p)$, is the convex combination of the p.i.t.'s of the Frêchet bound copulas.

(2) The BVE family of copulas (3) has a quantile function $\psi(p,u) = 1 - \frac{u-p}{1-(1-u)^{1-\lambda}}$, $p \le u \le u_0$, with u_0 the solution of (4). Using (8) we have

$$K_2(p) = p + 2(u_0 - p) - 2(1-\lambda)\int_p^{u_0} \frac{(1-u)^{-\lambda}(u-p)}{1-(1-u)^{1-\lambda}} du.$$

with density

$$\begin{aligned} k_2(p) &= -1 + \frac{2}{2-(2-\lambda)(1-u_0)^{1-\lambda}}[1 - (1-\lambda)\frac{u_0 - p}{(1-u_0)^\lambda + u_0 - 1}] \\ &\quad + 2(1-\lambda)\int_p^{u_0} \frac{du}{(1-u)^\lambda + u - 1} \end{aligned} \tag{9}$$

(3) For an Archimedian copula generated by a convex decreasing function ϕ on $(0,1]$ and vanishing at 1, we have $\frac{\partial C}{\partial u}(u,v) = \frac{\phi'(u)}{\phi'(C(u,v))}$. Replacing v by $\psi(p,u)$, we get $\frac{\partial C}{\partial u}(u, \psi(p,u)) = \frac{\phi'(u)}{\phi'(p)}$, which gives $K_2(p) = p - \frac{\phi(p)}{\phi'(p)}$. This result is derived otherwise by Genest et al (1993). Its density $k_2(p) = \frac{\phi(p)\phi''(p)}{[\phi'(p)]^2},\ p \in (0,1]$. Taking $\phi(u) = \frac{u^{-\lambda}-1}{\lambda},\ K_2(p) = p + p(1-p^\lambda)/\lambda$, and $k_2(p) = (1+\frac{1}{\lambda})(1-p^\lambda)$.

For higher dimensional copulas a similar technique is used to derive the pit's cdf. For simplicity we consider $s = 3$ first. Let (U_1, U_2, U_3) be a random vector distributed as the copula C, and $P_3 = C(U_1, U_2, U_3)$. Using the definition of the multivariate quantile function we have

Proposition 2

$$\begin{aligned} K_3(p) &= p + \int_p^1 \psi(p, u_1) du_1 + \\ &\quad \int_p^1 \left\{ \int_{\psi(p,u_1)}^1 P[U_3 \le \psi(p, u_1, u_2)/U_1 = u_1, U_2 = u_2] du_2 \right\} du_1. \end{aligned}$$

When $c(u_1,u_2)$ *exists,*

$$K_3(p) = p + \int_p^1 \psi(p,u_1)du_1 + \int_p^1 \left[\int_{\psi(p,u_1)}^1 \frac{\frac{\partial^2 C}{\partial u_1 \partial u_2}(u_1,u_2,\psi(p,u_1,u_2))}{c(u_1,u_2)} du_2\right] du_1, \qquad (10)$$

for $0 < p \leq 1$.

PROOF. We proceed sequentially by conditionning.

$$\begin{aligned} K_3(p) &= P[C(U_1,U_2,U_3) \leq p] \\ &= p + \textstyle\int_p^1 P[C(u_1,U_2,U_3) \leq p/U_1 = u_1]du_1 \\ &= p + \textstyle\int_p^1 \psi(p,u_1)du_1 + \\ &+ \textstyle\int_p^1 \left(\int_{\psi(p,u_1)}^1 P\left[C(u_1,u_2,U_3) \leq p/U_1 = u_1, U_2 = u_2\right] du_2\right) du_1. \end{aligned}$$

Using

$$\begin{aligned} &P[C(u_1,u_2,U_3) \leq p/U_1 = u_1, U_2 = u_2] \\ &\quad = \quad P[U_3 \leq \psi(p,u_1,u_2)/U_1 = u_1, U_2 = u_2], \end{aligned}$$

and when the density $c(u_1,u_2)$ exists,

$$P[U_3 \leq \psi(p,u_1,u_2)/U_1 = u_1, U_2 = u_2] = \frac{\frac{\partial^2 C}{\partial u_1 \partial u_2}(u_1,u_2,\psi(p,u_1,u_2))}{c(u_1,u_2)},$$

which completes the proof. ■

Some expressions of the pit's cdf for some multivariate families of copulas are given in Barbe *et al.* [1996].

Remark 1 From (10) $p + \int_p^1 \psi(p,u)du \leq K_3(p)$. Using $\psi(p,u) \geq p,$ implies $p(1+(1-p)) \leq K_3(p)$. However $p(1+(1-p))$ is the p.i.t.'s cdf of the trivariate upper Frêchet bound copula $C(u_1,u_2,u_3) = u_1 \wedge u_2 \wedge u_3$ (example 1 below). Generally, for $s \geq 2$ and the marginal density $c(u_1,...,u_{s-1})$ exists, (10) generalizes to

$$\begin{aligned} K_s(p) &= P[C(U_1,...,U_s) \leq p] \\ &= p + \int_p^1 P[C(u_1,U_2,...,U_s) \leq p/U_1 = u_1]du_1 \\ &= p + \int_p^1 \psi(p,u_1)du_1 + \int_p^1 \int_{\psi(p,u_1)}^1 \psi(p,u_1,u_2)du_2du_1 + ... + \\ &\quad + \int_p^1 ... \int_{\psi(p,u_1,..,u_{s-1})}^1 \frac{\frac{\partial^{s-1} C}{\partial u_1 ... \partial u_{s-1}}(u_1,...,u_{s-1},\psi(p,u_1,..,u_{s-1}))}{c(u_1,..,u_{s-1})} du_1..du_{s-1}. \end{aligned}$$

Since $\psi(p,u_1,...u_k) \geq p, \ \forall k < s,$

$$1-(1-p)^{s-1} \leq K_s(p), 0 < p \leq 1,$$

implying that

$$\lim_{s\to\infty} K_s(p) = 1, \ p > 0.$$

This means that a higher dimensional continuous random vector has a pit almost degenerate at 0.

Examples

(1) The upper Frêchet bound copula is $C(u_1, u_2, u_3) = u_1 \wedge u_2 \wedge u_3$ with quantile function $\psi(p, u_1, u_2) = p$ and $P[U_3 \leq p/U_1 = u_1, U_2 = u_2] = 0$. Using (10) $K_3(p) = p(2-p),\ 0 < p \leq 1$. Its density is $k_3(p) = 2(1-p)$. Likewise $1-(1-p)^{s-1}$ is the pit's cdf of the s-variate upper Frêchet copula $C(u_1, .., u_s) = u_1 \wedge ... \wedge u_s$.

(2) The independence copula is $C(u_1, u_2, u_3) = u_1 u_2 u_3$. The quantile function is $\psi(p, u_1, u_2) = \frac{p}{u_1 u_2}$, $\psi(p, u_1) = \frac{p}{u_1}$. The pit's cdf is $K_3(p) = p[1 + log(1/p) + \frac{1}{2} log^2(1/p)], p \in [0,1]$, with a density $k_3(p) = \frac{1}{2} log^2(p),\ 0 < p \leq 1$.

(3) The trivariate Archimedian copula generated by one parameter convex function ϕ satisfying $\frac{\partial^k}{\partial u^k}\phi^{-1}(u) \geq 0, \forall k \geq 1$ can be found in Barbe *et al.* [1996]. The trivariate Archimedian copula $C_{12,3}$, for example, generated by two convex functions $\phi_i(u) = (u^{-\lambda_i} - 1)/\lambda_i, i = 1,2$ provides

$$\begin{aligned} K_3(p) &= p + \int_p^1 \psi(p, u_1) du_1 + \\ & \frac{p^{1+\lambda_2}}{1+\lambda_1} \int_p^1 \int_{\psi(p,u_1)}^1 [u_1^{-\lambda_1} + u_2^{-\lambda_1} - 1]^{(\lambda_2+1)/\lambda_1} \times \\ & [(\lambda_2 - \lambda_1) + (1+\lambda_2) p^{1+\lambda_2} [u_1^{-\lambda_1} + u_2^{-\lambda_1} - 1]^{\lambda_2/\lambda_1}] du_2 du_1, \end{aligned}$$

where $\psi(p, u_1) = (p^{-\lambda_1} - u_1^{-\lambda_1} + 1)^{-1/\lambda_1}$, which does not seem to have a closed form expression.

3. Simulation

Algorithms for generating random samples from multivariate distributions are useful in Monte-Carlo studies of properties of multivariate statistical methods. When the univariate marginals are specified, the algorithm below uses the quantile of a conditional cdf. It is an immediate extension of Proposition 5 in Chakak *et al.* [2000].

Proposition 3 *Let U_1 and U_2 be two uniform(0,1) jointly distributed as a copula C. Let $V_1 = U_1$, and $V_2 = \inf\{W; \frac{\partial C}{\partial u_1}(U_1, W) \geq U_2\}$. Then (V_1, V_2) are jointly distributed as C, and consequently $P_2 = C(V_1, V_2)$ is distributed as K_2*

Examples

(1) For the family (2) with $\lambda \in (0,1)$,

$$\frac{\partial C_\lambda}{\partial u_1}(u_1, w) = \begin{cases} 1, & [u_1 < w, 0 < u_1 - (1-w)\frac{1-\lambda}{1+\lambda}] \\ \frac{1-\lambda}{1+\lambda}, & [w < u_1, 0 < w - (1-u_1)\frac{1-\lambda}{1+\lambda}]. \end{cases}$$

The set $[u_1 < w, 0 < u_1 - (1-w)\frac{1-\lambda}{1+\lambda}]$ is the triangle of vertices $(0,1), (\frac{1-\lambda}{2}, \frac{1-\lambda}{2}), (1,1)$. Likewise $[u_1 < w, 0 < u_1 - (1-w)\frac{1-\lambda}{1+\lambda}]$ is the triangle of vertices $(1,0)$, $(\frac{1-\lambda}{2}, \frac{1-\lambda}{2})$, $(1,0)$. The equation $w = 1 - \frac{1+\lambda}{1-\lambda}u_1$, and $w = \frac{1-\lambda}{1+\lambda}(1-u_1)$, represents the line joining $(0,1)$ and $(\frac{1-\lambda}{2}, \frac{1-\lambda}{2})$, and $(\frac{1-\lambda}{2}, \frac{1-\lambda}{2})$ and $(1,0)$ respectively. There are three cases for u_1 to consider:

(i) $u_1 \leq \frac{1-\lambda}{2}$, implies

$$\frac{\partial C_\lambda}{\partial u_1}(u_1, w) = \begin{cases} 0, & w < 1 - \frac{1+\lambda}{1-\lambda}u_1 \\ 1, & 1 - \frac{1+\lambda}{1-\lambda}u_1 < w \end{cases}.$$

Hence

$$\inf\{w; \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2\} = 1 - \frac{1+\lambda}{1-\lambda}u_1.$$

Then $(U_1, 1-\frac{1+\lambda}{1-\lambda}U_1)$ is distributed as C_λ and hence

$$P_2 = C_\lambda(U_1, 1-\frac{1+\lambda}{1-\lambda}U_1)) = 0.$$

(ii) $\frac{1-\lambda}{2} \leq u_1$, and $u_2 \leq \frac{1-\lambda}{1+\lambda}$, implies $\inf\{w; \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2\} = \frac{1-\lambda}{1+\lambda}(1-u_1)$. Then $(U_1, \frac{1-\lambda}{1+\lambda}(1-U_1))$ is distributed as C_λ, and

$$P_2 = C_\lambda\left(U_1, \frac{1-\lambda}{1+\lambda}(1-U_1)\right) = 0.$$

(iii) $\frac{1-\lambda}{2} \leq u_1$, and $\frac{1-\lambda}{1+\lambda} \leq u_2$, $\inf\{w; \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2\} = u_1$. Then (U_1, U_1) is distributed as C_λ, and

$$P_2 = C_\lambda(U_1, U_1) = \frac{2U_1}{1+\lambda} - \frac{1-\lambda}{1+\lambda}.$$

When $\lambda = 0$, $(U_1, 1-U_1)$ is distributed as C_0, and hence $P_2 = 0$, and when $\lambda = 1$, (U_1, U_1) is distributed as C_1, implying $P_2 = U_1$.

For $\lambda \in (0,1)$ the algorithm goes as follows:
Generate U_1, U_2 two independent uniform[0,1].

$$(V_1, V_2) = \begin{cases} (U_1, 1-\frac{1+\lambda}{1-\lambda}U_1), & \text{if } U_1 \leq \frac{1-\lambda}{2} \\ (U_1, \frac{1-\lambda}{1+\lambda}(1-U_1)), & \text{if } \frac{1-\lambda}{2} \leq U_1, U_2 \leq \frac{1-\lambda}{1+\lambda} \\ (U_1, U_1), & \text{if } \frac{1-\lambda}{2} \leq U_1, \frac{1-\lambda}{1+\lambda} \leq U_2, \end{cases}$$

is distributed as (2). The random variable

$$P_2 = \begin{cases} \frac{2U_1-1+\lambda}{1+\lambda} & \text{when } (U_1, U_2) \in [\frac{1-\lambda}{2}, 1] \times [\frac{1-\lambda}{1+\lambda}, 1] \\ 0, & \text{elsewhere} \end{cases}$$

is distributed as $K_2(p) = \lambda p + 1 - \lambda,\ 0 < p \leq 1, 0 < \lambda < 1$.
(2) For $0 < \lambda < 1$, the BVE is :

$$C_\lambda(u_1, u_2) = \begin{cases} u_1 + u_2 - 1 + (1-u_2)(1-u_1)^{1-\lambda} & u_1 \leq u_2 \\ u_1 + u_2 - 1 + (1-u_1)(1-u_2)^{1-\lambda} & u_2 \leq u_1. \end{cases}$$

Its first derivative is

$$\frac{\partial C_\lambda}{\partial u_1}(u_1, u_2) = \begin{cases} 1 - (1-\lambda)(1-u_2)(1-u_1)^{-\lambda}, & u_1 < u_2 \\ 1 - (1-u_2)^{1-\lambda}, & u_2 < u_1, \end{cases}$$

We have $\frac{\partial C_\lambda}{\partial u}(u, u+) = 1 - (1-\lambda)(1-u)^{1-\lambda}$ and $\frac{\partial C_\lambda}{\partial u}(u, u-) = 1 - (1-u)^{1-\lambda}$ which shows that $\frac{\partial C_\lambda}{\partial u_1}(u_1, w)$ is not continuous only at $w = u_1$. For fixed u_1, we consider the three situations:
(i) $u_2 < \frac{\partial C_\lambda}{\partial u_1}(u_1, u_1-) = 1 - (1-u_1)^{1-\lambda}$, implying $\inf\{w : \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2\} = 1 - (1-u_2)^{1/(1-\lambda)}$.
(ii) $\frac{\partial C_\lambda}{\partial u_1}(u_1, u_1-) < u_2 \leq \frac{\partial C_\lambda}{\partial u_1}(u_1, u_1+)$, implying $v_2 = \inf\{w : \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2\} = u_1$.
(iii) $u_2 \leq \frac{\partial C_\lambda}{\partial u_1}(u_1, u_1+)$, implying $v_2 = \inf\{w : \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2\} = 1 - \frac{1-u_2}{(1-\lambda)(1-u_1)^{-\lambda}}$.

Define the subsets $\mathcal{A} =[(u_1, u_2) : u_2 < 1-(1-u_1)^{1-\lambda}]$, $\mathcal{B} =[(u_1, u_2) : 1-(1-u_1)^{1-\lambda} < u_2 \leq 1-(1-\lambda)(1-u_1)^{1-\lambda}]$, $\mathcal{C} =[(u_1, u_2) : 1-(1-\lambda)(1-u_1)^{1-\lambda} \leq u_2]$.

The algorithm goes as follows:
Generate U_1, U_2 two independent uniform[0,1]

$$(V_1, V_2) = \begin{cases} (U_1, 1-(1-U_2)^{1/(1-\lambda)}), & \text{for } (U_1,U_2) \in \mathcal{A} \\ (U_1, U_1), & \text{for } (U_1,U_2) \in \mathcal{B} \\ (U_1, 1-\frac{1-U_2}{(1-\lambda)(1-U_1)^{-\lambda}}), & \text{for } (U_1,U_2) \in \mathcal{C} \end{cases}$$

is distributed as C_λ. The random variable

$$P_2 = \begin{cases} U_1-(1-U_2)^{1/(1-\lambda)} & \\ \quad +[(1-U_1)^{1-\lambda}(1-U_2)^{1/(1-\lambda)}] \wedge [(1-U_1)(1-U_2)], & (U_1,U_2) \in \mathcal{A} \\ 2U_1-1+(1-U_1)^{2-\lambda}, & (U_1,U_2) \in \mathcal{B} \\ U_1-\frac{(1-U_2)(1-U_1)^{\lambda}}{1-\lambda}+\frac{(1-U_1)(1-U_2)}{1-\lambda} \wedge \frac{(1-U_1)^{\lambda(1-\lambda)+1}(1-U_2)^{1-\lambda}}{(1-\lambda)^{1-\lambda}}, & (U_1,U_2) \in \mathcal{C}, \end{cases}$$

is distributed as (9).

Proposition 4 *Let U_1, U_2, U_3 be three independent uniform$(0,1)$, and C an absolutely continuous with a density c. Put $V_1 = U_1$, V_2 such that $\frac{\partial C}{\partial u_1}(V_1, V_2, 1) = U_2$, and V_3 satisfying $\frac{\partial^2 C}{\partial u_1 \partial u_2}(V_1, V_2, V_3) = c(U_1, U_2, 1)U_3$. Then V_1, V_2, V_3 are jointly distributed as C, and hence $C(V_1, V_2, V_3)$ is distributed as K_3.*

PROOF. From proposition 3, (V_1, V_2) is distributed as $C(u_1, u_2, 1)$.

$$\begin{aligned} &Pr[V_1 \le v_1, V_2 \le v_2, V_3 \le v_3] \\ &= \int_0^{v_1}\int_0^{v_2} Pr[U_3 c(u_1,u_2,1) \le \tfrac{\partial^2 C}{\partial u_1 \partial u_2}(u_1,u_2,v_3)] dC(u_1,u_2,1) \\ &= \int_0^{v_1}\int_0^{v_2} \tfrac{\partial^2 C}{\partial u_1 \partial u_2}(u_1,u_2,v_3) du_1 du_2 \\ &= C(v_1,v_2,v_3). \end{aligned}$$

Example

The trivariate Archimedian copula generated by one convex decreasing function on $[0,1]$ such that $\phi(1) = 0$, is $C(u_1,u_2,u_3) = \phi^{-1}[\phi(u_1)+\phi(u_2)+\phi(u_3)]$. The derivatives of C are $\frac{\partial C}{\partial u_1}(u_1,u_2,u_3) = \frac{\phi'(u_1)}{\phi'(C(u_1,u_2,u_3))}$ and $\frac{\partial^2 C}{\partial u_1 \partial u_2}(u_1,u_2,u_3) = -\frac{\phi'(u_1)\phi'(u_2)\phi''(C(u_1,u_2,u_3))}{[\phi'(C(u_1,u_2,u_3))]^3}$.
The algorithm goes as follows:
(1) Put $V_1 = U_1$.
(2) Solve in V_2, $\frac{\partial C}{\partial u_1}(V_1,V_2,1) = U_2$. This gives $V_2 = \phi^{-1}[\phi(W)-\phi(U_1)],\ W = \phi'^{-1}(\frac{\phi'(U_1)}{U_2})$.
(3) Solve in V_3, $\frac{\partial^2 C}{\partial u_1 \partial u_2}(V_1,V_2,V_3) = U_3 c(V_1,V_2,1)$ gives

$$\frac{[\phi'(C(V_1,V_2,1))]^3}{U_3\phi''(C(V_1,V_2,1))} = \frac{[\phi'(C(V_1,V_2,V_3)]^3}{\phi''(C(V_1,V_2,V_3))}.$$

We illustrate this algorithm with $\phi(u) = \frac{u^{-\lambda}-1}{\lambda}, \lambda > 0$, the convex decreasing function with $\phi(1) = 0$ generating the Clayton [1978] bivariate copula. The trivariate copula is $C(u_1,u_2,u_3) = [u_1^{-\lambda} + u_2^{-\lambda} + u_3^{-\lambda} - 2]^{-1/\lambda}$. Let (U_1,U_2,U_3) be three independent uniform$(0,1)$. Then
(1) consider $V_1 = U_1$.
(2) $V_2 = [U_1^{-\lambda}U_2^{-\lambda/(1+\lambda)} + 1 - U_1^{-\lambda}]^{-1/\lambda}$. Then $C(V_1,V_2) = U_1U_2^{1/(1+\lambda)}$ is distributed as $K_2(p) = p[1+\frac{1-p^{\lambda}}{\lambda}]$.
(3) $V_3 = [(U_3^{-\frac{\lambda}{(1+2\lambda)}} - 1)U_1^{-\lambda}U_2^{-\frac{\lambda}{(\lambda+1)}} + 1]^{-\frac{1}{\lambda}}$. Then (V_1,V_2,V_3) is distributed as $[v_1^{-\lambda} + v_2^{-\lambda} + v_3^{-\lambda} - 2]^{-\frac{1}{\lambda}}$, and hence $C(V_1,V_2,V_3) = U_1U_2^{\frac{1}{(1+\lambda)}}U_3^{\frac{1}{(1+2\lambda)}}$ is distributed as $K_3(p) = p[1+\frac{(1-p^{\lambda})}{\lambda}+\frac{(1+\lambda)}{2\lambda^2}(1-p^{-\lambda})^2p^{\lambda}]$ (see Barbe *et al.* [1996]).

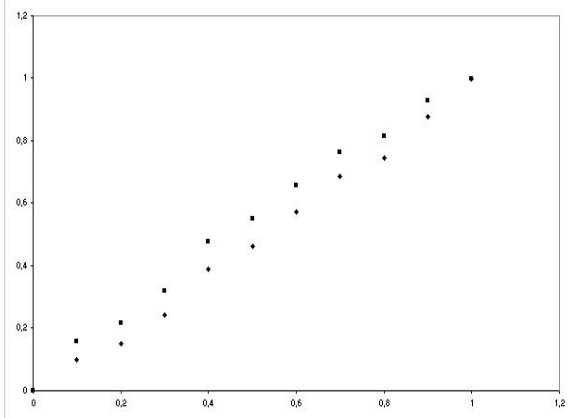

Figure 1. ml confidence intervals for (2)

4. Maximum likelihood estimation

We consider the three families (2), (3), and (5). Within the range of possible values of the dependence parameter, a random sample of size 500 of couples of independent uniform(0,1) is generated and transformed, according to proposition 4, to get independent random samples for each family of copulas.
(1) For the family (2), the likelihood function is

$$L = (1-\lambda)^{n_0}\lambda^{n-n_0},$$

where $n_0 = \#\{i : P_i = 0\}$. The maximum likelihood estimate is $\hat{\lambda} = 1 - n_0/n$, and $n - n_0$ is distributed as $Bin(n, \lambda)$. Therefore we have $E(\hat{\lambda}) = \lambda$ with variance $\frac{\lambda(1-\lambda)}{n}$. (See figure 1). The 95% asymptotic confidence intervals for the maximum likelihood estimates of the association parameter from (2) exhibit a good coverage in the range for the association parameter.

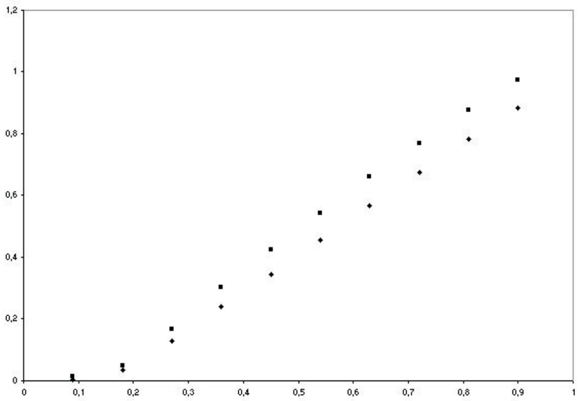

Figure 2. ml confidence intervals for (3)

(2) For the family (3), the likelihood function based on (9) is easily maximized numerically. The 95% asymptotic confidence intervals from (3) of the ml estimates clearly underestimates the true value of the association parameter. Moreover the asymptotic variance tend to increase with the association parameter. (See figure 2)

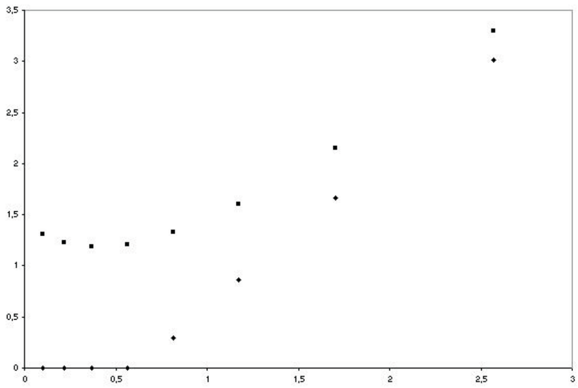

Figure 3. ml confidence intervals for (5)

(3) For the family of copulas (5), the likelihood function is

$$\mathcal{L} = nlog(1 + 1/\lambda) + \sum_{i=1}^{n} \log(1 - p_i^{\lambda}).$$

The likelihood equation is $\frac{n}{\lambda(1+\lambda)} + \sum \frac{p_i^{\lambda} log(p_i)}{(1-p_i^{\lambda})} = 0$ which has to be solved numerically. The 95% asymptotic confidence interval presents a good coverage for the association parameter within a reasonable range for association parameter, but the asymptotic variance based is fairly large for small values for the association parameter. (See figure 3)

In conclusion maximum likelihood estimation based on the pit is more reliable for singular families of distributions.

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A. Chakak and L. Imlahi
University Abdelmalek Essaadi
Faculty of Sciences
B.P. 2121 Tetouan
Morocco