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On the extension of measures

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Abstract. We give neccesary and sufficient conditions for a totally ordered by extension family $(\Omega, \Sigma_x, \mu_x)_{x\in X}$ of spaces of probability to have a measure μ which is an extension of all the measures μ_x. As an application we study when a probability measure on Ω has an extension defined on all the subsets of Ω.

Sobre la extensión de medidas

Resumen. Se estudian y se dan varias condiciones necesarias y suficientes para que dada una familia totalmente ordenada $(\Omega, \Sigma_x, \mu_x)_{x\in X}$ por extensión de espacios de probabilidades exista una medida μ que sea una extensión de todas las medidas μ_x. Como aplicación de ello se estudia cuando una medida de probabilidad sobre Ω tiene una extensión definida sobre todos los subconjuntos de Ω.

A *Banach measure* on a set Ω is a finite measure $\mu \neq 0$ on $\mathcal{P}(\Omega)$, the power set of Ω, such that $\mu(\omega) = 0$ for every $\omega \in \Omega$.

An *Ulam measure* on Ω is a Banach measure on Ω which takes values in the set $\{0, 1\}$.

A cardinal α is *real-measurable* if there exists a set Ω whose cardinal is α and such that there is a Banach measure on Ω.

A cardinal α is *2-measurable* if there exists a set Ω whose cardinal is α and there exists an Ulam measure on Ω.

Cardinals which are not 2-measurable are called *non measurable* and cardinals not *real-measurable* are called cardinals of zero measure.

Given a probability measure space (Ω, Σ, μ), we call as usual μ^* and μ_* the outer and inner measures associated to μ, i.e.,

$$\mu^*(A) = \inf\{\mu(X) : A \subset X \in \Sigma\} \text{ and}$$

$$\mu_*(A) = \sup\{\mu(X) : A \supset X \in \Sigma\}.$$

We call $(\Omega, \Sigma_0, \mu_0)$ to a fixed probability space.

Given two probability spaces $(\Omega, \Sigma_x, \mu_x)$ and $(\Omega, \Sigma_y, \mu_y)$, when we write $\mu_x \subset \mu_y$ we mean that μ_y is an extension of μ_x and, therefore, $\Sigma_x \subset \Sigma_y$ and $\mu_x^* \geq \mu_y^* \geq \mu_{y*} \geq \mu_{x*}$.

Proposition 1 *Let $(\Omega, \Sigma_x, \mu_x)_{x\in X}$ be a totally ordered by extension or inclusion family of spaces of probability such that $\mu_0 \subset \mu_x$ for every $x \in X$. Let us consider*

$$\mu_*(A) = \sup\{\mu_{x*}(A) : x \in X\}$$

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and

$$\mu^*(A) = \inf\{\Sigma_n \tau(A_n) : \cup_n A_n \supset A\},$$

where

$$\tau(A) = \inf\{\mu_x^*(A) : x \in X\}$$

and A and all of the A_n are subset of Ω. Then, the following properties are all equivalent:

(i) There exists an extension λ of the measures μ_x, $x \in X$.

(ii) $\mu_* \leq \mu^*$.

(iii) $\mu_*(A) \leq \mu^*(A)$ for every $A \in \mathcal{S}_\sigma$, where $\mathcal{S} = \cup_{x\in X}\Sigma_x$.

(iv) $\mu_*(A) = \mu^*(A)$ for every $A \in \mathcal{S}_\sigma$.

PROOF. First of all, let us note that μ_* and μ^* are, respectively, an inner measure and an outer measure (see [3]).

(i) $\Rightarrow$ (ii). Let us suppose (i). Then, since $\mu_{x*} \leq \lambda_* \leq \lambda^* \leq \mu_x^*$ for every $x \in X$, we get that $\mu_* \leq \lambda_* \leq \lambda^* \leq \tau$ and, hence, $\Sigma_n \tau(A_n) \geq \Sigma_n \lambda^*(A_n) \geq \lambda^*(A)$ if $\cup_n A_n \supset A$. It follows now that $\mu^* \geq \lambda^*$ and $\mu^* \geq \mu_*$.

(ii) $\Rightarrow (iii)$. Obvious.

(iii) $\Rightarrow (iv)$. Indeed, if $A \in \mathcal{S}_\sigma$, there exists a disjoint sequence (S_n) in $\mathcal{S}$ such that $\cup_n S_n = A$ and, hence, $\Sigma_n \mu_*(S_n) \leq \mu_*(A) \leq \mu^*(A) \leq \Sigma_n \mu^*(S_n)$. Since $\mu_*(S_n) = \tau(S_n) = \mu^*(S_n)$ for every $n \in \mathbb{N}$, we get that (iv) holds.

(iv) $\Rightarrow$ (i). Let us suppose (iv). Let $\lambda^*(A) = \inf\{\mu^*(H) : A \subset H \in \mathcal{S}_\sigma\}$. Then, λ^* is an outer measure and, for every $\epsilon > 0$ and for every $E \subset \Omega$, there exists $H \in \mathcal{S}_\sigma$ such that $\mu^*(H) < \lambda^*(E) + \epsilon$ and $E \subset H$. Therefore, if $A \in \mathcal{S}$, we get

$$\lambda^*(E \cap A) + \lambda^*(E \setminus A) \leq \lambda^*(H \cap A) + \lambda^*(H \setminus A) = \mu^*(H \cap A) + \mu^*(H \setminus A) =$$

$$\mu_*(H \cap A) + \mu_*(H \setminus A) \leq \mu_*(H) = \mu^*(H) < \lambda^*(E) + \epsilon.$$

It follows from here that every $A \in \mathcal{S}$ is λ^*-measurable. Moreover, $\lambda^*(A) = \mu^*(A)$ for every $A \in \mathcal{S}$ and μ^* is an extension of the measures μ_x, so we get that the restriction λ of λ^* on the σ-algebra of the λ^*-measurable sets is an extension of the measures μ_x. Now we can easily prove that $\lambda^* = \mu^*$ and that μ^* is a regular outer measure. (For every $A \subset \Omega$, $\mu_*(A) + \tau(\Omega \setminus A) = 1$ and $\lambda_*(A) + \lambda^*(\Omega \setminus A) = 1$, hence$\lambda_* = \mu_*$ if and only if $\mu^* = \tau$). ■

It is clear that $\mathcal{S}$ is an algebra. In what follows, Ω, Σ_x, μ_x, τ, μ_*, μ^* and $\mathcal{S}$ will mean the same as in Proposition 1.

Proposition 2 *If the restriction of τ to $\mathcal{S}$ is a (countably additive) measure, then $\mu_* \leq \mu^*$.*

PROOF. Let $\lambda^*(A) = \inf\{\sum_n \tau(A_n) : \cup_n A_n \supset A,\ A_n \in \mathcal{S}\}$. Then it can be easily proved that λ^* is an outer measure such that every $A \in \mathcal{S}$ is λ^*-measurable. On the other hand, if (A_n) is a sequence in $\mathcal{S}$ such that $\cup_n A_n \supset A \in \mathcal{S}$ and $S_n = A_n \setminus \cup_{k<n} A_k$, we get

$$\sum_n \tau(A_n) \geq \sum_n \tau(S_n \cap A) = \tau(A),$$

because the restriction of τ to $\mathcal{S}$ is a measure and, therefore, $\lambda^*(A) = \tau(A)$ for every $A \in \mathcal{S}$ and λ^* is an extension of the measures μ_x. It follows that the restriction of λ^* to the σ-algebra of the λ^*-measurable sets is an extension of the measures μ_x and it follows from Proposition 1 that $\mu_* \leq \mu^*$. ■

Now it can be easily proved, using Proposition 1, that

$$\lambda^*(A) = \inf\{\mu^*(H) : A \subset H \in \mathcal{S}_\sigma\},$$

and, hence, $\lambda^* = \mu^*$.

Proposition 3 *Every set $A \in \mathcal{S}$ is μ^*-measurable.*

PROOF. If $A \in \Sigma_x$ and $E \subset \Omega$ we have that $\mu_x^*(E \cap A) + \mu_x^*(E \setminus A) = \mu_x^*(E)$, so taking limits in x we get that $\tau(E \cap A) + \tau(E \setminus A) = \tau(E)$ for every $A \in \mathcal{S}$ and $E \subset \Omega$. For every $\epsilon > 0$ there exists a sequence (A_n) of subsets of Ω such that $\sum_n \tau(A_n) < \mu^*(E) + \epsilon$ and $\cup_n A_n \supset E$, so we get

$$\mu^*(E \cap A) + \mu^*(E \setminus A) \leq \sum_n \tau(A_n \cap A) + \sum_n \tau(A_n \setminus A) = \sum_n \tau(A_n) < \mu^*(E) + \epsilon,$$

so,

$$\mu^*(E \cap A) + \mu^*(E \setminus A) = \mu^*(E),$$

and, therefore, every $A \in \mathcal{S}$ is μ^*-measurable. ■

Proposition 4 *If $\mu^*(\Omega) = 1$, then $\mu_* \leq \mu^*$.*

PROOF. If $A \in \Sigma_x$, we get $\mu^*(A) \leq \mu_x(A)$, $\mu^*(\Omega \setminus A) \leq \mu_x(\Omega \setminus A)$ and

$$1 \leq \mu^*(A) + \mu^*(\Omega \setminus A) \leq \mu_x(A) + \mu_x(\Omega \setminus A) = 1,$$

so, $\mu^*(A) = \mu_x(A)$ and μ^* is an extension of the measures μ_x. Now, Proposition 3 tells us that every $A \in \mathcal{S}$ is μ^*-measurable, so we get that the restriction λ of μ^* to the σ-algebra of the μ^*-measurable sets is an extension of the measures μ_x, so Proposition 1 implies that $\mu_* \leq \mu^*$. ■

Proposition 5 *If μ_* is μ^*-continuous, that is, if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu^*(A) < \delta$ implies $\mu_*(A) < \epsilon$, then $\mu_* \leq \mu^*$.*

PROOF. Let (A_n) be a sequence in $\mathcal{S}$ such that $(A_n) \searrow \emptyset$. Then, since every A_n is μ^*-measurable according to Proposition 3, we get that $\mu^*(A_n) \to 0$ and, hence, $\tau(A_n) = \mu_*(A_n) \to 0$, because μ_* is μ^*-continuous. It follows that the restriction of τ to $\mathcal{S}$ is a measure and it follows from Proposition 2 that $\mu_* \leq \mu^*$. ■

Proposition 6 *If $\tau(\cup_n A_n) \leq \sum_n \tau(A_n)$ for any sets $A_n \in \mathcal{S}$, then $\mu^* = \tau$.*

PROOF. Let

$$\lambda^*(A) = \inf\{\sum_n \tau(A_n) : \cup_n A_n \supset A, A_n \in \mathcal{S}\}.$$

Then, we get from our hypothesis that

$$\lambda^*(A) = \inf\{\tau(H) : A \subset H \in \mathcal{S}_\sigma\} \geq \tau(A)$$

for every set $A \subset \Omega$ and we also get that restriction of τ to $\mathcal{S}$ is a measure. By Proposition 2 $\lambda^* = \mu^*$, so we finally get that $\mu^* = \tau$. ■

Proposition 7 *The following properties are equivalent:*

(i) For any increasing sequence $(A_n) \subset \mathcal{S}$ and $A = \cup_n A_n$ we have $\tau(A) = \lim_n \tau(A_n)$.

(ii) For every sequence $A_n \searrow \emptyset$ in $\mathcal{S}_\sigma$ we get $\tau(A_n) \to 0$.

(iii) τ is μ^*-continuous, that is, for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu^*(A) < \delta$ implies $\tau(A) < \epsilon$.

(iv) $\mu^* = \tau$.

PROOF. (i) ⇒ (iv). Let us suppose (i). Let (A_n) be a sequence in $\mathcal{S}$. Then we have

$$\tau(\cup_n A_n) = \lim_n \tau(\cup_{k<n} A_k) \leq \sum_n \tau(A_n),$$

so, Proposition 6 implies that $\mu^* = \tau$

(ii) $\Rightarrow$ (i). Let us suppose (ii). Let (A_n) be an increasing sequence in $\mathcal{S}$ and $A = \cup_n A_n$. Since

$$\tau(A_n) \leq \tau(A) \leq \tau(A_n) + \tau(A \setminus A_n),$$

it follows that $\lim_n \tau(A_n) = \tau(A)$, because $A \setminus A_n \searrow \emptyset$ and $A \setminus A_n \in \mathcal{S}_\sigma$.

(iii) $\Rightarrow$ (ii). Let us suppose (iii). Let (A_n) be a decreasing sequence in $\mathcal{S}_\sigma$ such that $\cap_n A_n = \emptyset$. Then $\mu^*(A_n) \to 0$ because, according to Proposition 3, every A_n is μ^*-measurable and, hence, $\tau(A_n) \to 0$.

(iv) $\Rightarrow$ (iii). It is obvious. ■

Remark 1

(1). Property (iii) can not be replaced by "$\mu_* \leq \mu^*$ and $\mu^*(A) = 0$ implies $\tau(A) = 0$". To see this, let $\Omega = [0,1]$, $X = \mathbb{N}$, λ the Lebesgue measure of Ω, $\mathcal{Z} = \{Z : \lambda^*(Z) = 0\}$, Σ_n the σ-algebra generated by the interval $[\frac{k-1}{2^n}, \frac{k}{2^n}) \subset \Omega$ and $\mathcal{Z}$ and μ_n the restriction of λ to Σ_n. Then, λ is an extension of the measures μ_n, $\mu^* = \lambda^*$ holds and $\tau(Z) = 0$ for every $Z \in \mathcal{Z}$. Therefore, $\mu^*(A) = 0$ implies $\tau(A) = 0$ and, yet, $\mu^* \neq \tau$, because, if A is an open dense set in Ω with Lebesgue measure $\mu^*(A) < 1$ we have $\tau(A) = \mu_n^*(A) = 1$.

(2). $\mu_* \leq \mu^*$ does not follow from "$\mu^*(A) = 0$ implies $\tau(A) = 0$". To see this, let $\Omega = \mathbb{N}$, $X = \mathbb{N}$, let Σ_n be the σ-algebra generated by the subsets of $M_n = \{1, 2, \ldots, n\}$ and (c_k) a positive sequence such that $\sum_k c_k < 1$. Let $\mu_n(A) = \sum_{k \in A} c_k$ when $A \subset M_n$ and $\mu_n(A) = 1 - \sum_{k \notin A} c_k$ when the complementary set $A^c \subset M_n$. Then, $\tau(A) = \sum_{k \in A} c_k$ when $A \subset \mathbb{N}$ is a finite set and $\tau(A) = 1 - \sum_{k \notin A} c_k$ when $A \subset N$ is an infinite set and $\mu^*(A) = \sum_{k \in A} c_k$ for every $A \subset \mathbb{N}$. Then, $\mu^*(A) = 0$ implies $A = \emptyset$ and $\tau(A) = 0$, but $\mu^*(\mathbb{N}) < 1 = \mu_*(\mathbb{N})$. In this case μ^* is regular, since it is even a measure.

(3). If, for every sequence (x_n) in X there exists an $x \in X$ such that $x_n \leq x$ for every $n \in \mathbb{N}$, then it follows from Proposition 7 that $\mu^* = \tau$. Moreover $\sigma(\mathcal{S}) = \mathcal{S}_\sigma = \mathcal{S}$, when $\sigma(\mathcal{S})$ is the σ-algebra generated by $\mathcal{S}$.

Proposition 8 *μ^* is a regular outer measure.*

PROOF. For every set $A \subset \Omega$ and for every $\epsilon > 0$ there exists $x \in X$ such that $\mu_x^*(A) < \tau(A) + \epsilon$. Moreover, there exists $B \in \Sigma_x$ such that $\mu_x^*(B) = \mu_x^*(A)$ and $A \subset B$, so $\tau(B) < \tau(A) + \epsilon$. Then, for every sequence (A_n) of subsets of Ω there exists a sequence (B_n) in $\mathcal{S}$ such that $\tau(B_n) < \tau(A_n) + \frac{\epsilon}{2^n}$ and $A_n \subset B_n$. Therefore

$$\mu^*(A) = \inf\{\sum_n \tau(A_n) : \cup_n A_n \supset A\} \geq$$

$$\geq \inf\{\sum_n \tau(B_n) : \cup_n B_n \supset A, B_n \in \mathcal{S}\} - \epsilon \geq$$

$$\geq \inf\{\sum_n \mu^*(B_n) : \cup_n B_n \supset A, B_n \in \mathcal{S}\} - \epsilon \geq$$

$$\geq \inf\{\mu^*(\cup_n B_n) : \cup_n B_n \supset A, B_n \in \mathcal{S}\} - \epsilon =$$

$$\geq \inf\{\mu^*(H) : A \subset H \in \mathcal{S}_\sigma\} - \epsilon \geq \mu^*(A) - \epsilon.$$

It follows immediately that

$$\mu^*(A) = \inf\{\sum_n \tau(B_n) : \cup_n B_n \supset A, B_n \in \mathcal{S}\} = \inf\{\mu^*(H) : A \subset H \in \mathcal{S}_\sigma\}$$

and now Proposition 3 implies that μ^* is regular. In this way we complete the result obtained in the proof of Proposition 1. ■

Remark 2 It follows from this last proposition that, if λ is the restriction of μ^* to the σ-algebra of the μ^*-measurable sets, then $\mu^* = \lambda^*$, but it can happen that $\mu_* \neq \lambda_*$.

Proposition 9 *Let us consider the following conditions:*

(i) For every decreasing sequence $(A_n) \searrow \emptyset$ of subsets of Ω we have

$$\lim_n \mu_0^*(A_n) = 0.$$

(ii) For every disjoint sequence $(A_n) \searrow \emptyset$ of subsets of Ω we have

$$\lim_n \mu_0^*(A_n) = 0.$$

(iii) For every disjoint sequence $(A_n) \searrow \emptyset$ of subsets of Ω we have

$$\sum_n \mu_0^*(A_n) < \infty.$$

Then, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

PROOF. (iii) $\Rightarrow$ (ii). It is obvious.

(ii) $\Rightarrow$ (i). Let (A_n) be a decreasing sequence such that $\lim_n A_n = \emptyset$ and $\mu_0^*(A_n) > \epsilon > 0$ for every $n \in \mathbb{N}$. The outer regularity of μ_0^* implies that $\lim_k \mu_0^*(A_n \setminus A_k) = \mu_0^*(A_n) > \epsilon$ and, therefore, for every $n \in \mathbb{N}$ there exists $k > n$ such that $\mu_0^*(A_n \setminus A_k) > \epsilon$ and it is obvious that using this result we can construct a disjoint sequence (B_n) such that $\mu_0^*(B_n) > \epsilon$ and $\inf_n \mu_0^*(B_n) \geq \epsilon > 0$.

Proposition 10 *In the conditions of Proposition 9, we can state in Proposition 1 that $\mu^* = \tau$ and, hence, there exists a measure μ which is an extension of all the measures μ_x, $x \in X$*

PROOF. From condition (i) in Proposition 9, it follows that $(A_n) \searrow \emptyset$ in $\mathcal{S}_\sigma$ implies that $\tau(A_n) \to \emptyset$ and, hence, $\mu^* = \tau$, according to Proposition 7. ■

Proposition 11 *If (Ω, Σ, μ) is a maximal probability space, in the sense that every extension λ of μ coincides with μ, then Σ is the σ-algebra $\mathcal{P}(\Omega)$ of all the subsets of Ω.*

PROOF. If $\mu^*(A) = 1$, then $\nu^*(E) = \mu^*(A \cap E)$ $(E \subset \Omega)$ is an outer measure such that its associated measure ν is an extension of μ and A is a ν-measurable set. From the maximality of μ, it follows that $A \in \Sigma$. If $\mu^*(A) < 1$ and B is a μ-measurable covering of A, then $\mu^*(A \cup (\Omega \setminus B)) = 1$ and, therefore, $A \cup (\Omega \setminus B) \in \Sigma$ and $A \in \Sigma$. It follows that $\Sigma = \mathcal{P}(\Omega)$ and the proposition is proved. ■

Corollary 1 *If $(\Omega, \Sigma_0, \mu_0)$ verifies one of the conditions of Proposition 9, then there exists a probability space (Ω, Σ, μ) which extends $(\Omega, \Sigma_0, \mu_0)$ and such that Σ is the σ-algebra $\mathcal{P}(\Omega)$.*

PROOF. It follows from Zorn's Lemma and Propositions 10 and 11: Note that if $(\Omega, \Sigma_x, \mu_x)_{x \in X}$ is a totally ordered family of extensions of $(\Omega, \Sigma_0, \mu_0)$, Proposition 10 states the existence of a measure which is an extension of all the measures μ_x. The corollary follows now from Proposition 11. ■

Remark 3

(1) Corollary 1 can also be proved using the Theorem of Hahn-Banach: By this theorem, there exists a finitely additive extension μ of μ_0 to $\Sigma = \mathcal{P}(\Omega)$. But $\lim_n \mu_0^*(A_n) = 0$ for every sequence $A_n \searrow \emptyset$, so $\lim_n \mu(A_n) = 0$ and, hence, μ is countably additive.

(2) If Card(Ω) has zero measure and (Ω, Σ, μ_0) is a fuzzy probability space, then there exists a disjoint sequence (A_n) such that $\inf \mu_0^*(A_n) > 0$.

Remark 4 **Remark.** The techniques used allow us to construct, using the good order of $\mathcal{P}(\Omega)$, a well ordered family $(\Omega, \Sigma_\alpha, \mu_\alpha)_{\alpha \in A}$ (indexed by ordinal numbers) of finite and different measure spaces and another family $(\mu_\alpha^*)_{\alpha \in A}$ of outer measures such that:

(i) μ_α is the restriction of μ_α^* to the σ-algebra Σ_α of the μ_α^*-measurable sets, and every μ_α^* is regular.

(ii) If $\alpha < \beta$, then $\Sigma_\alpha \subset \Sigma_\beta$ and $\mu^*_\alpha \geq \mu^*_\beta$.
(iii) The first measure μ_0 is an arbitrary complete finite measure and $\mu_{\alpha+1}$ is an extension of μ_α.
(iv) If $\mu^*_\alpha(\Omega) = \mu^*_\beta(\Omega)$ and $\alpha < \beta$, then μ_β is an extension of μ_α.
(v) If $\mu^*_\alpha(\Omega) > \mu^*_\beta(\Omega)$ for every $\alpha < \beta$, then μ^*_β is built based on the measures $(\mu_\alpha)_{\alpha<\beta}$ as in Proposition 1. This technique can also be used in general when β is a limit ordinal. For the other ordinals $\beta > 0$ with a predecessor $\beta - 1$, we can use the method of Proposition 11 so that that $\mu^*_\beta(A) = \mu^*_{\beta-1}(A \cap E)$ with $\mu^*_{\beta-1}(E) = \mu^*_{\beta-1}(\Omega)$ and $E \notin \Sigma_{\beta-1}$, where E is the first set in the given well ordering of $\mathcal{P}(\Omega)$ with such properties, in case it exists.
(vi) The family $(\mu_\alpha)_{\alpha\in A}$ has a last measure μ defined on $\mathcal{P}(\Omega)$.

If $(\Omega, \Sigma_0, \mu_0)$ is a fuzzy probability space and Card(Ω) has zero measure, then $\mu = 0$.

A measure μ is called *ultracomplete* if it is defined on $\mathcal{P}(\Omega)$.

It there is no extension of the Lebesgue measure λ to all the subsets of $[0, 1]$, then there is no non atomic ultracomplete probability measure μ. To see this, let us suppose that there is one such measure μ. Then, there exists an increasing family $(E_x)_{x\in X}$, $X = [0, 1]$, such that $\mu(E_x) = x$. Let $Z_x = E_x \setminus \cup_{y<x} E_y$. Then, it can be easily proved that

$$\nu(A) = \mu(\cup_{x\in A} Z_x)$$

is a measure on $\mathcal{P}([0, 1])$ which is an extension of the Lebesgue measure λ, because

$$\nu((a, b)) = \mu(\cup_{x\in(a,b)} Z_x) = \mu(E_b \setminus E_a) = b - a.$$

It can be easily proved that there exists an ultracomplete extension of the Lebesgue measure λ if and only if $c = 2^{\aleph_0}$ is real-measurable. Let us recall that, according to a result of Ulam ([5]), it follows from the Continuum Hypothesis that c has zero measure.

It follows that, if c has zero measure, then the last measure μ of the previous process is a purely atomic measure and, therefore, μ_0 is purely atomic if μ is an extension of μ_0, as follows from:

Proposition 12 *If μ is purely atomic extension of μ_0, then μ_0 is also purely atomic.*

PROOF. It is enough to prove that, if μ_0 is not atomic, then μ is not atomic. To see this, let us suppose that μ_0 is not atomic and let A be an atom of μ. Then, there exists an increasing family $(A_x)_{x\in I}$, $I = [0, 1]$, of μ_0-measurable sets such that $\mu_0(A_x) = x$. Then, the function $f(x) = \mu(A \cap A_x)$ is a continuous function taking only the values 0 and $\mu(A) \neq 0$. The last part of the proof remains true when μ is the measure associated to μ^* with the notation of Proposition 1 or when μ is the last measure of the previous process. ■

Proposition 13 *If there exists an purely atomic ultracomplete extension μ of the probability measure μ_0, then there exists a process which finishes in a purely atomic measure ν which is an ultracomplete extension of μ_0 and which has a disjoint and complete system of atoms formed by atoms of μ.*

PROOF. Let (A_n) be a disjoint sequence of atoms of μ such that $\cup_n A_n = \Omega$. We proceed by transfinite induction. Let us suppose that we have defined $(\mu_\alpha)_{\alpha<\beta}$ so that μ is an extension of μ_α for every $\alpha < \beta$. If β is a limit ordinal, it is clear that μ is an extension of μ_β. Let us suppose that β has a predecessor $\beta - 1$. Then, if there exists $E \notin \Sigma_{\beta-1}$ such that $\mu(E) = 1$, then $\mu^*_{\beta-1}(E) = 1$ and we take $\mu^*_\beta(A) = \mu^*_{\beta-1}(A \cap E) \geq \mu(A \cap E) = \mu(A)$ and, therefore, μ is an extension of μ_β. This first part of the process, where we use the given good order of $\mathcal{P}(\Omega)$, finishes in an ordinal β such that $\mu_\beta(A) = 0$ for every A such that $\mu(A) = 0$.

Since μ is an extension of μ_β, it follows from Proposition 12 that μ_β is purely atomic, but it can happen that all the sets A_n are not atoms of μ_β. Let (B_n) be a disjoint and complete system of atoms of μ_β. By the previous property, we can suppose all of the B_n to be union of some of the A_k. To see this, let $M_k = \{h : \mu(A_h \cap B_k) \neq 0\}$ and let $Z_k = \cup_{h\neq M_k} A_h \cap B_k$. Then $\mu(Z_k) = 0$. Therefore, if $B'_k = B_k \setminus Z_k$, then B'_k is an atom of μ_β and we have

$$\mu(\cup_{h\in M_k} A_h \setminus B'_k) = \sum_{h\in M_k} \mu(A_h \setminus A_h \cup B'_k) = 0.$$

Hence, $B_k = \cup_{h\in M_k} A_h$ is also an atom of μ_β. It can be easily proved that the sequence (B_k) is disjoint and its union is Ω.

If $A_{n_1} \subset B_1$, then B_1 is μ_β-measurable covering of A_{n_1}. Therefore, if we call $E_1 = A_{n_1} \cup (\Omega \setminus B_1)$, we have $\mu^*_\beta(E_1) = 1$, and we can define $\mu^*_{\beta+1}(A) = \mu^*_\beta(A \cap E_1)$, so that $\mu_{\beta+1}$ is an extension of μ_β and $\mu_{\beta+1}(A_{n_k}) = \mu_\beta(B_1)$ and $\mu^*_{\beta+1}(A_k) = \mu^*_\beta(A_k) = \mu_\beta(B_n)$ when $A_k \subset B_n$ and $n > 1$. Now we can repeat the process taking $A_{n_2} \subset B_2$, $E_2 = A_{n_2} \cup (\Omega \setminus B_2)$ and $\mu^*_{\beta+2}(A) = \mu^*_{\beta+1}(A \cap E_2)$. Then we get that $\mu_{\beta+2}$ is an extension of $\mu_{\beta+1}$ and $\mu_{\beta+2}(A_{n_k}) = \mu_\beta(B_k)$ for every $k \le 2$ and $\mu^*_{\beta+2}(A_k) = \mu^*_\beta(A_k) = \mu_\beta(B_n)$ when $A_k \subset B_n$ and $n > 2$. With the same reasonings we can construct $\mu_{\beta+h}$ with the property that it is an extension of $\mu_{\beta+h-1}$ and it verifies $\mu_{\beta+h}(A_{n_k}) = \mu_\beta(B_k)$ for every $k \le h$ and

$\mu^*_{\beta+h}(A_k) = \mu^*_\beta(A_k) = \mu_\beta(B_n)$ when $A_k \subset B_n$ and $n > h$. Every A_{n_k} is an atom of $\mu_{\beta+k}$ and every subset E of A_{n_k} is $\mu_{\beta+k}$-measurable. To see this, let us note that either $\mu(E) = 0$ or $\mu(A_{n_k} \setminus E) = 0$ and, therefore, either $\mu_{\beta+k}(E) = 0$ or $\mu_{\beta+k}(A_{n_k} \setminus E) = 0$, which proves the previous statement, since A_{n_k} is a $\mu_{\beta+k}$-measurable set.

This process can finish in a measure $\mu_{\beta+n}$; also, the measures $\mu_{\beta+k}$ can be equal, but both difficulties can be overcome, and, in the worst case, we can suppose that the process does not finish like that. Then, using the same notation as in Proposition 1, if $E_k \subset A_{n_k}$ and $\tau(E_k) = \tau(A_{n_k}) = \tau(B_k)$, we have

$$\tau(\cup_{k\in M} E_k) \le \tau(\cup_{k\in M} B_k) = \mu_\beta(\cup_{k\in M} B_k) =$$

$$= \sum_{k\in M} \mu_\beta(B_k) = \sum_{k\in M} \tau(E_k).$$

It follows immediately that

$$\tau(\cup_{k\in M} E_k) = \sum_{k\in M} \tau(E_k)$$

and

$$\tau(\cup_{k\in\mathbb{N}} A_{n_k}) = \sum_{k\in\mathbb{N}} \tau(A_{n_k}) = \sum_{k\in\mathbb{N}} \mu_\beta(B_k) = 1.$$

Moreover, if $E_k \subset A_{n_k}$ and $\tau(E_k) = 0$, then $\mu(E_k) = 0$ and, therefore,

$$\tau(\cup_{k\in M} E_k) = \mu(\cup_{k\in M} E_k) = 0.$$

Moreover, $\tau(\Omega \setminus \cup_k A_{n_k}) = 0$, because, for every h,

$$\mu^*_{\beta+h}(\Omega \setminus \cup_k A_{n_k}) \le \mu_{\beta+h}(\Omega \setminus \cup_{k\le h} A_{n_k}) = \mu_\beta(\Omega \setminus \cup_{k\le h} B_k).$$

Hence, τ is an ultracomplete measure and $\mu_{\beta+\omega} = \tau$, and we can take $\nu = \tau$. ∎

Remark 5 It can be proved that the last measure μ_β is independent of the well order of $\mathcal{P}(\Omega)$ that we choose. Indeed, if μ'_γ is the analogous measure corresponding to another well order of $\mathcal{P}(\Omega)$, and we suppose that μ'_γ is an extension of μ_α for every $\alpha < \alpha_0$ ($\alpha_0 \le \beta$) we get that μ'_γ is an extension of μ_{α_0}. This is clear if α_0 is a limit ordinal, and if α_0 has a predecessor $\alpha_0 - 1$, we have $\mu^*_{\alpha_0}(A) = \mu^*_{\alpha_0-1}(A \cap E)$, where $\mu(E) = 1$. Then, $\mu^*_{\alpha_0}(A) \ge \mu'^*_\gamma(A \cap E) = \mu'^*_\gamma(A)$ because $\mu'_\gamma(E) = 1$ and, therefore, μ'_γ is an extension of μ_{α_0}. It follows that μ'_γ is an extension of μ_β. Similarly μ_β is an extension of μ'_γ and, hence, $\mu_\beta = \mu'_\gamma$. The same proof shows that Σ_β is the σ-algebra Σ generated by Σ_0 and the sets of null μ-measure. Therefore, μ_β is the restriction of μ to Σ. Now it is easy to prove that if μ_β is different to μ, then no process starting in μ_0 will finish in μ. So, if a process starting in μ_0 finishes in μ, then $\mu_\beta = \mu$. All of this remains true even if μ is not purely atomic.

Proposition 14 *If μ_0 is a probability measure and there exists a process starting in μ_0 that finishes in an extension μ of μ_0, then every atom of μ_0 is an atom of μ.*

PROOF. Let us suppose that Ω is an atom of μ_0. Let $(\Omega, \Sigma_\alpha, \mu_\alpha)_{\alpha \leq \beta}$ be the family of probability spaces of the process and let Σ'_α be the σ-algebra of the sets $A \in \Sigma_\alpha$ with measure $\mu_\alpha(A) \in \{0, 1\}$. Then, $\Sigma'_\alpha = \Sigma_\alpha$ for every $\alpha \leq \beta$. Clearly $\Sigma'_0 = \Sigma_0$ and, if $\Sigma'_\alpha = \Sigma_\alpha$ for every $\alpha < \alpha_0$ ($\alpha_0 \leq \beta$) then $\Sigma'_{\alpha_0} = \Sigma_{\alpha_0}$. Indeed, if α_0 has a predecessor $\alpha_0 - 1$ then $\mu^*_{\alpha_0}(A) = \mu^*_{\alpha_0 - 1}(A \cap E)$, with $\mu^*_{\alpha_0-1}(E) = 1$ and, therefore, $\Sigma'_{\alpha_0} = \Sigma_{\alpha_0}$ because $\mu^*_{\alpha_0}$ takes values in $\{0, 1\}$. If α_0 is a limit ordinal then we also have $\Sigma'_{\alpha_0} = \Sigma_{\alpha_0}$ because $\mathcal{S} = \cup_{\alpha < \alpha_0} \Sigma_\alpha \subset \Sigma'_{\alpha_0} \subset \Sigma_{\alpha_0}$ and Σ_{α_0} is the σ-algebra generated by $\mathcal{S}$ (this follows from Proposition 1). Hence, $\mu = \mu^*_\beta$ takes values in $\{0, 1\}$ and it follows that Ω is an atom of μ. It is clear that the proposition follows now, because, if A is a μ_0-measurable set and there exists a process starting in μ_0 which finishes in μ, then there exists a process starting on the induced measure μ_{0A} which finishes in μ_A. ■

Corollary 2 *If μ_0 is a purely atomic probability measure, then there exists a process starting in μ_0 and finishing in an extension μ of μ_0 if and only if there exists an purely atomic ultracomplete extension of μ_0.*

Proposition 15 *Let μ_0 be a probability measure, μ an ultracomplete extension of μ, Σ the σ-algebra generated by Σ_0 and the sets with null ν-measure and let ν be the restriction of μ to Σ. Then there exists a process starting in μ_0 and finishing in μ if and only if $\nu = \mu$.*

PROOF. It is enough to use the Remark following Proposition 13. ■

Proposition 16 *If μ_0 is a non atomic probability measure and there exists a process finishing in a measure $\mu \neq 0$, then the cardinal $c = 2^{\aleph_0}$ is real measurable and, therefore, if follows from the Continuum Hypothesis that $\mu = 0$, according to [5].*

PROOF. As we have seen in Proposition 12, μ is a non atomic measure and, therefore, there is an ultracomplete extension of the Lebesgue measure on the interval $[0, \mu(\Omega)]$ and c is a real-measurable cardinal. ■

Proposition 17 *If μ_0 is a non atomic probability measure, built using only (ZF) and the axiom of choice, like the Lebesgue measures, and there exists a process starting in μ_0 and finishing in μ, then $\mu = 0$.*

PROOF. First of all, the Continuum Hypothesis is independent of (ZF) and the axiom of choice, according to the well known result of P. J. Cohen. On the other hand, in the construction of μ and in Proposition 18 we have only used (ZF) and the axiom of choice. Then, if $\mu \neq 0$, the negation of (CH) would follow from (ZF) and the axiom of choice, in contradiction to the result Cohen. Therefore $\mu = 0$. ■

Remark 6 If c is a real-measurable cardinal we can not avoid in the previous proposition the condition μ_0 is built using only (ZF) and the axiom of choice. Indeed, then there exists an ultracomplete non atomic probability measure μ_0 and, obviously, every process starting in μ_0 finishes in $\mu = \mu_0 \neq 0$.

Proposition 18 *If μ_0 is a purely atomic probability measure and there exists a process finishing in μ then, for every atom A of μ_0 we have $\mu(A) = \mu_0(A)$ or $\mu(A) = 0$. In the first case, the induced measure μ_A is an extension of μ_{0A} and, therefore, A is also an atom of μ.*

PROOF. We notice that there exists a process starting in the induced measure μ_{0A} and finishing in μ_A, as in Proposition 14. So, it is enough to consider the case when Ω is an atom of μ_0. Let Ω be an atom of μ_0 and $(\mu_\alpha)_{\alpha \leq \gamma}$ the family of the measures of the process, and let us suppose that μ is not an extension of μ_0. Then there exists a first ordinal α_0 such that $\mu_{\alpha 0}$ is not an extension of μ_0. It is clear that α_0 is a limit ordinal. It follows from the proof of Proposition 14 that $\mu^*_\alpha(E) \in \{0, 1\}$ for every $E \subset \Omega$ and $\alpha < \alpha_0$. It follows also that $\tau(E) = \inf_{\alpha < \alpha_0} \mu^*_\alpha(E) \in \{0, 1\}$ and, therefore, $\mu^*_{\alpha_0}(E) \in \{0, 1\}$. Since μ_{α_0} is not an extension of μ_0 it follows that $\mu_{\alpha_0}(\Omega) = 0$ and $\mu(\Omega) = 0$. ■

Corollary 3 *If μ_0 is a purely atomic probability measure and (A_n) is a complete and disjoint system of atoms of μ_0 and $\{A_n : n \in \mathbb{M}\}$ is the set of the A_n which have the property that the induced measure μ_{0A_n} has an ultracomplete extension μ_n, then there exists a process starting in μ_0 and that finishes in the measure $\mu(A) = \sum_{n \in M} \mu_n(A \cap A_n)$.*

PROOF. It follows from Propositions 13, 14 and 18, taking into account that if B_1 and B_2 are two disjoint μ_0-measurable sets and there exists a process starting in μ_{0B_i} and finishing in ν_i, then there exists a process starting in $\mu_{0(B_1 \cup B_2)}$ and finishing in the measure $\nu(A) = \nu_1(A \cap B_1) + \nu_2(A \cap B_2)$, where $A \subset B_1 \cup B_2$. ■

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