RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **95 (**2), 2001, pp. 171–180
Análisis Matemático / Mathematical Analysis

Biduality in (LF)-spaces

K. D. Bierstedt and J. Bonet

Dedicated to Professor Manuel Valdivia on the occasion of his 70th birthday

Abstract. In Section 1, abstract results on preduals and on the biduality of (LF)-spaces are proved. Let $E = \mathrm{ind}_n E_n$ denote an (LF)-space and put $H = \mathrm{ind}_n H_n$ for a sequence of Fréchet subspaces H_n of E_n with $H_n \subset H_{n+1}$. We investigate under which conditions E is canonically (topologically isomorphic to) the inductive bidual $(H'_b)'_i$ or (even) the strong bidual of H. The abstract results are applied in Section 2, mainly to weighted (LF)-spaces of holomorphic functions, but also to two other examples.

Bidualidad en Espacios (LF)

Resumen. En la Sección 1 se prueban resultados abstractos sobre preduales y sobre bidualidad de espacios (LF). Sea $E = \mathrm{ind}_n E_n$ un espacio (LF), ponemos $H = \mathrm{ind}_n H_n$ para una sucesión de subespacios de Fréchet H_n de E_n con $H_n \subset H_{n+1}$. Investigamos bajo qué condiciones el espacio E es canónicamente (topológicamente isomorfo a) el bidual inductivo $(H'_b)'_i$ o (incluso) al bidual fuerte de H. Los resultados abstractos se aplican en la Sección 2, especialmente a espacios (LF) ponderados de funciones holomorfas, pero también a otros ejemplos.

The line of research on biduality which led to a series of papers of which this is the most recent one, started in 1986 with a concrete question of J. Duncan on certain spaces of entire functions in one complex variable. The first article [8] in the series mainly stayed in the framework of weighted Banach spaces of holomorphic functions, but developed a functional analytic approach which could be generalized. In [3], the present authors started to study biduality from a more abstract point of view, both in Fréchet and (LB)-spaces. And while it soon became clear that there existed other applications for the abstract results, the main, motivating examples still were weighted Fréchet and (LB)-spaces of holomorphic functions. Our studies continued in Section 1 of [5], in which some of the results of [3] could be improved.

In the meantime, two important new developments provided a better understanding of the structure of (LF)-spaces: In [21], [22], D. Vogt investigated regularity properties and acyclicity in (LF)-spaces in the spirit of Palamodov and of Retakh's condition (M), as well as their consequences for Köthe (LF)-sequence spaces; his results had interesting applications to solution operators for linear partial differential equations. On the other hand, J. Wengenroth [23], [24] proved the rather surprising theorem that several strong regularity conditions like sequential retractivity and bounded retractivity coincide with Retakh's condition (M) for arbitrary (LF)-spaces.

These developments renewed the interest in (LF)-function spaces. As a first step towards a thorough investigation of weighted (LF)-spaces of holomorphic functions, we extended Vogt's results from Köthe

Presentado por Manuel Valdivia.
Recibido: 14 de Septiembre 2001. *Aceptado*: 20 de Octubre 2001.
Palabras clave / Keywords: biduality, (LF)-spaces, spaces of holomorphic functions, weighted inductive limits.
Mathematics Subject Classifications: MSC 2000, primary: 46A13, secondary: 46A04, 46A20, 46A45, 46E10, 46E50, 46M40.

(LF)-sequence spaces to weighted (LF)-spaces of continuous functions in [4], often providing different proofs by further developing and using "projective description". As a next step in the program, it is natural to return to biduality questions and to see how far our former results can be utilized in the more general context of (LF)-spaces. As we had expected, some difficulties appear at this stage, and the situation is not as 'easy' as it had been in the (LB)-case. But the present results do indeed yield (as we believe) rather convincing applications to weighted (LF)-spaces of holomorphic functions, and this is their main justification. – We should like to mention at this point that many important questions on projective description for weighted (LF)-spaces of holomorphic functions have remained open so far, and we hope to return to this matter some time in the not too distant future.

The organization of the article is as follows: In Section 1, the abstract results are presented. The general framework is introduced in subsection (a). Theorem 1 in subsection (b) shows that (LF)-spaces satisfying certain natural conditions are inductive duals of reduced projective limits of sequences of complete barrelled (DF)-spaces. Corollary 1 in subsection (c) is our main result on biduality in (LF)-spaces, in the setting mentioned in the abstract. Subsection (d) deals with the question when the larger space E is not only the inductive, but even the strong bidual of its subspace H.

In part A. of Section 2, the results of Section 1 are applied in the context of weighted (LF)-spaces of holomorphic functions. At the end of the paper, as further examples, some consequences for spaces of entire functions of uniformly bounded type on a complex Fréchet space and for Köthe (LF)-sequence spaces are added. – The present article exploits ideas, methods, and results from our former work on biduality, and this makes some of the proofs appear shorter than they would really be if all the details were given.

Much of the terminology relevant in the context of (LF)-spaces will be explained as we go along, at the place where it is needed, but sometimes only references will be given. For background on general locally convex spaces we refer the reader to [20]; many notions and results on inductive limits can be found in the survey article [2]. For weighted (LF)-spaces of continuous and holomorphic functions see [4]; in particular, we refer to this article for the definitions and the concept of projective description.

1. Abstract results

(a) General setting

Let $(E,\tau) = \mathrm{ind}_n(E_n,\tau_n)$ be an (LF)-space; i.e., the inductive limit of an increasing sequence of Fréchet spaces E_n, where we assume that the inductive limit topology τ is Hausdorff. We denote by $i_n : E_n \to E$ and $i_{n,n+1} : E_n \to E_{n+1}$ the canonical inclusions. For each n let $(U_{n,k})_k$ be a basis of closed absolutely convex 0-neighborhoods in E_n. We assume without loss of generality that

$$U_{n,k+1} \subset U_{n,k} \subset U_{n+1,k} \text{ for each } n,k \in I\!N.$$

Moreover, we suppose that the following condition holds:

(BBC) *There is a locally convex (necessarily coarser) Hausdorff topology* $\tilde{\tau}$ *on* E *such that every bounded subset* B *of* (E,τ) *is contained in an absolutely convex bounded subset* C *of* (E,τ) *which is* $\tilde{\tau}$*-compact.*

By Grothendieck's factorization theorem this implies that $\mathrm{ind}_n E_n$ is regular. We denote by $\mathcal{B}$ a basis of $\tilde{\tau}$-compact absolutely convex bounded subsets of (E,τ). For $n \in I\!N$ we put $\tilde{\tau}_n := \tilde{\tau}|E_n$. Finally, we assume that the following two conditions hold for each n:

$(BBC)_n$ *Every bounded subset of* (E_n,τ_n) *is contained in an absolutely convex bounded subset of* (E_n,τ_n) *which is* $\tilde{\tau}_n$*-compact.*

$(CNC)_n$ *For each* k *the 0-neighborhood* $U_{n,k}$ *of* E_n *is* $\tilde{\tau}_n$*-closed.*

Observe that if $(BBC)_n$ is satisfied for each n, then (BBC) holds if and only if $\mathrm{ind}_n E_n$ is regular.

(b) Predual

Our first aim is to construct a predual F of (E,τ) which can be represented as a projective limit of the preduals of the spaces (E_n,τ_n). To do this we define F as the space which consists of all the elements $u \in E'$ such that the restriction of u to every bounded subset of (E,τ) is $\tilde{\tau}$-continuous. The space F is endowed with the Hausdorff locally convex topology of uniform convergence on the elements of $\mathcal{B}$, and for $B \in \mathcal{B}$, we denote by p_B the seminorm defined by $p_B(u) := \sup_{x\in B}|u(x)|$ for each $u \in F$. In fact, F is a topological subspace of E'_b . By Mujica's theorem (cf. [18], [19]) in the form stated in [3], F is a complete locally convex space, and the evaluation map $J : E \to F'$, $J(x)(u) := u(x)$ for $x \in E$, $u \in F$, is a topological isomorphism from E onto the inductive dual $F'_i := \mathrm{ind}_U (F')_{U^\circ}$ of F, where U runs through a basis of 0-neighborhoods in E.

We fix n and denote by $\mathcal{B}_n$ a basis of absolutely convex $\tilde{\tau}_n$-compact (hence τ_n-bounded) subsets of E_n. We define F_n as the space of all $u \in E'_n$ the restriction of which to every bounded subset of (E_n,τ_n) is $\tilde{\tau}$-continuous, and we endow it with the locally convex topology of uniform convergence on the elements of $\mathcal{B}_n$; thus, F_n is a topological subspace of the strong dual $(E_n)'_b$ of (E_n,τ_n). By [3], the evaluation map $J_n : E_n \to F'_n$, defined by $J_n(x)(u) := u(x)$ for all $x \in E_n$, $u \in F_n$, is a topological isomorphism of (E_n,τ_n) onto $(F_n)'_b$; moreover, F_n is a complete barrelled (DF)-space, which is even a boundedly retractive (LB)-space (hence bornological) if (E_n,τ_n) is quasinormable (see [3, 5.(d)]).

The maps $(i_n)^t : E' \to (E_n)'$ and $(i_{n,n+1})^t : (E_{n+1})' \to (E_n)'$ are strongly continuous with $(i_n)^t(F) \subset F_n$ and $(i_{n,n+1})^t(F_{n+1}) \subset F_n$. The continuous restrictions of $(i_n)^t$ to F and of $(i_{n,n+1})^t$ to F_{n+1} are denoted by $\pi_n : F \to F_n$ and $\pi_{n,n+1} : F_{n+1} \to F_n$, respectively. Then the following equalities hold:

(a) $\pi_n = \pi_{n,n+1}\pi_{n+1}$ on F,
(b) $Ji_n = (\pi_n)^t J_n$ on E_n,
(c) $J_{n+1}i_{n,n+1} = (\pi_{n,n+1})^t J_n$ on E_n.

As an example, we check (b) and fix $x \in E_n$. For all $u \in F$ we have

$$\begin{aligned}[(\pi_n)^t(J_n(x))](u) &= [J_n(x)](\pi_n(u)) = [\pi_n(u)](x)\\ &= [(i_n)^t(u)](x) = u(i_n(x)) = [J(i_n(x))](u).\end{aligned}$$

In particular, (b) and (c) imply that π_n and $\pi_{n,n+1}$ have dense range.

We denote by $\tilde{F}$ the projective limit $\mathrm{proj}_n(F_n,\pi_{n,n+1})$ which, by definition, equals

$$\{(u_n)_n;\ u_n \in F_n \text{ and } \pi_{n,n+1}(u_{n+1}) = u_n \text{ for each } n\},$$

endowed with the topology induced by the product of the spaces $(F_n)_n$. We prove that the map

$$\pi : F \to \tilde{F},\ \pi(u) = (\pi_n(u))_n \text{ for } u \in F,$$

is a topological isomorphism, and hence F and $\tilde{F}$ coincide canonically. In fact, it is easy to see that π is well-defined, linear and continuous. On the other hand, if $u \in F$ satisfies $\pi_n(u) = 0$ for each n, its restriction to each E_n is 0, hence $u = 0$, and π is injective. To see that π is surjective, we take $v = (v_n)_n \in \tilde{F}$ and define u on E by setting $u(x) = v_n(x)$ if $x \in E_n$. Clearly u is well-defined, continuous on (E,τ), an element of F by the regularity of $\mathrm{ind}_n\, E_n$, and $\pi(u) = v$ holds. We finally show that π is open. Given a bounded set B in (E,τ), we find n such that B is bounded in (E_n,τ_n). If $v = (v_n)_n \in \tilde{F}$ satisfies $p_B(v_n) \le 1$, then $p_B(\pi^{-1}(v)) \le 1$ since $[\pi^{-1}(v)](x) = v_n(x)$ for all $x \in B$. This and [3, 5.(d)] complete the proof of the following result.

Theorem 1 *(a) If $(E,\tau) = \mathrm{ind}_n(E_n,\tau_n)$ is an (LF)-space that satisfies (BBC), $(BBC)_n$ and $(CNC)_n$, then there is a reduced projective sequence $(F_n,\pi_{n,n+1})_n$ of complete barrelled (DF)-spaces such that E coincides canonically with the inductive dual F'_i of the projective limit $F = \mathrm{proj}_n F_n$.*

(b) If the Fréchet spaces (E_n, τ_n) are quasinormable, then the spaces F_n are boundedly retractive (LB)-spaces.

Note that by [3, Corollary 2.(b)] the condition

(CNC) *τ has a 0-neighborhood base of absolutely convex $\tilde{\tau}$-closed sets*

implies that F'_i equals the strong dual F'_b, and hence that E is topologically isomorphic to F'_b. By [3, Remark 4.], it is also known that $E = F'_b$ holds if $E = \operatorname{ind}_n E_n$ is a boundedly retractive (LB)-space. However, a similar result is not true for arbitrary (LF)-spaces; see the discussion in subsection (d) below.

(c) Biduality

We introduce the following additional setting to study biduality. For each n we denote by H_n a closed subspace of (E_n, τ_n), and we assume that $H_n \subset H_{n+1}$ for each n. We put $H := (H, \tau') := \operatorname{ind}_n(H_n, \tau_n)$. Clearly (H, τ') is a Hausdorff (LF)-space which is continuously injected in (E, τ), but we do not assume a priori that it is a topological subspace of (E, τ) . Let us denote by $R : F \to H'_b$ the restriction map. If i_H is the injection from H into E, R coincides with the restriction to F of the transpose $(i_H)^t : E'_b \to H'_b$. Therefore $R : F \to H'_b$ is well-defined, linear and continuous. We investigate when R is a topological isomorphism onto; in case this is satisfied, we have $E = (H'_b)'_i$ canonically. We recall that a monomorphism is a topological isomorphism onto its range.

Proposition 1 *(1) The map R is a monomorphism if and only if for every bounded subset B of E there is a bounded subset C of H such that $B \subset \overline{C}^{\tilde{\tau}}$.*

(2) If R is a monomorphism, then the following conditions are equivalent:

(a) $R(F)$ is dense in H'_b,

(b) R is onto,

(c) $R^t : H'' \to F'$ is injective,

(d) for every continuous linear form u on (H, τ') and for every bounded subset C of H, the restriction of u to C is $\tilde{\tau}$-continuous.

(3) If R is a topological isomorphism, then (H, τ') is a topological subspace of (E, τ).

PROOF. (1) follows exactly as in the proof of [3, Theorem 6]. (Alternatively, it can also be proved as an application of Grothendieck's homomorphism theorem, cf. [16, 32.5(1)].)

(2) Assume that R is a monomorphism. Since F is complete, R is onto if and only if it has dense range in H'_b. It is well-known that $R(F)$ is dense in H'_b if and only if (c) holds. Now (b) clearly implies (d), and (d) $\Rightarrow$ (b) can be shown as in the proof of [3, Theorems 6 and 7].

(3) Assume that R is an isomorphism. Then $R^t : (H'_b)'_i \to F'_i$ is an isomorphism which extends the inclusion i_H. Since H is an (LF)-space, the inclusion $H \to (H'_b)'_b$ is a topological isomorphism into, hence $H \to (H'_b)'_i$ has closed graph. Since both spaces are (LF)-spaces, the latter inclusion is continuous. Hence $H \to E = (H'_b)'_i$ is a monomorphism. ■

Corollary 1 *If for each n and for each bounded subset B of (E_n, τ_n) there is a bounded subset C of H_n such that $B \subset \overline{C}^{\tilde{\tau}_n}$, then R is a topological isomorphism onto – and hence $E = (H'_b)'_i$ holds canonically – if the following condition is satisfied:*
$(*)$ *For each m, for each continuous linear form u on H_m, and for each bounded subset C of H_m, the restriction of u to C is $\tilde{\tau}$-continuous.*

PROOF. Since $\operatorname{ind}_n E_n$ is regular under our general assumptions, the first condition here implies the condition which appears in Proposition 1.(1). Hence R is a monomorphism. To conclude, we check that 1.(2)(d) holds and fix $u \in H'$ and a bounded subset C of H. Since $\mathrm{ind}_n E_n$ is regular, there is m such that C is bounded in E_m. By the first condition of the corollary we find a bounded subset A of H_m such that C is contained in the closure D of A for the topology $\tilde{\tau}$. By condition $(*)$, the restriction of u to A is $\tilde{\tau}$-continuous. Since A is absolutely convex, there is a unique extension w of u to D which is $\tilde{\tau}$-uniformly continuous. It is enough to show that w and u coincide on C. We fix $x \in C \subset H$. There is $p \geq m$ such that $x \in H_p$. Since $A \cup \{x\}$ is bounded in H_p, u is $\tilde{\tau}$-continuous on this set. Let $(a_i)_i$ be a net in A which $\tilde{\tau}$-converges to x. Then we have $w(x) = \lim_i u(a_i) = u(x)$, and the proof is complete. ■

In our examples in the next section it will turn out that the assumptions of Corollary 1 are satisfied. In these examples, the condition in Proposition 1.(2)(d) would be difficult to check while $(*)$ in Corollary 1 can be verified easily.

(d) When is E the strong bidual of H?

To discuss this question, we consider the following situation: Let $(E, \tau) = \mathrm{ind}_n(E_n, \tau_n)$ be an (LF)-space which satisfies the assumptions of Theorem 1, and let $F = \mathrm{proj}_n F_n$ be its predual. Let H_n be a closed subspace of (E_n, τ_n) with $H_n \subset H_{n+1}$ for each n. We assume that $R : F \to H'_b$ is a topological isomorphism onto. Then (H, τ') is a topological subspace of (E, τ) by Proposition 1 .(3).

From now on, we suppose that (E, τ) satisfies *condition (M) of Retakh* or, equivalently by [23], that $\mathrm{ind}_n(E_n, \tau_n)$ is *sequentially retractive* or *boundedly retractive*. Then $(H, \tau') = \mathrm{ind}_n(H_n, \tau_n)$ also satisfies condition (M), and hence it is a complete (LF)-space with the strict Mackey convergence condition which is acyclic. Since each (H_n, τ_n) is distinguished in view of [3, Remark on page 120], it follows from [22, Lemma 4.2] that $H'_b = \mathrm{proj}(H_n)'_b$ holds topologically and that this space is ultrabornological. Consequently the predual F of E is ultrabornological, hence barrelled, and quasinormable. Moreover, since (H, τ') satisfies condition (M), the space $F = H'_b$ is the projective limit of a projective spectrum of (LB)-spaces of strong P-type in the sense of Vogt [21]; see also [10].

Proposition 2 *Assume that the (LF)-space $(E, \tau) = \mathrm{ind}_n(E_n, \tau_n)$ is boundedly retractive and satisfies the conditions of Theorem 1. Let F be the predual of E, and let H be a subspace of E with the present conditions.*

(a) Then for every bounded subset B of (E, τ) there is $n \in \mathbb{N}$ such that B is contained in E_n and the topologies τ, $\beta(E, F)$ and τ_n coincide on B.

(b) $E = F'_b = (H'_b)'_b$ holds if and only if F'_b is bornological or, equivalently, if $\beta(E, F)$ is the strongest locally convex topology coinciding with itself on the bounded sets.

According to our comments before the statement of the proposition, the space F is barrelled and quasinormable. Thus, F'_b satisfies the strict Mackey convergence condition. This condition and the fact that $E = F'_i$ is boundedly retractive imply (a).

Since $F = H'_b$ and $E = F'_i$, E coincides with $F'_b = (H'_b)'_b$ if and only if F'_b is bornological. The other equivalence in (b) follows from (a). ■

If, under the conditions of Proposition 2, $E = \mathrm{ind}_n E_n$ is an (LB)-space, then $E = F'_b$ follows from [18] and [3, Remark 4]. Moreover, $E = F'_b$ also holds if F is a Schwartz space, by the classical result of Laurent Schwartz that the strong dual of any complete Schwartz space is bornological. Note that we have $(E, \tau) = F'_b$ and $F = (E, \tau)'_b$ topologically if each (E_n, τ_n) is reflexive. But then (E, τ) is reflexive as well, and $(H, \tau') = (E, \tau)$.

An example which shows that $E = F'_b$ need not hold in the general situation of Proposition 2 was given in [9]. In fact, Grothendieck [14, p. 121] had asked if the strong bidual of a strict (LF)-space coincides topologically with the inductive limit of the strong biduals. This problem was solved in the negative in

[9], and the same counterexample can be used in the present setting. In [10], a partial positive answer to our question is presented: If all the steps (E_n, τ_n) of the (LF)-space E are quasinormable, then F is the projective limit of a projective spectrum of boundedly retractive (LB)-spaces of strong P-type, and then [10, Theorem 1] implies that $E = F_b'$.

2. Examples and applications

A. Weighted (LF)-spaces of holomorphic functions

Let G be an open subset of $\mathbb{C}^N$. For each n let $V_n = (v_{n,k})_k$ be a sequence of strictly positive continuous functions, called *weights*, on G such that the sequence $\mathcal{V} = (V_n)_n$ satisfies the inequalities $v_{n+1,k} \le v_{n,k} \le v_{n,k+1}$ on G for all n, k. We define the Fréchet spaces

$$E_n = HV_n(G) = \{f \in H(G);\ \sup_{z \in G} v_{n,k}(z)|f(z)| < \infty \text{ for each } k\},\ n \in \mathbb{N}.$$

Without loss of generality, let us suppose that the sets

$$U_{n,k} := \{f \in HV_n(G);\ v_{n,k}|f| \le 1 \text{ on } G\},\ k \in \mathbb{N},$$

form a basis of 0-neighborhoods in $HV_n(G)$. For each n, $H_n = H(V_n)_0(G)$ denotes the closed subspace of $HV_n(G)$ of all the functions f such that $v_{n,k}|f|$ vanishes at infinity on G for each k.

The *weighted inductive limits of spaces of holomorphic functions* are defined, as usual (cf. [4]), by $E = \mathcal{V}H(G) := \mathrm{ind}_n HV_n(G)$ and $H = \mathcal{V}_0H(G) := \mathrm{ind}_n H(V_n)_0(G)$. We assume that the inductive limit $\mathcal{V}H(G)$ is regular; this holds for example if the sequence $\mathcal{V}$ satisfies the *condition* (ωQ) of [4]. In the sequel, $\tilde{\tau}$ will denote the compact open topology (on $H(G)$). Since $\mathcal{V}H(G)$ is regular, every bounded subset of $\mathcal{V}H(G)$ is contained and bounded in some Fréchet space $HV_n(G)$. Hence it is contained in a $\tilde{\tau}$-compact subset of $HV_n(G)$. (Compare with [3].) It is easy to see that the sets $U_{n,k}$ are closed in $(HV_n(G), \tilde{\tau})$ for each n and k. Therefore the conditions (BBC), $(BBC)_n$ and $(CNC)_n$ of Section 1 are satisfied for the space $E = \mathcal{V}H(G)$.

Proposition 3 *If the (LF)-space $\mathcal{V}H(G)$ is regular, then there is a projective sequence $(F_n, \pi_{n,n+1})_n$ of complete barrelled (DF)-spaces such that $\mathcal{V}H(G) = (\mathrm{proj}_n F_n)_i'$ holds topologically in a canonical way.*

Remark 1 If $\overline{\tau}_n$ denotes the finest locally convex topology on $HV_n(G)$ which coincides with the compact open topology $\tilde{\tau}$ on the bounded sets, then $F_n = (HV_n(G), \overline{\tau}_n)_b'$ for each $n \in \mathbb{N}$. (See [5, Proposition 1.3].)

The argument in [3, pp. 123] (which is due to [8]) implies that for each m, for every continuous linear form u on $H(V_m)_0(G)$ and for every bounded subset C of $H(V_m)_0(G)$, the restriction of u to C is continuous for the compact open topology.

We recall that a weight v on a balanced set $G \subset \mathbb{C}^N$ is said to be *radial* if $v(\lambda z) = v(z)$ for all $z \in G$ and all complex numbers λ of modulus 1.

Proposition 4 *Let G be a balanced open subset of $\mathbb{C}^N$ and let $\mathcal{V} = (v_{n,k})_{n,k}$ be a (double) sequence of strictly positive continuous radial weights on G satisfying our general assumptions. Suppose that $H(V_1)_0(G)$ contains the polynomials and that the (LF)-space $\mathcal{V}H(G)$ is regular. Then the following assertions hold:*

(a) $\mathcal{V}_0H(G)$ has the bounded approximation property, and the polynomials are dense in $\mathcal{V}_0H(G)$.

(b) $\mathcal{V}_0H(G)_b'$ is the projective limit of a sequence of complete barrelled (DF)-spaces.

(c) $(\mathcal{V}_0H(G)_b')_i' = \mathcal{V}H(G)$ holds canonically, and $\mathcal{V}_0H(G)$ is a topological subspace of $\mathcal{V}H(G)$. If every step $HV_n(G)$ is quasinormable and if $\mathcal{V}_0H(G)$ is boundedly retractive, then $(\mathcal{V}_0H(G)_b')_b' = \mathcal{V}H(G)$ holds topologically.

PROOF. Part (a) is a particular case of [5, Theorem 1.6.(a)]. In our present situation, [5, Theorem 1.5.(c)] can be applied to show that for each n and each bounded subset B of $HV_n(G)$ there is a bounded subset C of $H(V_n)_0(G)$ such that B is contained in the closure of C in the compact open topology. Parts (b) and (c) now follow from Propositions 1 and 3 and Corollary 1. As remarked above, the last statement in part (c) is a consequence of [10, Theorem 1]. ■

The biduality of $\mathcal{V}_0H(G)$ and $\mathcal{V}H(G)$ has consequences for the projective hulls $H\overline{V}_0(G)$ and $H\overline{V}(G)$ of these spaces and for projective description. For the definition of the system $\overline{V}$ of weights on G, of the corresponding spaces $H\overline{V}_0(G)$ and $H\overline{V}(G)$ of holomorphic functions, as well as for the importance of projective description, we refer to [4].

Proposition 5 *Under the hypotheses of Proposition 4, assume that $\mathcal{V}H(G)$ has a basis of 0-neighborhoods which are closed for the compact open topology. If $\mathcal{V}_0H(G)$ is a topological subspace of $H\overline{V}_0(G)$, then $\mathcal{V}H(G)$ is a topological subspace of $H\overline{V}(G)$, too.*

PROOF. In the notation of [3], the space $\mathcal{V}H(G)$ satisfies condition (CNC), and [3, 1.2] implies that this space is isomorphic to the strong dual of $F = (\mathcal{V}_0H(G))'_b$. Let U be an absolutely convex 0-neighborhood in $\mathcal{V}H(G)$. There is a bounded subset B of F such that the polar Z of B in $\mathcal{V}H(G)$ is contained in U. Since $\mathcal{V}_0H(G)$ is a topological subspace of $H\overline{V}_0(G)$, we can find a continuous radial and strictly positive weight $\overline{v} \in \overline{V}$ such that the bipolar U_0 of

$$U_1 := \{f \in \mathcal{V}_0H(G)\ ;\ \overline{v}|f| \le 1 \text{ on } G\}$$

is contained in Z, and hence in U. To complete the proof it is enough to show that

$$W := \{f \in \mathcal{V}H(G)\ ;\ \overline{v}|f| \le 1 \text{ on } G\}$$

is contained in U_0. To see this, fix $f \in W$. There are n and a continuous radial and strictly positive weight w on G with $\sup_{z\in G} w(z)v_{n,k}(z) < \infty$ for every k such that $|f| \le w$ on G (see [5, remark before Corollary 1.7]). For each j, let $C_j : H(G) \to H(G)$ be the operator given by the jth Cesàro mean of the partial sums of the homogeneous Taylor expansion about 0 (cf. [5, Section 1]). We have $|C_jf| \le w$ and $\overline{v}|C_jf| \le 1$ on G for each j. Accordingly, the sequence $(C_jf)_j$ is bounded in $H(V_n)_0(G)$, hence in $\mathcal{V}_0H(G)$, and it converges to f for the compact open topology $\tilde{\tau}$. By the argument in the proof of part (a) of the remark after [3, Theorem 6], the finest locally convex topology $\overline{\tau}$ which coincides with $\tilde{\tau}$ on the bounded subsets of $\mathcal{V}H(G)$ is finer than the topology $\sigma(\mathcal{V}H(G), F)$. This implies that f belongs to the closure of the set U_1 in the topology $\sigma(\mathcal{V}H(G), F)$. The proof is complete by the bipolar theorem. ■

From the present point of view, the following application of Proposition 5 to weighted (LB)-spaces of holomorphic functions may be the most important one. We refer the reader to [7] for the definition of *regularly decreasing* sequences $\mathcal{V} = (v_n)_n$ on G.

Corollary 2 *Let G be a balanced open subset of $\mathbb{C}^N$. Let $\mathcal{V} = (v_n)_n$ be a regularly decreasing sequence of strictly positive radial weights on G such that $H(v_1)_0(G)$ contains the polynomials. "Then projective description holds for $\mathcal{V}_0H(G)$ if and only if it holds for $\mathcal{V}H(G)$", which means: $\mathcal{V}_0H(G)$ is a topological subspace of $H\overline{V}_0(G)$ if and only if $\mathcal{V}H(G)$ coincides algebraically and topologically with $H\overline{V}(G)$.*

PROOF. Since $\mathcal{V}$ is regularly decreasing and we are in the (LB)-case (when $\mathcal{V}H(G)$ always coincides with $H\overline{V}(G)$ algebraically), the conditions of Proposition 5 are satisfied, see [6, Theorem 6.(2)(ii)]. This yields the implication '⇒'. The other implication was already known by [5, Theorem 1.6(d)]. ■

B. Spaces of entire functions of uniformly bounded type

We denote now by E a complex Fréchet space with an increasing fundamental sequence $(p_n)_n$ of seminorms such that the unit balls U_n of p_n form a basis of 0-neighborhoods in E. We denote by E_n the

Banach space which is the completion of the space $(E/\ker p_n, \tilde{p}_n)$ and by $\pi_n : E \to E_n$ the canonical map. An entire function g on E is called *of uniformly bounded type*, and we write $g \in H_{ub}(E)$, if there is n such that for all m the function g is bounded on mU_n. For each n we set

$$G_n := \{g \in H_{ub}(E) \ ; \ g(mU_n) \text{ is bounded for each } m\},$$

endowed with its canonical metrizable locally convex topology.

If X is a Banach space, we denote by $H_b(X)$ the Fréchet space of all the entire functions on X which are bounded on the bounded sets. It is easy to see that for each $n \in I\!\!N$ the map $\Phi : H_b(E_n) \to G_n$, $\Phi(\varphi) := \varphi \circ \pi_n$, defines a linear topological isomorphism onto. Hence, G_n is a quasinormable Fréchet space by [1], and $H_{ub}(E)$, canonically topologized as $\mathrm{ind}_n G_n$, is an (LF)-space. The regularity of $H_{ub}(E)$ was investigated in [13], [17]. In particular, this space is regular if E is a quojection or if E satisfies the condition (QNo) of Peris.

We denote by $\tilde{\tau}$ the compact open topology. Since E is a k-space, the space $(H(E), \tilde{\tau})$ is semi-Montel by [11, 3.37]. Clearly each space $H_b(E_n)$ satisfies the conditions (CNC) and (BBC); see [3, 3.D.(b)] or [12]. Therefore, if $H_{ub}(E)$ is regular, all the assumptions of our Theorem 1 are satisfied, and we have the following result. (Observe that the predual of each step $H_b(E_n)$ is an (LB)-space by [3, 1.5].)

Proposition 6 *Let E be a complex Fréchet space such that the (LF)-space $H_{ub}(E)$ is regular. Then there is a reduced projective sequence $(F_n)_n$ of complete (LB)-spaces such that $H_{ub}(E)$ coincides canonically with the inductive dual of the projective limit $\mathrm{proj}_n F_n$.*

C. (LF)-sequence spaces

The same method as in part A. above yields consequences for the biduality of Köthe (LF)-sequence spaces as considered in [22]. This corresponds to weighted (LF)-spaces of continuous functions on a discrete space, cf. [4].

Let I be an index set. For each n let $V_n = (v_{n,k})_k$ be a (double) sequence of strictly positive weights on I such that the sequence $\mathcal{V} = (V_n)_n$ satisfies $v_{n+1,k} \le v_{n,k} \le v_{n,k+1}$ on I for all n, k. We define the Fréchet spaces

$$E_n = \lambda_\infty(V_n) := \{x = (x(i)); \ \sup_{i \in I} v_{n,k}(i)|x(i)| < \infty \text{ for each } k\}, \ n \in I\!\!N.$$

Without loss of generality, let us suppose that the sets

$$U_{n,k} := \{f \in \lambda_\infty(V_n); \ v_{n,k}|x| \le 1 \text{ on } I\}, \ k \in I\!\!N,$$

form a basis of 0-neighborhoods in $\lambda_\infty(V_n)$. For each n, $H_n = \lambda_0(V_n)$ denotes the closed subspace of all the functions x such that $v_{n,k}|x|$ tends to 0 on I for each k.

The weighted (LF)-sequence spaces are defined, as usual (cf. [22]), by $E = k_\infty(\mathcal{V}) := \mathrm{ind}_n \lambda_\infty(V_n)$ and $H = k_0(\mathcal{V}) := \mathrm{ind}_n \lambda_0(V_n)$. The inductive limit E is regular (or complete) if and only if the sequence $\mathcal{V}$ satisfies the condition (ωQ) of [22]; see [4, 2.4, 2.7]. In the sequel, $\tilde{\tau}$ will denote the topology of pointwise convergence on I. If $\mathcal{V}$ has (ωQ), the conditions (BBC), $(BBC)_n$ and $(CNC)_n$ of Section 1 are satisfied for the space E. In the present setting, Section 2 of [4] gives a concrete description of the predual F of E, which is the strong dual of H as a sequence space; also compare with [22, Section 5]. In fact, if $\mathcal{V}$ has condition (ωQ), then F is an ultrabornological projective limit of complete (LB)-spaces, and the topology $\beta(E, F)$ can be described by weighted sup-seminorms; cf. [4, 2.2]. The following result now follows from [4, 4.5], [22, Section 5] and Corollary 1 above. Part (b) of the proposition below should be compared with the counterexample of [10] and with our comments at the end of Section 1: Indeed it shows that certain pathologies cannot occur in the case of sequence spaces.

Proposition 7 *(a) If* $\mathcal{V}$ *satisfies condition* (ωQ) *(or, equivalently,* $k_\infty(\mathcal{V})$ *is regular), then we have* $k_\infty(\mathcal{V}) = (k_0(\mathcal{V})'_b)'_i$.

(b) If $\mathcal{V}$ *satisfies condition (Q) of Vogt [22] (or, equivalently,* $k_\infty(\mathcal{V})$ *satisfies condition (M) of Retakh), then* $k_\infty(\mathcal{V}) = (k_0(\mathcal{V})'_b)'_b$.

Acknowledgement. The second-named author was partially supported by DGESIC Project PB97-0333. The present research started during his visit at the University of Paderborn in 1998, for which the support of the A.v. Humboldt Foundation is gratefully acknowledged. The article was finished during a stay of the first-named author at Universidad Politécnica de Valencia in the year 2000, supported by a grant of the Spanish Ministerio de Educación y Cultura (Ref.: SAB 1999-0035).

References

[1] Ansemil, J. M., Ponte, S.(1981). An example of a quasinormable Fréchet function space which is not a Schwartz space, *Advances in Functional Analysis, Holomorphy and Approximation Theory*, Springer Lecture Notes in Math. **843**, 1–8.

[2] Bierstedt, K. D. (1988). An introduction to locally convex inductive limits, *Funct. Anal. Appl.*, 35–133.

[3] Bierstedt,K. D., Bonet, J.(1992). Biduality in Fréchet and (LB)-spaces, *Progress in Functional Analysis*, North-Holland Math. Studies **170**, Amsterdam, 113–133.

[4] Bierstedt, K. D., Bonet, J. (1994). Weighted (LF)-spaces of continuous functions, *Math. Nachr.* **165**, 25–48.

[5] Bierstedt,K. D., Bonet, J., Galbis, A. (1993). Weighted spaces of holomorphic functions on bounded domains, *Michigan Math. J.* **40**, 271–297.

[6] Bierstedt, K. D., Meise, R. (1986). Weighted inductive limits and their projective descriptions, *Doğa Tr. J. Math.* **10**, 1, 54–82.

[7] Bierstedt, K. D., Meise, R., Summers, W. H. (1982). A projective description of weighted inductive limits, *Trans. Amer. Math. Soc.* **272**, 107–160.

[8] Bierstedt, K. D., Summers, W. H. (1993). Biduals of weighted spaces of analytic functions, *J. Austral. Math. Soc.* **54**, 70–79.

[9] Bonet, J., Dierolf, S. (1988). A note on the biduals of strict (LF)-spaces, *Results Math.* **13**, 23–32.

[10] Bonet, J., Dierolf, S., Wengenroth, J. Strong duals of projective limits of (LB)-spaces, *Czech. Math. J.* (to appear).

[11] Dineen, S. (1981). *Complex Analysis in Locally Convex Spaces*, North-Holland Math. Studies **57**, Amsterdam.

[12] Galindo, P., D. García, D., Maestre, M. (1992). Holomorphic mappings of bounded type, *J. Math. Anal. Appl.* **166**, 236–246.

[13] Galindo, P., García, D., Maestre, M. (1993). Entire functions of bounded type on Fréchet spaces, *Math. Nachr.* **161**, 185–198.

[14] Grothendieck, A. (1954). Sur les espaces (F) et (DF), *Summa Brasil.* **3**, 57–122.

[15] Jarchow, H. (1981). *Locally Convex Spaces*, B.G. Teubner.

[16] Köthe, G. (1969 and 1980). *Topological Vector Spaces I and II*, Springer-Verlag, Berlin.

[17] Mangino, E. (1997). *Productos tensoriales de espacios de Fréchet, (DF) y (LF)*, Doctoral Thesis, Valencia.

[18] Mujica, J. (1984). A completeness criterion for inductive limits of Banach spaces, *Functional Analysis, Holomorphy and Approximation Theory*, North-Holland Math. Studies **86**, Amsterdam, 319–329.

[19] Mujica, J., L. Nachbin, L. (1992).Linearization of holomorphic mappings on locally convex spaces, *J. Math. Pures Appl.* **71**, 543–560.

[20] Pérez Carreras, P., Bonet, J. (1987). *Barrelled Locally Convex Spaces*, North-Holland Math. Studies **131**.

[21] Vogt, D. (1987). *Lectures on Projective Spectra of (DF)-spaces*, Seminar Lectures, AG Funktionalanalysis, Düsseldorf-Wuppertal.

[22] Vogt, D. (1992). Regularity properties of (LF)-spaces, *Progress in Functional Analysis*, North-Holland Math. Studies **170**, 57–84.

[23] Wengenroth, J. (1995). *Retractive (LF)-spaces*, Dissertation, Trier.

[24] Wengenroth, J. (1996) Acyclic spectra of Fréchet spaces, *Studia Math.* **120**, 247–258.

Klaus D. Bierstedt
FB 17, Mathematik
Univ. Paderborn
D-33095 Paderborn
Germany
klausd@uni-paderborn.de

José Bonet
Dpto. Matemática Aplicada
E.T.S. Arquitectura
Universidad Politécnica de Valencia
E-46071 Valencia
Spain
jbonet@mat.upv.es