RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **95** (1), 2001, pp. 145–153
Matemática Aplicada / Applied Mathematics

A conjecture on multivariate polynomial interpolation

J. M. Carnicer and M. Gasca

Abstract. The generalization of Lagrange and Newton univariate interpolation formulae is one of the topics of multivariate polynomial interpolation. Two classes of geometric configurations of points in the plane, suitable for the use of those formulas, were given by Chung and Yao in 1978 for the Lagrange formula, and by Gasca and Maeztu in 1982 for the Newton formula. The latter authors conjectured that every configuration of the first class belongs to the second class and proved that the converse is not true. In 1990 J. R. Busch proved the conjecture for polynomials of degree not greater than 4, showing the difficulty of extending his reasoning to higher degree. In this paper we prove the same result using different arguments with similar difficulties, in the hope that these arguments could shed more light to the problem.

Una conjetura sobre interpolación polinómica en varias variables

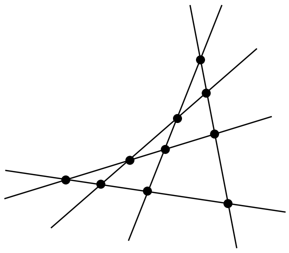
Resumen. La generalización de las fórmulas de interpolación de Lagrange y Newton a varias variables es uno de los temas habituales de estudio en interpolación polinómica. Dos clases de configuraciones geométricas particularmente interesantes en el plano fueron obtenidas por Chung y Yao en 1978 para la fórmula de Lagrange y por Gasca y Maeztu en 1982 para la de Newton. Estos últimos autores conjeturaron que toda configuración de la primera clase es de la segunda, y probaron que el recíproco no es cierto. En 1990 J. R. Busch probó la conjetura para polinomios de grado no mayor que 4, viendo la dificultad de extender su razonamiento a grado superior. En este trabajo damos otra demostración del mismo resultado con otros argumentos que muestran similar dificultad pero ofrecen alguna esperanza de generalización.

1. Introduction

Multivariate polynomial interpolation is a much more difficult problem than the corresponding univariate one. Tensor product constructions can be reduced to univariate interpolation problems but the points must be on a rectangular grid. For arbitrarily distributed nodes, it is not clear how to obtain a subspace of polynomials suitable for interpolation. Even assuming that the number of points in the set of nodes equals the dimension of a previously chosen space, the existence and uniqueness of the solution depends on the geometrical distribution of the points. The relevance of all these questions can be seen for example in the recent surveys [7], [10].

An interesting and standard problem is the study of distributions of points suitable for interpolation in the subspace of polynomials of total degree not greater than n. The bivariate case is the simplest and most important one for its application to the construction of surfaces.

Presentado por Baltasar Rodríguez-Salinas
Recibido: 13 de Julio 2001. *Aceptado*: 10 de Octubre 2001.
Palabras clave / Keywords: multivariate polynomial interpolation, geometric characterization
Mathematics Subject Classifications: 41A05, 65D05, 41A63.



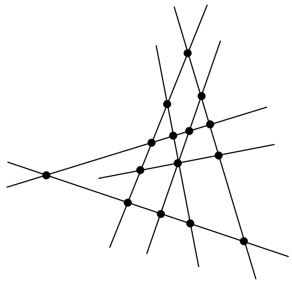


Figure 1. Two examples of sets satisfying GC_3 and GC_4 respectively.

Let us denote by $\Pi_n(\mathbf{R}^2)$ the set of bivariate polynomials of total degree not greater than n with dimension $(n+2)(n+1)/2$. A set of nodes X is said $\Pi_n(\mathbf{R}^2)$-unisolvent if the Lagrange interpolation problem of finding a polynomial $p \in \Pi_n(\mathbf{R}^2)$ with prescribed arbitrary values on X has always a unique solution. A necessary and sufficient condition for a given set of nodes X with cardinal $|X| = (n+2)(n+1)/2$ to be $\Pi_n(\mathbf{R}^2)$-unisolvent is that X is not contained in an algebraic curve of degree n. This condition leads in practice to solve the linear interpolation problem as a linear system of equations. However, as in the univariate case, one tries to avoid this viewpoint looking for a simple formula which solves the problem. In the univariate case, the simplest interpolation formulae are the Lagrange and Newton formulae, and their successful extension to multivariate problems is always one of the topics of this theory.

A *Lagrange formula* in $\Pi_n(\mathbf{R}^2)$ for a set of nodes X is

$$p = \sum_{x \in X} f(x) l_x,$$

where $l_x \in \Pi_n(\mathbf{R}^2)$ is a Lagrange polynomial associated to the node x,

$$l_x(x) = 1, \qquad l_x(y) = 0, \quad \forall y \in X \setminus \{x\}, \tag{1}$$

and $f(x)$ is the prescribed value of p on x.

Let us remark that the Lagrange functions l_x, $x \in X$ are linearly independent. So, if a Lagrange polynomial (1) in $\Pi_n(\mathbf{R}^2)$ exists for each node $x \in X$, then $|X| \le \dim \Pi_n(\mathbf{R}^2) = (n+1)(n+2)/2$.

The most interesting approach for simple multivariate Lagrange formulae was provided by Chung and Yao, who in 1977 [5] introduced a geometric characterization (GC) for a set X whose associated Lagrange polynomials are products of polynomials of first degree. In the bivariate case, the GC condition can be stated as follows.

Definition 1 *Let $X \subseteq \mathbf{R}^2$, $|X| = (n+2)(n+1)/2$. The set X satisfies the geometric characterization GC_n if for each $x \in X$, there exist lines $L_1^x, \dots, L_n^x$ such that*

$$X \setminus \{x\} \subseteq L_1^x \cup \cdots \cup L_n^x, \quad x \notin L_1^x \cup \cdots \cup L_n^x.$$

We say that the lines $L_1^x, \dots, L_n^x$ are used *by the node $x \in X$. The set of lines used by $x \in X$ will be denoted by Γ_x and $\Gamma_X := \bigcup_{x \in X} \Gamma_x$ is the set of lines used by some node. The set of nodes in X using a line of the plane L will be denoted by X_L:*

$$X_L = \{x \in X \mid L \in \Gamma_x\}.$$

Let us remark that $L \in \Gamma_x$ if and only if $x \in X_L$ and that X_L is nonempty if and only if $L \in \Gamma_X$.

By abuse of notation, if L is a line we shall also use the letter L for denoting a polynomial of first degree such that $L(x) = 0$ is the equation of the line L.

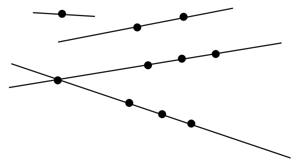


Figure 2. An example of a set satisfying (3)

Clearly the GC_n condition implies the existence of Lagrange polynomials in $\Pi_n(\mathbf{R}^2)$

$$l_x = \frac{L_1^x \cdots L_n^x}{L_1^x(x) \cdots L_n^x(x)} \tag{2}$$

and, since $|X| = (n+2)(n+1)/2$, X is $\Pi_n(\mathbf{R}^2)$-unisolvent and the Lagrange functions are unique. Therefore the set of lines Γ_x used by a node is uniquely determined by $x \in X$.

A different approach was considered in 1982 by Gasca and Maeztu [6], giving an easy way of obtaining bivariate *Newton formulae*

$$p = \sum_{j=1}^{N} c_j \phi_j, \qquad \phi_j(x_i) = 0, \quad \forall i < j$$

to deal with Lagrange and Hermite bivariate interpolation problems. Here we restrict ourselves to Lagrange problems. The basic functions ϕ_j are, in [6], product of polynomials of first degree. In that paper, the authors showed that their approach, applied to the Lagrange problem, produced the interpolation space $\Pi_n(\mathbf{R}^2)$ if and only if the nodes form a *system of order* n, which means that the nodes X are distributed on lines $L_0, L_1, \ldots, L_n$ so that

$$|L_i \cap X \setminus (L_0 \cup \cdots \cup L_{i-1})| = n + 1 - i, \quad i = 0, \ldots, n. \tag{3}$$

In other words, $n+1$ points lie on a line, n of the remaining points lie on another line, and so on. The last line contains a point which is not on the previous lines. Condition (3) had been previously used by several authors, but not in connection with Newton formulae. In 1948, Radon [11] constructed cubature formulae based on it. It appears also in a paper by Guenter and Roetman [9] in 1970, for decomposing a multivariate interpolation problem in two simpler problems. For more details on this condition, see the survey [8].

Figure 2 shows an example of a set of points in the plane satisfying the condition (3). Let us observe that the sets in Figure 1 also satisfy condition (3). However, the set of Figure 2 does not satisfy GC_3.

Proposition 1 *Let $X \subseteq \mathbf{R}^2$ and assume that there exist lines $L_0, L_1, \ldots, L_n$ such that (3) holds. Then X is $\Pi_n(\mathbf{R}^2)$-unisolvent.*

The proof can be found in [6], see also [4]. The authors in [6] also showed that there exist set of nodes X satisfying (3) but not GC_n. However, they could not find sets of nodes satisfying the GC_n condition but not (3). They conjectured that (3) must hold for every set of points satisfying GC_n.

Why is this conjecture interesting? Condition GC_n provides an elegant characterization of sets which are $\Pi_n(\mathbf{R}^2)$-unisolvent and give rise to very simple Lagrange formulae with basic functions product of polynomials of first degree. Chung and Yao gave in [5] some nice examples of distributions of points satisfying GC_n, but not a systematic classification of all sets which satisfy that condition. In a precedent paper [3] we have looked for a better understanding of it and a starting point for a classification. On the other hand, condition (3) is an extremely simple condition for $\Pi_n(\mathbf{R}^2)$-unisolvence which gives rise to very simple Newton formula with basic functions product of polynomials of first degree. Therefore it is important to know if the sets satisfying GC_n are always included among the class of those which satisfy (3).

2. Preliminary results and statement of the problem

In Proposition 2.1 of [3], several auxiliary properties of the GC condition were stated and proven. The following Proposition contains some of these properties which will be repeatedly used in this paper.

Proposition 2 *Let X be a set of nodes satisfying the GC_n condition. Then*

(a) A set of r lines cannot contain more than $r(2n+3-r)/2$ points of X. In particular, no line contains more than $n+1$ points.

(b) A line L containing $n+1$ points of X must be in Γ_X and it is used by each point not lying on it ($n \geq 1$).

(c) Two lines, each containing $n+1$ points of X, cannot be parallel and meet at a point $x \in X$.

(d) Three lines, each containing $n+1$ points of X, cannot be concurrent.

We shall also use the following result:

Proposition 3 *Let X be a set of nodes satisfying GC_n, L a line and Y the set of nodes which neither lie on L nor use L*

$$Y := \{y \in X \setminus L \mid L \notin \Gamma_y\} = X \setminus (L \cup X_L). \tag{4}$$

Then one has:

(a) If $|L \cap X| = n+1$, the set Y is empty and $X \setminus L$ satisfies GC_{n-1}.

(b) If $|L \cap X| < n+1$, the set Y is nonempty and cannot be contained in a $\Pi_{n-1}(\mathbf{R}^2)$-unisolvent set.

PROOF.

(a) By Proposition 2 (b), $X_L = X \setminus L$ and so $Y = \emptyset$. Take any $y \in X \setminus L$. Since X is GC_n, there exist n (unique) lines $L_1, \ldots, L_{n-1}, L_n \in \Gamma_X$ which are used by y and L must be one of the L_i's. We may assume without loss of generality that $L_n = L$. The lines $L_1, \ldots, L_{n-1}$ satisfy

$$(X \setminus L) \setminus \{y\} \subseteq L_1 \cup \cdots \cup L_{n-1}, \quad y \notin L_1 \cup \cdots \cup L_{n-1}$$

and since y is any point of $X \setminus L$, this set satisfies the GC_{n-1} condition.

(b) Let us assume that $Y = \emptyset$, that is, L is used by all points in $X \setminus L$. Let $l_x \in \Pi_n(\mathbf{R}^2)$ be the Lagrange polynomial corresponding to $x \in X \setminus L$ in the interpolation problem defined by the set of nodes X. Then $L(x)l_x/L \in \Pi_{n-1}(\mathbf{R}^2)$ is a Lagrange polynomial corresponding to x for the interpolation problem on $X \setminus L$ with $\Pi_{n-1}(\mathbf{R}^2)$ as interpolation space. Since the Lagrange polynomials are linearly independent, $|X \setminus L| \leq \dim \Pi_{n-1}(\mathbf{R}^2) = n(n+1)/2$. Then $|L \cap X| \geq n+1$ and, by Proposition 2 (a), $|L \cap X| = n+1$.

So, if $|L \cap X| < n+1$, the set Y is nonempty. Let us assume that Y is contained in a $\Pi_{n-1}(\mathbf{R}^2)$-unisolvent set. For each $y \in Y$, there exists $p \in \Pi_{n-1}(\mathbf{R}^2)$ such that p vanishes on $Y \setminus \{y\}$ but $p(y) \neq 0$. The polynomial pL has degree not greater than n. Taking into account that X is $\Pi_n(\mathbf{R}^2)$-unisolvent, we can use the Lagrange formula to obtain

$$pL = \sum_{x \in L \cap X} p(x)L(x)l_x + \sum_{x \in Y} p(x)L(x)l_x + \sum_{x \in X_L} p(x)L(x)l_x.$$

The first term of the right hand side is zero because $L(x) = 0$ for all $x \in L \cap X$. The second term reduces to $p(y)L(y)l_y$ because $p(x) = 0$ for all $x \in Y \setminus \{y\}$. Then we have

$$l_y = \frac{1}{p(y)L(y)}\Big(pL - \sum_{x \in X_L} p(x)L(x)l_x\Big).$$

Since all the points in X_L use the line L, each l_x in the sum inside the brackets must contain L as a factor. So, L divides l_y, which implies that y uses L, contradicting the definition of Y. ■

Proposition 2 analyzes some properties of the lines containing $n+1$ nodes of a GC_n set. The following shows some properties of the lines containing at least n nodes.

Proposition 4 *Let X be a set of nodes satisfying* GC_n *and let*

$$\mathcal{L} := \{L \mid L \text{ is a line of } \mathbf{R}^2, |L \cap X| \geq n\}.$$

Then we have

(a) For all $x \in X$, $|\{L \in \mathcal{L} \mid x \in L\}| \leq 4$, .

(b) $|\mathcal{L}| \leq 2n + 6 + 4/n$, and, if equality holds, then for all $x \in X$ one has $|\{L \in \mathcal{L} \mid x \in L\}| = 4$.

(c) If $L_1 \in \Gamma_x$ and L_2 is a line of $\mathcal{L}$ such that $L_1 \cap L_2 \cap X = \emptyset$ and $x \notin L_2$, then $L_2 \in \Gamma_x$.

(d) If $\Gamma_X \cap \mathcal{L} \neq \emptyset$, $n \geq 1$, then there exists a line $L \in \Gamma_X$ such that $|X_L| > n/2$.

PROOF.

(a) If there exist 5 different lines $L_i \in \mathcal{L}$, $x \in L_i$, $i = 1, 2, 3, 4, 5$, then

$$|(L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5) \cap X| \geq 5n - 4,$$

contradicting Proposition 2 (a) with $r = 5$.

(b) Let us define the set

$$S := \{(x, L) \mid x \in X, L \in \mathcal{L}, x \in L\}.$$

By (a), one has

$$n|\mathcal{L}| \leq |S| \leq 4|X|$$

and then

$$|\mathcal{L}| \leq \frac{4}{n} \frac{(n+2)(n+1)}{2} = 2n + 6 + \frac{4}{n}.$$

If equality holds, then $|S| = 4|X|$.

(c) If $|L_2 \cap (X \setminus \{x\})| \geq n$ and $L_2 \notin \Gamma_x$ then each of the n lines in Γ_x must contain exactly one node of L_2. Since $L_1 \in \Gamma_x$, we would have that $L_1 \cap L_2 \cap X \neq \emptyset$.

(d) Let $M \in \Gamma_x \cap \mathcal{L}$ for some $x \in X$. If $|M \cap X| = n + 1$, then, by Proposition 2 (b), $X_M = X \setminus M$. Therefore $|X_M| = (n+1)n/2 > n/2$. Otherwise, if $|M \cap X| = n$, let $L_1, \ldots, L_n$ be the n lines joining x with each of the nodes in $M \cap X$. By (c), each of the points in $X \setminus M$ uses at least one of the $n + 1$ lines $M, L_1, \ldots, L_n$, and then there exists a line $L \in \{M, L_1, \ldots, L_n\}$ such that

$$|X_L| \geq \frac{|X \setminus M|}{n+1} = \frac{n}{2} + \frac{1}{n+1} > n/2. \blacksquare$$

In [6] the following conjecture was stated.

Conjecture 1 *Let X be a set satisfying the* GC_n *condition. Then there exists a line L in the plane such that $|L \cap X| = n + 1$.*

Conjecture 1 is equivalent to saying that all GC_n sets satisfy (3). This equivalence is a consequence of Proposition 3 (a).

Conjecture 1 is trivial for $n = 1$. In the case $n = 2$, we have 6 points. For each of these points, the five remaining ones must lie on two lines and one of the lines must contain at least 3 nodes, and no more by Proposition 2 (a). This proves the conjecture for $n = 2$.

The case $n = 3$ was checked by Gasca and Maeztu as mentioned in [6], analyzing all possible combinatorial configurations of the alignments of 9 points on 3 lines. In the preprint [1], Busch gave a proof of the cases $n = 3$ and $n = 4$. However, in the paper [2], only the case $n = 4$ was detailed. Busch recognized in [2] that his method cannot be extended for $n > 4$ because the number of different cases to be analyzed is much higher. His proof is based in 5 lemmas, where he shows the following properties for a set X satisfying GC_4:

- Lemma 1 of [2]: If a set of three lines intersect another set of three lines at 9 points of X and a cubic polynomial vanishes on 8 of the 9 points, this polynomial must also vanish at the remaining point.
- Lemma 2 of [2]: If for some $x \in X$ there exists $L \in \Gamma_x$ with $|L \cap X| = 4$, then there exist three nodes in X using the same line.
- Lemma 3 of [2]: Let x, y be distinct nodes such that $|\Gamma_x \cap \Gamma_y| = 1$, and let L be the line used by x and y. Then $|L \cap X| \geq 4$. If a third node $z \in X$ uses L then $|L \cap X| = 5$.
- Lemma 4 of [2]: Let x, y be distinct nodes such that $|\Gamma_x \cap \Gamma_y| = 2$. Then at least one line of $\Gamma_x \cap \Gamma_y$ contains five nodes.
- Lemma 5 of [2]: Let x, y, z be distinct nodes such that there exists $L \in \Gamma_x \cap \Gamma_y \cap \Gamma_z$ and assume that for each $M \in \Gamma_x$ one has $|M \cap X| < 5$. Then there exist three lines L_1, L_2, L_3 used by the three nodes x, y, z. Each of the lines L_1, L_2, L_3 contains exactly 4 nodes and two of the lines have no node in common. If another node w uses some of the lines L_1, L_2, L_3, it uses the three lines.

From the previous lemmas, Busch showed that, if no line contains 5 nodes, one always gets a contradiction.

Our present paper is devoted to provide an alternative proof of the cases $n = 3$ and $n = 4$ trying to shed more light on the conjecture.

3. Solution of the conjecture for $n = 3$ and $n = 4$

Let us first consider the case $n = 3$.

Theorem 1 *Let X be a set of nodes satisfying* GC_3. *Then there exist 4 points in X which are collinear.*

PROOF. Assume that no line of the plane contains 4 points of X. Let $\mathcal{L}$ be the set of lines such that $|L \cap X| = 3$. By Proposition 4 (b), $|\mathcal{L}| \leq 13$.

For any $x \in X$, the remaining 9 points of X lie on three lines $L_1^x, L_2^x, L_3^x \in \mathcal{L}$. Let us define

$$Q := \{(x, L) \mid x \in X, L \in \mathcal{L} \cap \Gamma_x\}.$$

If each line $L \in \mathcal{L} \cap \Gamma_X$ would be used exactly by one point of X, then $13 \geq |\mathcal{L}| \geq |Q| = 3|X| = 30$ which is a contradiction. So, at least one line $L \in \mathcal{L}$ is used by two points $x_1 \neq x_2$ in X. By Proposition 3 (b), the set

$$Y := \{y \in X \setminus L \mid L \notin \Gamma_y\}$$

cannot be contained in a $\Pi_2(\mathbf{R}^2)$-unisolvent set. However, neither x_1 nor x_2 belong to Y and so this set contains no more than 5 points, with no 4 of them collinear. Therefore Y is contained in a $\Pi_2(\mathbf{R}^2)$-unisolvent set of the type of condition (3) and we get a contradiction. ■

Let us now consider the case $n = 4$. Let $X \subset \mathbf{R}^2$ be a set satisfying GC_4 and from now on we shall assume that no line contains 5 points of X, obtaining finally a contradiction.

Let us define

$$\mathcal{L} := \{L \mid L \text{ is a line of } \mathbf{R}^2, |L \cap X| = 4\}. \tag{5}$$

The following lemma shows some properties of X under the above assumptions.

Lemma 1 *Let X be a set satisfying* GC_4 *and assume that no line of the plane contains 5 nodes. Let $\mathcal{L}$ be the set (5). Then we have*

(a) If $L_1, L_2, L_3 \in \mathcal{L}$ and $L_i \cap L_j \cap X = \emptyset$ for all $i \neq j$ in $\{1, 2, 3\}$, then the three nodes in $X \setminus (L_1 \cup L_2 \cup L_3)$ are not collinear and each of them uses the lines L_1, L_2, L_3.

(b) For each $x \in X$ *one has* $|\mathcal{L} \cap \Gamma_x| \geq 2$.

(c) If $L \in \mathcal{L}$, *then* $|X_L| \leq 3$.

(d) If $L_1 \in \mathcal{L} \cap \Gamma_{x_1} \cap \Gamma_{x_2} \cap \Gamma_{x_3}$, *where* x_1, x_2, x_3 *are distinct nodes in* X, *then there exist lines* $L_2, L_3 \in \mathcal{L}$ *such that* $L_i \cap L_j \cap X = \emptyset$ *for all* $i \neq j$ *in* $\{1, 2, 3\}$, $\Gamma_{x_1} \cap \Gamma_{x_2} \cap \Gamma_{x_3} = \{L_1, L_2, L_3\}$ *and* $X_{L_1} = X_{L_2} = X_{L_3} = \{x_1, x_2, x_3\}$.

PROOF.

(a) The 3 points in $X \setminus (L_1 \cup L_2 \cup L_3)$ are not collinear. Otherwise X would be contained in a set of 4 lines and would not be $\Pi_4(\mathbf{R}^2)$-unisolvent. If M is the line joining two of the points in $X \setminus (L_1 \cup L_2 \cup L_3)$, then the remaining point uses the lines M, L_1, L_2, L_3.

(b) For any $x \in X$, Γ_x is a set of 4 lines containing 14 nodes. If no line contains 5 nodes, then at least two lines of Γ_x belong to $\mathcal{L}$.

(c) If a line $L \in \mathcal{L}$ is used by more than three points, then the set $Y = X \setminus (L \cup X_L)$ would have less than 8 nodes and, since no five of them are collinear, Y is contained in a $\Pi_3(\mathbf{R}^2)$-unisolvent set of the type (3), contradicting Proposition 3 (b).

(d) Let L_1, L_2, L_3, L_4 be the lines associated to x_1. Since $L_1 \in \mathcal{L}$, $|L_1 \cap X| = 4$. Let us define $Z := (X \setminus L_1) \setminus \{x_1, x_2, x_3\}$ and let $d_2 := |L_2 \cap Z|$, $d_3 := |(L_3 \cap Z) \setminus L_2|$, $d_4 := |(L_4 \cap Z) \setminus (L_2 \cup L_3)|$. Taking into account that no line contains 5 nodes and reordering the lines L_2, L_3, L_4, we may assume that $0 \leq d_4 \leq d_3 \leq d_2 \leq 4$. Since Z contains 8 nodes, $d_2 + d_3 + d_4 = 8$ and $d_4 \leq 2$.

If $d_3 \leq 3$ then we would have $d_i \leq 6 - i$, $i = 2, 3, 4$ and, by Proposition 1, Z would be contained in a $\Pi_3(\mathbf{R}^2)$-unisolvent set of the type (3). This contradicts the fact that, by Proposition 3 (b), Z cannot be contained in a $\Pi_3(\mathbf{R}^2)$-unisolvent set. Hence $d_3 = 4$ and consequently $d_2 = d_3 = 4$, $d_4 = 0$. Then we have that $L_2, L_3 \in \mathcal{L}$ and $L_i \cap L_j \cap X = \emptyset$ for all $i \neq j$ in $\{1, 2, 3\}$. By Proposition 4 (c), $L_2, L_3 \in \Gamma_{x_1} \cap \Gamma_{x_2} \cap \Gamma_{x_3}$ and $|\Gamma_{x_1} \cap \Gamma_{x_2} \cap \Gamma_{x_3}| \geq 3$. If $|\Gamma_{x_1} \cap \Gamma_{x_2} \cap \Gamma_{x_3}| = 4$ then $\Gamma_{x_1} = \Gamma_{x_2} = \Gamma_{x_3}$ and the Lagrange polynomials associated to x_1, x_2, x_3 would be linearly dependent, a contradiction. So $\Gamma_{x_1} \cap \Gamma_{x_2} \cap \Gamma_{x_3} = \{L_1, L_2, L_3\}$. Moreover, we have shown that $\{x_1, x_2, x_3\} \subseteq X_{L_i}$, $i = 1, 2, 3$, and by (c) we get $X_{L_1} = X_{L_2} = X_{L_3} = \{x_1, x_2, x_3\}$. ■

Let us define

$$X_i := \{x \in X \mid |\Gamma_x \cap \mathcal{L}| = i\}, \quad \mathcal{L}_i := \{L \in \mathcal{L} \mid |X_L| = i\}$$

for each i. Clearly, for all $x \in X$ one has $|\Gamma_x \cap \mathcal{L}| \leq 4$ and, by Lemma 1 (b), $|\Gamma_x \cap \mathcal{L}| \geq 2$. Moreover, by Lemma 1 (c), for all $L \in \mathcal{L}$, $|X_L| \leq 3$. Then we have

$$X = X_2 \cup X_3 \cup X_4, \quad \mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3.$$

By Proposition 4 (b), $|\mathcal{L}| \leq 15$ and so we obtain

$$|\mathcal{L}_0| + |\mathcal{L}_1| + |\mathcal{L}_2| + |\mathcal{L}_3| = |\mathcal{L}| \leq 15 = |X| = |X_2| + |X_3| + |X_4|. \tag{6}$$

Let us define an equivalence relation in $\mathcal{L}_3$:

$$L \sim M \Leftrightarrow X_L = X_M, \quad L, M \in \mathcal{L}_3.$$

By Lemma 1 (d), each equivalence class in $\mathcal{L}_3$ consists of 3 lines.

Let us also define an equivalence relation in $X_3 \cup X_4$:

$$x \sim y \Leftrightarrow \Gamma_x \cap \mathcal{L}_3 = \Gamma_y \cap \mathcal{L}_3, \quad x, y \in X_3 \cup X_4.$$

Recall that any point $x \in X_3 \cup X_4$ uses 3 or 4 lines in $\mathcal{L}$.

The set of points $x \in X_3 \cup X_4$ such that $\Gamma_x \cap \mathcal{L}_3 = \emptyset$, form an equivalence class. For all the rest of the points $x \in X_3 \cup X_4$ we have $\Gamma_x \cap \mathcal{L}_3 \neq \emptyset$. Then there exists $L_1 \in \Gamma_x \cap \mathcal{L}_3$ with $x \in X_{L_1} = \{x_1, x_2, x_3\}$. By Lemma 1 (d), there exist lines $L_2 \neq L_3$ different from L_1 such that $\Gamma_{x_1} \cap \Gamma_{x_2} \cap \Gamma_{x_3} \cap \mathcal{L}_3 = \{L_1, L_2, L_3\}$ and $\{x_1, x_2, x_3\} = X_{L_1} = X_{L_2} = X_{L_3}$. So, $x_2, x_3 \in X_3 \cup X_4$ and $\{L_1, L_2, L_3\} \subseteq \Gamma_{x_i} \cap \mathcal{L}_3$ for $i = 1, 2, 3$.

Let us assume that for some x_i, $i \in \{1,2,3\}$, $\Gamma_{x_i} \cap \mathcal{L}_3 = \{L_1, L_2, L_3, L_4\}$, where $L_4 \in \mathcal{L}_3$. Then, $X_{L_4} = \{x_i, x_4, x_5\}$ and, by Lemma 1 (d), there exist lines $L_5, L_6 \in \mathcal{L}_3$, different from L_4, such that $X_{L_4} = X_{L_5} = X_{L_6} = \{x_i, x_4, x_5\}$. Therefore $L_5, L_6 \in \Gamma_{x_i} \cap \mathcal{L}_3 = \{L_1, L_2, L_3, L_4\}$. Then L_5 and L_6 are in $\{L_1, L_2, L_3\}$ and we obtain that $X_{L_4} = \{x_i, x_4, x_5\} = \{x_1, x_2, x_3\}$. So $L_1 \sim L_2 \sim L_3 \sim L_4$ and since the equivalence classes of lines in $\mathcal{L}_3$ are formed by 3 elements we get that $L_4 \in \{L_1, L_2, L_3\}$ and then $\Gamma_{x_i} \cap \mathcal{L}_3 = \{L_1, L_2, L_3\}$.

So $\Gamma_{x_1} \cap \mathcal{L}_3 = \Gamma_{x_2} \cap \mathcal{L}_3 = \Gamma_{x_3} \cap \mathcal{L}_3 = \{L_1, L_2, L_3\}$ and we deduce that x_1, x_2, x_3 are equivalent. By Lemma 1 (c), no other point can be equivalent to them, and consequently the equivalence class is formed by exactly 3 elements.

In summary, the equivalence classes in $X_3 \cup X_4$ are formed by 3 points (which use the same 3 lines of $\mathcal{L}_3$) except for the class of points which do not use any line of $\mathcal{L}_3$.

We can now define an injective map from $\mathcal{L}_3/\sim \longrightarrow (X_3 \cup X_4)/\sim$ associating to each equivalence class $\{L_1, L_2, L_3\}$ the set $X_{L_1} = X_{L_2} = X_{L_3} = \{x_1, x_2, x_3\}$. So we deduce that

$$|\mathcal{L}_3| \leq |X_3 \cup X_4|. \tag{7}$$

Let us now define

$$Q := \{(x, L) \mid x \in X, L \in \mathcal{L} \cap \Gamma_x\}.$$

We can compute the cardinal of this set either counting the number of lines used by each point or counting the number of points using a given line:

$$|Q| = 2|X_2| + 3|X_3| + 4|X_4| = |\mathcal{L}_1| + 2|\mathcal{L}_2| + 3|\mathcal{L}_3| \tag{8}$$

Using (6) and (7) we can substitute $|X_2| = 15 - |X_3| - |X_4|$, $|\mathcal{L}_2| \leq 15 - |\mathcal{L}_0| - |\mathcal{L}_1| - |\mathcal{L}_3|$ and $|\mathcal{L}_3| \leq |X_3| + |X_4|$ to obtain

$$30 + |X_3| + 2|X_4| = |Q| \leq 30 - 2|\mathcal{L}_0| - |\mathcal{L}_1| + |\mathcal{L}_3| \leq 30 - 2|\mathcal{L}_0| - |\mathcal{L}_1| + |X_3| + |X_4|,$$

which means that $|X_4| + 2|\mathcal{L}_0| + |\mathcal{L}_1| \leq 0$, that is,

$$X_4 = \emptyset, \quad \mathcal{L}_0 = \mathcal{L}_1 = \emptyset. \tag{9}$$

Therefore no point uses 4 lines of $\mathcal{L}$ and any line in $\mathcal{L}$ is used by at least 2 points of X. Taking this into account (6), (7) and (8) can be rewritten as

$$|\mathcal{L}_2| + |\mathcal{L}_3| \leq 15 = |X_2| + |X_3|, \quad |\mathcal{L}_3| \leq |X_3|, \quad 2|X_2| + 3|X_3| = 2|\mathcal{L}_2| + 3|\mathcal{L}_3|,$$

and from these inequalities, we deduce that

$$|X_3| = 2|X_2| + 3|X_3| - 30 = 2|\mathcal{L}_2| + 3|\mathcal{L}_3| - 30 \leq |\mathcal{L}_3| \leq |X_3|.$$

Then all inequalities of this chain must be equalities and we obtain

$$|X_3| = |\mathcal{L}_3|, \quad |X_2| = |\mathcal{L}_2|. \tag{10}$$

From here, it follows that $|\mathcal{L}| = |\mathcal{L}_2| + |\mathcal{L}_3| = |X_2| + |X_3| = 15$, and therefore equality in Proposition 4 (b) holds. So we deduce that each point of X is the intersection of exactly 4 lines of $\mathcal{L}$.

Since $X = X_2 \cup X_3$ we know that each node in X uses at least two lines of $\mathcal{L}$. By Proposition 4 (d), there exists a line L such that $|X_L| > 2$, that is, $\mathcal{L}_3$ is nonempty. Let $L_1 \in \mathcal{L}_3$ and $\{x_1, x_2, x_3\} := X_{L_1}$. By Lemma 1 (d), there exist lines $L_2, L_3 \in \mathcal{L}$ such that $L_i \cap L_j \cap X = \emptyset$ and $\Gamma_{x_1} \cap \Gamma_{x_2} \cap \Gamma_{x_3} = \{L_1, L_2, L_3\}$.

Let W_1, W_2, W_3, W_4 be the four lines in $\mathcal{L}$ such that $x_1 \in W_i$, $i = 1, 2, 3, 4$. By Lemma 1 (a) x_1, x_2 and x_3 are not collinear, which means that for each $j \in \{1, 2, 3, 4\}$ either $x_2 \notin W_j$ or $x_3 \notin W_j$. On the other hand, $L_i \in \Gamma_{x_2} \cap \Gamma_{x_3}$ and, if $L_i \cap W_j \cap X = \emptyset$, Proposition 4 (c) would imply that either $W_j \in \Gamma_{x_2}$ or $W_j \in \Gamma_{x_3}$. Then either x_2 or x_3 would use four lines in $\mathcal{L}$: L_1, L_2, L_3, W_j, contradicting the fact that

$X_4 = \emptyset$. Therefore one has $L_i \cap W_j \cap X \neq \emptyset$, $i = 1,2,3$, $j = 1,2,3,4$. So the 4 nodes of each W_j are x_1 and $L_i \cap W_j$, $i = 1,2,3$, and since neither x_2 nor x_3 lie on $L_1 \cup L_2 \cup L_3$ (because x_2, x_3 use the lines L_1, L_2, L_3) we get that x_2, x_3 are the two unique points of the set

$$X \setminus (W_1 \cup W_2 \cup W_3 \cup W_4).$$

Since each of the points x_1, x_2, x_3 uses the lines L_1, L_2, L_3, the Lagrange polynomials (2) corresponding to those points in the interpolation problem defined by X contain the factor $L_1 L_2 L_3$. Each polynomial of degree less than or equal to 4 vanishing on $X \setminus \{x_1, x_2, x_3\}$ is, by the Lagrange formula, a linear combination of those Lagrange polynomials and therefore contains the factor $L_1 L_2 L_3$. The product of the four lines $W_1 W_2 W_3 W_4$ vanishes in all points of $X \setminus \{x_2, x_3\}$ and consequently $L_1 L_2 L_3$ divides $W_1 W_2 W_3 W_4$. This implies that each L_i divides some W_j, which is impossible.

So we have seen that the initial assumption that no line contains 5 nodes on a GC_4 set of nodes leads to a contradiction, which means that we have proved the following theorem.

Theorem 2 *Let X be a set of nodes satisfying* GC_4. *Then there exist 5 points in X which are collinear.*

As we have seen, the conjecture holds for $n \leq 4$ but there is a strongly increasing difficulty in the proof, which is very relevant passing from $n = 3$ to $n = 4$. As in [2], it seems very complicated to extend the arguments used above to $n > 4$ but we feel that some of these arguments could help.

Acknowledgement. Research partially supported by the Spanish Research Grant BFM2000–1253

References

[1] Busch, J. R., Nota sobre la interpolación de Lagrange en $\mathbf{R}^2$, *preprint*.

[2] Busch, J. R. (1990). A note on Lagrange interpolation in $\mathbf{R}^2$, *Rev. Un. Mat. Argentina*, **36**, 33–38.

[3] Carnicer, J. M. and Gasca, M. (2000). Planar Configurations with Simple Lagrange Formula, in *Mathematical Methods in CAGD: Oslo 2000*, Tom Lyche and Larry L. Schumaker (eds.), 55–62. Vanderbilt University Press, Nashville, TN.

[4] Cheney, W. and Light, W. (2000). *A Course in Approximation Theory*, Brooks/Cole Publishing Company, Pacific Grove CA.

[5] Chung, K. C. and Yao, T. H. (1977). On lattices admitting unique Lagrange interpolation, *SIAM J. Numer. Anal.* **14**, 735–743.

[6] Gasca, M. and Maeztu, J. I. (1982). On Lagrange and Hermite interpolation in $\mathbf{R}^n$, *Numer. Math.* **39**, 1–14.

[7] Gasca, M. and Sauer, T. (2000). Polynomial interpolation in several variables, *Adv. Comput. Math.*, **12**, 377–410.

[8] Gasca, M. and Sauer, T. (2000). On the history of multivariate polynomial interpolation, *J. Comput. Appl. Math.* **122**, 23–35.

[9] Guenter, R. B. and Roetman, E. L. (1970). Some observations on interpolation in higher dimensions, *Math. Comp.* **24**, 517–521.

[10] Lorentz, R. A. (2000). Multivariate Hermite interpolation by algebraic polynomials: a survey, *J. Comput. Appl. Math.* **122**, 167–201.

[11] Radon, J. (1948). Zur mechanishen Kubatur, *Monatshefte Math. Physik* **52**, 286–300.

J. M. Carnicer and M. Gasca
Depto. Matemática Aplicada, Universidad de Zaragoza
Edificio de Matemáticas, Planta 1ª. 50009 Zaragoza, Spain