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Linear parabolic problems involving measures

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Abstract. We develop a general solvability theory for linear evolution equations of the form $\dot{u} + Au = \mu$ on $\mathbb{R}^+$, where $-A$ is the infinitesimal generator of a strongly continuous analytic semigroup, and μ is a bounded Banach-space-valued Radon measure. It is based on the theory of interpolation-extrapolation spaces and the Riesz representation theorem for such measures.

The abstract results are illustrated by applications to second order parabolic initial value problems. In particular, the case where Radon measures occur on the Dirichlet boundary can be handled, which is important in control theory and has not been treated so far.

We also give sharp estimates under various regularity assumptions. They form the basis for the study of semilinear parabolic evolution equations with measures to be studied in a forthcoming paper jointly with P. Quittner.

Problemas lineales parabólicos involucrando medidas

Resumen. Desarrollamos una teoría general para la resolución de ecuaciones lineales de evolución de la forma $\dot{u} + Au = \mu$ sobre $\mathbb{R}^+$, donde $-A$ es el generador infinitesimal de un semigrupo analítico fuertemente continuo y μ es una medida de Radón con valores en un espacio de Banach. Utilizamos la teoría de interpolación-extrapolación de espacios y el teorema de representación de Riesz para tales medidas.

Los resultados abstractos son ilustrados mediante aplicaciones a problemas de valor inicial parabólicos de segundo orden. En particular, el caso importante en teoría de control en el que las medidas de Radón aparecen sobre la frontera Dirichlet puede ser contemplado pese a no haber sido tratado hasta ahora.

Damos también precisas estimaciones bajo diversas hipotesis de regularidad. Estos resultados constituyen la base para el estudio de ecuaciones semilineales parabólicas de evolución involucrando medidas que será abordado en un próximo trabajo conjunto con P. Quittner.

1. Introduction

In this paper we study linear parabolic evolution equations of the form

$$\dot{u} + Au = \mu \qquad \text{on} \quad \mathbb{R}^+, \tag{1}$$

where A is the negative infinitesimal generator of an analytic semigroup U on some Banach space E, and μ is a bounded E-valued Radon measure on $\mathbb{R}^+$. Equation (1) is to be understood in a weak sense. More

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precisely, a solution is a locally integrable E-valued function u on $\mathbb{R}^+$ such that

$$\int_0^\infty \{-\langle\dot\varphi, u\rangle + \langle A^\top\varphi, u\rangle\}\, dt = \int_0^\infty \varphi\, d\mu \tag{2}$$

for all smooth E'-valued functions with compact support in $\mathbb{R}^+ := [0, \infty)$, and $A^\top$ being the dual of A (with respect to a suitable duality pairing). It is shown that (1) possesses a unique solution and that it is given, as can be expected, by $U \star \mu$, the convolution of the semigroup U with the vector measure μ.

These facts, although looking natural, are far from obvious. For example, since U is strongly continuous only, it is by no means clear how to define $U \star \mu$ in general. Furthermore, the validity of (2) depends on a careful choice of the space E and the operator A. As a rule, neither E nor A is the one appearing in the original concrete formulation of problem (1), but is derived from the latter by interpolation-extrapolation techniques. This has already been done in [10] for elliptic problems. But here the situation is more complicated since we deal with evolution problems.

We investigate in detail problem (1) and prove basic existence, regularity, and continuity theorems. They are fundamental for the study of semilinear parabolic evolution equations involving measures carried out in [11]. However, the results of this paper are of interest for their own sake as well. They apply, in particular, to linear parabolic boundary value problems with measure data. In the following, we describe some of these applications for second order model problems. More general cases are treated in Section 7.

Throughout this paper, Ω is a nonempty subdomain of $\mathbb{R}^n$ with a compact boundary Γ. If $\Gamma \neq \emptyset$ then it is supposed to be smooth and lying locally on one side of Ω. We denote by γ the trace operator, by $\vec{\nu}$ the outward pointing unit normal, and by ∂_ν the corresponding normal derivative on Γ, if $\Gamma \neq \emptyset$. In the latter case, we assume that $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$ and Γ_0 and Γ_1 being open (hence closed) in Γ. Of course, either Γ_0 or Γ_1 may be empty.

In the following all implicit or explicit references to Γ or Γ_j and all formulas containing Γ or Γ_j, respectively, have to be neglected if the corresponding boundary is empty.

We write $W_q^s := W_q^s(\Omega)$ for the usual Sobolev-Slobodeckii spaces for $s \in \mathbb{R}$ and $1 \leq q \leq \infty$ so that $W_q^0 = L_q$. If $1 \leq q < \infty$ then we denote by $\mathring{W}_q^s$ the closure of $\mathcal{D}$ in W_q^s, where $\mathcal{D} := \mathcal{D}(\Omega)$ is the space of smooth functions with compact support in Ω. Then $\mathring{W}_q^s = W_q^s$ for all $s \in \mathbb{R}$ if $\Gamma = \emptyset$, and for $s < 1/q$ otherwise. Furthermore, $W_q^{-s} = (\mathring{W}_{q'}^s)'$ for $s \geq 0$ and $1 < q < \infty$, with respect to the L_q-duality pairing induced by

$$\langle u, v\rangle := \int_\Omega uv\, dx, \qquad u, v \in \mathcal{D}.$$

We also set $\mathcal{D}(\overline{\Omega}) := \{u\,|\,\overline{\Omega}\ ;\ u \in \mathcal{D}(\mathbb{R}^n)\}$, etc. These spaces are given their usual topologies.

Let X be a σ-compact metric space, that is, a locally compact metric space which can be written as a countable union of compact subsets. Then $C_0(X)$ is the space of all continuous functions on X vanishing at infinity, endowed with the maximum norm. Moreover, $C_0(X)'$, the dual space of $C_0(X)$, is identified with the space $\mathcal{M}(X)$ of bounded Radon measures on X with respect to the duality pairing

$$\langle \mu, u\rangle := \langle \mu, u\rangle_{C_0(X)} := \int_X u\, d\mu, \qquad (\mu, u) \in \mathcal{M}(X) \times C_0(X).$$

We also put

$$\mathcal{M}_{\mathrm{loc}}(X \times \mathbb{R}^+) := \bigcap_{T>0} \mathcal{M}\big(X \times [0, T]\big)$$

and give this space its natural Fréchet space topology. The same applies to $L_{q,\mathrm{loc}}(\mathbb{R}^+, W_q^s)$, etc.

Now suppose that

$$(\mu_\Omega, \mu_\Gamma) \in \mathcal{M}_{\mathrm{loc}}(\Omega \times \mathbb{R}^+) \times \mathcal{M}_{\mathrm{loc}}(\Gamma \times \mathbb{R}^+) \tag{3}$$

and

$$\mathrm{dist}\big(\mathrm{supp}(\mu_{\Omega,s}), \Gamma \times \mathbb{R}^+\big) > 0, \tag{4}$$

where $\mu_{\Omega,s}$ is the singular part of μ_Ω in its Lebesgue decomposition with respect to Lebesgue's measure $dx\,dt$ on $\Omega \times \mathbb{R}^+$. Consider the model problem

$$\left.\begin{aligned} \partial_t u - \Delta u &= \mu_\Omega && \text{in} && \Omega \times \mathbb{R}^+, \\ u &= \mu_0 && \text{on} && \Gamma_0 \times \mathbb{R}^+, \\ \partial_\nu u &= \mu_1 && \text{on} && \Gamma_1 \times \mathbb{R}^+, \end{aligned}\right\} \tag{5}$$

where $(\mu_0, \mu_1) := \mu_\Gamma$ with $\mu_j \in \mathcal{M}_{\mathrm{loc}}(\Gamma_j \times \mathbb{R}^+)$.

Suppose that $1 < q < \infty$. By a (weak) L_q-solution of (5) we mean a function $u \in L_{1,\mathrm{loc}}(\mathbb{R}^+, L_q)$ satisfying

$$-\int_0^\infty \langle \partial_t \varphi + \Delta\varphi, u\rangle\, dt = \int_{\Omega\times\mathbb{R}^+} \varphi\, d\mu_\Omega + \int_{\Gamma_1\times\mathbb{R}^+} \varphi\, d\mu_1 - \int_{\Gamma_0\times\mathbb{R}^+} \partial_\nu\varphi\, d\mu_0 \tag{6}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^+, \mathcal{D}_{\mathcal{B}})$, where

$$\mathcal{D}_{\mathcal{B}} := \{\, \varphi \in \mathcal{D}(\overline{\Omega})\ ;\ \varphi = 0 \text{ on } \Gamma_0,\ \partial_\nu\varphi = 0 \text{ on } \Gamma_1 \,\}.$$

In particular, an L_q-solution is a distributional solution of the first equation of (5).

Observe that (6) is obtained from (5) by multiplying the first equation by φ, integrating over $\overline{\Omega} \times \mathbb{R}^+$, integrating by parts, and employing the boundary conditions — formally, of course. Here we have used assumption (4) to guarantee that $\int_{\Omega\times\mathbb{R}^+} \varphi\, d\mu$ is well-defined for $\varphi \in \mathcal{D}(\mathbb{R}^+, \mathcal{D}_{\mathcal{B}})$ since, a priori, it makes sense only for $\varphi \in \mathcal{D}(\mathbb{R}^+, \mathcal{D})$.

For this simple model problem our general results imply the following existence, uniqueness, continuity, and positivity result. Clearly, $q' := q/(q-1)$ is the exponent dual to q.

Theorem 1

- *Suppose that*
$$0 \le \sigma < 1 - n/q'. \tag{7}$$
Then problem (5) *possesses a unique* L_q*-solution u, and*
$$u \in L_{p,\mathrm{loc}}(\mathbb{R}^+, W_q^\sigma) \tag{8}$$
for each $p \ge 1$ *satisfying*
$$\frac{2}{p} + \frac{n}{q} > n + 1 + \sigma. \tag{9}$$
- *If* $\mu_0 = 0$ *then* (8) *is true for*
$$0 \le \sigma < 2 - n/q', \quad \frac{2}{p} + \frac{n}{q} > n + \sigma. \tag{10}$$
- *In either case the map* $(\mu_\Omega, \mu_\Gamma) \mapsto u$ *is linear and continuous from the subspace of*
$$\mathcal{M}_{\mathrm{loc}}(\Omega \times \mathbb{R}^+) \times \mathcal{M}_{\mathrm{loc}}(\Gamma \times \mathbb{R}^+)$$
defined by (4) *to* $L_{p,\mathrm{loc}}(\mathbb{R}^+, W_q^\sigma)$.
- *If* μ_Ω *and* μ_Γ *are positive measures then u is positive as well.*
- *u is independent of* $q \in \big(1, n/(n-1)\big)$ *in the general case, and, if* $\mu_0 = 0$*, of* $q \in \big(1, n/(n-2)\big)$*, where* $n/(n-2)$ *has to be replaced by* ∞ *if* $n = 1$.

PROOF. This is a special case of Theorem 13. ■

Remarks 1 **(a)** Assumption (7) implies $(n-1)q < n$, whereas it follows from the first inequality in (10) that $(n-2)q < n$. Also observe that we can choose σ close to 1 in the general case, resp. close to 2 if $\mu_0 = 0$, provided p and q are close to 1.
(b) If

$$\mu_0 = 0 \tag{11}$$

then the second inequality in (10) implies

$$u \in L_{p,\mathrm{loc}}(\mathbb{R}^+, W_q^1) \qquad \text{for} \quad \frac{2}{p} + \frac{n}{q} > n+1. \tag{12}$$

Furthermore, it follows from (8), (10), and Sobolev's embedding theorem that

$$u \in L_{p,\mathrm{loc}}\big(\mathbb{R}^+, C_0(\overline{\Omega}) \cap C^\rho(\overline{\Omega})\big), \qquad 0 \le \rho < 1,$$

if $n = 1$ and $\mu_0 = 0$.
(c) If $\mu_0 = 0$ then the left-hand side of (6) can be replaced by

$$\int_0^\infty \big\{ -\langle \partial_t \varphi, u \rangle + \langle \nabla \varphi, \nabla u \rangle \big\}\, dt,$$

that is, by the expression occurring in the standard formulation of weak solutions for the heat equation.
(d) If μ_Ω and μ_Γ have better regularity properties then the same is true for u.
(e) Theorem 1 remains valid if $-\Delta$ is replaced by an elliptic operator of the form

$$\mathcal{A}u := -\nabla \cdot (\boldsymbol{a} \nabla u) + \vec{a} \cdot \nabla u + a_0 u$$

and ∂_ν on Γ_1 by a boundary operator $\mathcal{B}_1$ of the form $\partial_{\boldsymbol{\nu}} u + bu$, respectively, where $\partial_{\boldsymbol{\nu}}$ is the conormal derivative on Γ_1, and $\mathcal{A}$ and $\mathcal{B}_1$ have appropriately smooth coefficients. It remains also valid if $\boldsymbol{a}$, $\vec{a}$, a_0, and b are $(N \times N)$-matrix-valued, that is, if $(\mathcal{A}, \gamma_0, \mathcal{B}_1)$ is a normally elliptic system, γ_0 being the trace operator on Γ_0. ■

Linear parabolic boundary value problems involving measures occur naturally in control theory, often formulated as final value problems for the formally adjoint problem (eg., [12], [14], [23], [18], [24], [25]). By choosing linear combinations of measures of the form $\mu \otimes \delta_t$ with $\mu \in \mathcal{M}(\overline{\Omega})$ and δ_t being the Dirac measure supported at $t \in \mathbb{R}^+$, so-called 'impulsive' systems can be incorporated into the framework of this paper as well. (We refer to [21] and the references given there, for example, for an idea on impulsive differential equations.)

Problem (5) (with general operators $\mathcal{A}$ and $\mathcal{B}$ of the form given above, but *not* for systems) has already been studied by several authors, always imposing the condition that μ_Γ does not charge Γ_0, that is, condition (11) is satisfied. More precisely, given (11) and a bounded domain, it has been shown by Casas [14] and Raymond [23] (also see [24], [25]) that (5) possesses a unique solution satisfying (12) (with the additional restriction $p, q < 2$). However, the class of test functions φ for (6) in those papers is larger than $\mathcal{D}(\mathbb{R}^+, \mathcal{D}_\mathcal{B})$ (and defined in a somewhat ad hoc manner). Thus our uniqueness assertion is more general. In addition, we get the much more precise regularity and continuity results exhibited in Theorem 1, which are of importance for studying nonlinear problems.

Given assumption (11), one has also to refer to the earlier work of Lasiecka [20] (absolutely continuous measures in an L_2-setting) and Baras and Pierre [13] (the case $\Gamma = \Gamma_0$). (We do not comment on the rather large literature on nonlinear equations involving measures since this will be done in [11].)

Although problems with measures on the Dirichlet boundary are quite natural, occurring, for example, in control theory when point-wise boundary observations are employed, this paper is the first one to deal with them in the parabolic case. (For the elliptic counterpart see [10].)

Our general results apply to problems involving more singular distributions than measures as well. For example, suppose that

$$\rho_j \in \mathcal{M}_{\mathrm{loc}}(\Omega \times \mathbb{R}^+), \quad \operatorname{dist}\big(\operatorname{supp}(\rho_{j,s}), \Gamma \times \mathbb{R}^+\big) > 0, \qquad 1 \le j \le n. \tag{13}$$

Consider the model problem

$$\left.\begin{aligned} \partial_t u - \Delta u &= \mu_\Omega + \textstyle\sum_j \partial_j \rho_j && \text{in} \quad \Omega \times \mathbb{R}^+, \\ u &= \mu_0 && \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \\ \partial_\nu u &= \mu_1 && \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \end{aligned}\right\} \tag{14}$$

where $\partial_j \rho_j$ is the distributional derivative of ρ_j in the j-th coordinate direction, of course. By an L_q-solution of (14) we mean a function $u \in L_{1,\mathrm{loc}}(\mathbb{R}^+, L_q)$ satisfying (6) with the term

$$-\sum_{j=1}^{n} \int_{\Omega \times \mathbb{R}^+} \partial_j \varphi \, d\rho_j$$

being added to the right-hand side.

Theorem 2 *Let assumptions* (3), (4), *and* (13) *be satisfied. Given condition* (7), *problem* (14) *possesses a unique L_q-solution satisfying* (8) *und* (9). *It is independent of q, and the map*

$$\big((\mu_\Omega, \rho_1, \ldots, \rho_n), \mu_0, \mu_1\big) \mapsto u$$

is continuous from the subspace of

$$\big(\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{M})\big)^{n+1} \times \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, \Gamma_0) \times \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, \Gamma_1)$$

defined by (4) *and* (13) *into* $L_{p,\mathrm{loc}}(\mathbb{R}^+, L_q)$.

PROOF. This is a special case of Remark 3(b). ■

Note that $\sum_j \partial_j \rho_j$ can be interpreted as distributions of dipoles in $\Omega \times \mathbb{R}^+$. Similar dipole distributions can be admitted on the Neumann boundary Γ_1 as well.

If $\Omega = \mathbb{R}^n$ then we can allow much more singular distributions. Indeed, given $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, W_q^{s-2})$ for some $s < 2 - m - n/q'$ with $m \in \mathbb{N}$, the differential equation

$$\partial_t u - \Delta u = \mu \qquad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+$$

possesses a unique distributional solution

$$u \in L_{1,\mathrm{loc}}(\mathbb{R}^+, W_q^\sigma), \qquad -m - \frac{n}{q'} \le \sigma < 2 - m - \frac{n}{q'},$$

and

$$u \in L_{p,\mathrm{loc}}(\mathbb{R}^+, W_q^\sigma), \qquad \frac{2}{p} + \frac{n}{q} > m + n + \sigma.$$

As mentioned in the beginning, the results for parabolic boundary value problems are special instances of our general results for (1). Parabolic evolution equations with singular data of a very general nature have lately been studied by G. Lumer (e.g., [22] and the references therein) by completely different techniques and applying to concrete problems of a different nature.

Convolutions involving operators, vector measures in particular, have been studied by many authors (see [15] and the references therein). However, those results do not seem to apply to our situation since the semigroup U is strongly continuous only. To define $U \star \mu$ we rely on duality theory and on the theory of interpolation-extrapolation spaces developed by the author (see [4, Chapter V] for an account).

In Section 2 we briefly collect some results on vector measures. In the next section we derive estimates for solutions of linear parabolic evolution equations of the form

$$\dot{u} + Au = f(t) \quad \text{in} \quad (0,T], \qquad u(0) = x$$

under various assumptions on f. These estimates form the basis, in Section 4, for the definition of $U \star \mu$ in suitable interpolation-extrapolation spaces. In that section we also prove a trace lemma which we need for piecing together local solutions in [11]. Furthermore, we derive an important and natural Green's formula.

Section 6 contains our general results for problem (1). Besides of establishing existence and uniqueness, we prove sharp regularity and continuity theorems under various restrictions on the data. These facts will be fundamental for our study of semilinear problems in [11].

Although the results of Section 6 apply to a variety of problems, we restrict ourselves in this paper to illustrating the applicability of some of them to second order parabolic boundary problems involving measures. We also leave it to the reader to translate the regularity results of Section 6 to concrete boundary value problems. Finally, we refrain from giving applications to higher order problems and systems or problems with dynamic boundary conditions. Some of those applications might be taken up in later publications.

We point out that one of the most crucial results in Section 7 is Theorem 10, giving a precise description of abstract extrapolation spaces in terms of standard spaces of distributions on Ω and Γ. This theorem also clarifies the setting of [10].

For the sake of simplicity we have restricted ourselves to the case of a constant operator A. Building on some estimates derived in [4], it is not too difficult to extend the abstract theory of this paper to non-autonomous equations of the form $\dot{u} + A(t)u = \mu$. Since some points are more technical and are likely to obscure the simple ideas, this will be done somewhere else.

Finally, many thanks go to P. Quittner for reading preliminary versions of this paper and helpful comments.

2. Vector measures

Let X be a σ-compact metric space and E a Banach space with norm $|\cdot|$. Denote by $\mathcal{B}_X$ the Borel σ-algebra of X. By an E-valued measure on X we mean a σ-additive map $\mu : \mathcal{B}_X \to E$ satisfying $\mu(\emptyset) = 0$. For such a (vector) measure μ we define its variation $|\mu| : \mathcal{B}_X \to \mathbb{R}^+ \cup \{\infty\}$ by

$$|\mu|(B) := \sup_{\pi(B)} \sum_{A \in \pi(B)} |\mu(A)|, \qquad B \in \mathcal{B}_X,$$

where the supremum is taken over all partitions $\pi(B)$ of B into a finite number of pair-wise disjoint Borel subsets. Then $|\mu|$ is a positive Borel measure on X. The (vector) measure μ is said to be of bounded variation if

$$\|\mu\|_{\mathcal{M}} := |\mu|(X) < \infty.$$

We write $\mathcal{M}(X,E) := \bigl(\mathcal{M}(X,E), \|\cdot\|_{\mathcal{M}}\bigr)$ for the normed vector space of all E-valued measures of bounded variation so that $\mathcal{M}(X,\mathbb{C}) = \mathcal{M}(X)$.

Let E_0, E_1, and E_2 be Banach spaces and suppose that

$$E_1 \times E_2 \to E_0, \quad (e_1, e_2) \mapsto e_1 \bullet e_2 \tag{15}$$

is a continuous bilinear map of norm at most one, a multiplication. We denote by $\mathcal{B}(X,E_1)$ the closure in $B(X,E_1)$, the Banach space of all bounded maps from X into E_1, of the linear subspace $S(X,E_1)$ of all simple functions

$$u = \sum_{B \in \pi(X)} \chi_B e_B, \qquad e_B \in E_1, \tag{16}$$

where χ_B is the characteristic function of B. If $u \in S(X, E_1)$ is given by (16) and $\mu \in \mathcal{M}(X, E_2)$ then we put

$$\int_X u\, d\mu := \sum_{B \in \pi(X)} e_B \bullet \mu(B) \in E_0.$$

It follows that

$$S(X, E_1) \times \mathcal{M}(X, E_2) \to E_0, \quad (u, \mu) \mapsto \int_X u\, d\mu$$

is a well-defined bilinear map satisfying

$$\Big| \int_X u\, d\mu \Big|_{E_0} \leq \sum_{B \in \pi(X)} |e_B|\, |\mu(B)| \leq \sum_{B \in \pi(X)} |e_B|\, |\mu|\, (B) \leq \|u\|_\infty\, \|\mu\|_{\mathcal{M}}.$$

Hence it possesses a bilinear continuous extension over $\mathcal{B}(X, E_1) \times \mathcal{M}(X, E_2)$ of norm at most one, again denoted by the same symbol.

Recall that $C_0(X, E)$ is the space of all continuous E-valued functions on X vanishing at infinity, endowed with the maximum norm. It is a closed linear subspace of $BUC(X, E)$, the Banach space of bounded and uniformly continuous E-valued functions on X, hence a Banach space, and $C_0(X, \mathbb{C}) = C_0(X)$.

Since $C_0(X, E_1)$ is a closed linear subspace of $\mathcal{B}(X, E_1)$, we obtain by restriction a well-defined multiplication

$$C_0(X, E_1) \times \mathcal{M}(X, E_2) \to E_0, \quad (u, \mu) \mapsto \int_X u\, d\mu, \tag{17}$$

and $\int_X u\, d\mu$ is said to be the integral of u over X with respect to the (vector) measure μ (and multiplication (1.1)). Moreover,

$$\Big| \int_X u\, d\mu \Big| \leq \int_X |u|\, d|\mu| \leq \|u\|_\infty\, \|\mu\|_{\mathcal{M}}. \tag{18}$$

Now suppose that $E_1 := E$, $E_2 := E'$, and $E_0 := \mathbb{C}$. Then (17) implies that $u \mapsto \int_X u\, d\mu$ is a continuous linear form on $C_0(X, E)$ satisfying (18). The converse is also true, that is, the (generalized) Riesz representation theorem holds:

$$C_0(X, E)' = \mathcal{M}(X, E') \tag{19}$$

with respect to the duality pairing

$$\langle \mu, u \rangle := \langle \mu, u \rangle_{C_0(X,E)} := \int_X u\, d\mu, \qquad (\mu, u) \in \mathcal{M}(X, E') \times C_0(X, E). \tag{20}$$

Thus, in particular, $\mathcal{M}(X, E')$ is a Banach space.

Now we give some general examples of vector-valued measures. They will be of importance in connection with evolution equations.

Here and below, J stands for a perfect interval, that is, an interval containing more than one point. We denote by $\mathcal{L}(E_0, E_1)$ the Banach space of bounded linear operators from E_0 into E_1, and $\mathcal{L}(E) := \mathcal{L}(E, E)$. Finally, we write $E_1 \hookrightarrow E_0$ if E_1 is continuously injected in E_0, and $E_1 \stackrel{d}{\hookrightarrow} E_0$ if E_1 is dense in E_0 as well. In later sections we use these notations also in cases where E_0 and E_1 are locally convex spaces.

Examples 1 **(a)** Let dt be Lebesgue's measure on J. Given $u \in L_1(J, E)$,

$$(u\, dt)(B) := \int_B u\, dt, \qquad B \in \mathcal{B}_J,$$

defines an E-valued measure $u\, dt \in \mathcal{M}(J, E)$, and $\|u\, dt\|_{\mathcal{M}} = \|u\|_1$. Hence

$$L_1(J, E) \to \mathcal{M}(J, E), \quad u \mapsto u\, dt$$

is a linear isometry. By means of it we identify $L_1(J,E)$ with a closed linear subspace of $\mathcal{M}(J,E)$.
PROOF. For this we refer to [19]. ■
(b) Denote by δ_t the Dirac measure with support at $t \in J$ and let $e \in E$ be given. Then $\mu := e \otimes \delta_t$ belongs to $\mathcal{M}(J,E)$ and $\langle \mu, u\rangle_{C_0(J,E')} = \langle u(t), e\rangle_E$ for $u \in C_0(J,E')$.
(c) Let F be a Banach space and $T \in \mathcal{L}(E,F)$. Then

$$(\mu \mapsto T\mu) \in \mathcal{L}\big(\mathcal{M}(X,E), \mathcal{M}(X,F)\big).$$

In particular, $\mathcal{M}(X,E) \hookrightarrow \mathcal{M}(X,F)$ if $E \hookrightarrow F$.
(d) It is not difficult to prove that $C_0(J \times X) = C_0\big(J, C_0(X)\big)$ by means of the 'canonical' identification

$$u(t,x) := u(t)(x), \qquad (t,x) \in J \times X,$$

for $u \in C_0\big(J, C_0(X)\big)$. From this it follows that $\mathcal{M}(J \times X) = \mathcal{M}\big(J, \mathcal{M}(X)\big)$ with respect to the 'canonical' identification

$$\mathcal{M}(J \times X) \ni \nu \longleftrightarrow \mu_\nu \in \mathcal{M}\big(J, \mathcal{M}(X)\big),$$

given by

$$\langle \mu_\nu, u\rangle_{C_0(J,C_0(X))} = \langle \nu, u\rangle_{C_0(J\times X)} \tag{21}$$

for $u \in C_0\big(J, C_0(X)\big) = C_0(J \times X)$. ■

Now suppose that E is reflexive and preordered by a closed convex cone P. Let P' be the dual cone of P inducing the natural (dual) preorder of E'. Then $\mu \in \mathcal{M}(X,E)$ is said to be positive, in symbols: $\mu \geq 0$, if

$$\langle \mu, u\rangle \geq 0, \qquad u \in C^+(X,E') := C(X,P').$$

Observe that, in the scalar case, this definition is consistent with the usual notion of a positive measure.

For the general theory of vector measures and the corresponding integration we refer to [17] (also see [16], [19], and [18]), where proofs for the above facts can be found. A direct short proof of the generalized Riesz representation theorem is given in [8, Theorem VI.2.2.4].

3. Preliminary estimates

Let (X_0, X_1) be a densely injected Banach couple, that is, X_0 and X_1 are Banach spaces such that $X_1 \overset{d}{\hookrightarrow} X_0$. For $0 < \theta < 1$ and $1 \leq q \leq \infty$ we denote by $(\cdot,\cdot)_{\theta,q}$ the real, by $(\cdot,\cdot)^0_{\theta,\infty}$ the continuous, and by $[\cdot,\cdot]_\theta$ the complex interpolation functor of exponent θ. Then we set

$$X_{\theta,q} := (X_0,X_1)_{\theta,q}, \quad X^0_{\theta,\infty} := (X_0,X_1)^0_{\theta,\infty}, \quad X_{[\theta]} := [X_0,X_1]_\theta.$$

It follows that

$$X_1 \overset{d}{\hookrightarrow} X_{\theta,q} \overset{d}{\hookrightarrow} X_{\theta,p} \overset{d}{\hookrightarrow} X^0_{\theta,\infty} \hookrightarrow X_{\theta,\infty} \overset{d}{\hookrightarrow} X_{\vartheta,1} \overset{d}{\hookrightarrow} X_{[\vartheta]} \overset{d}{\hookrightarrow} X^0_{\vartheta,\infty} \overset{d}{\hookrightarrow} X_0 \tag{22}$$

for $1 \leq q < p < \infty$ and $0 < \vartheta < \theta < 1$. (See [4, Section I.2] for a summary of interpolation theory.) To avoid tedious distinctions we also put

$$X_{j,q} := X^0_{j,\infty} := X_{[j]} := X_j, \qquad j \in \{0,1\}, \quad 1 \leq q \leq \infty.$$

Let X be a Banach space. Then $\mathcal{H}(X)$ is the set of all densely defined closed linear operators A in X such that $-A$ generates a strongly continuous analytic semigroup, denoted by $\{\, e^{-tA}\ ;\ t \geq 0 \,\}$, on X, that is, in $\mathcal{L}(X)$. We set

$$\mathcal{H}(X_1,X_0) := \mathcal{L}(X_1,X_0) \cap \mathcal{H}(X_0).$$

It follows that $A \in \mathcal{H}(X_0)$ belongs to $\mathcal{H}(X_1, X_0)$ iff $X_1 \doteq D(A)$, where $\doteq$ means equivalent norms, and $D(A)$ is the domain of A equipped with its graph norm. (For this and more details on $\mathcal{H}(X_1, X_0)$ and analytic semigroups we refer to [4, Section I.1].)

Henceforth, c denotes positive constants whose values may differ from occurrence to occurrence, but they are always independent of all free variables in a given situation. We set $J_T := [0, T]$ and $\dot{J}_T := (0, T]$ for $T > 0$. Finally, throughout this chapter T is a positive real number.

Now we suppose that

- (X_0, X_1) is a densely injected Banach couple and $A \in \mathcal{H}(X_1, X_0)$.

We set $U(t) := e^{-tA}$ for $t \geq 0$. It is a well-known fact from semigroup theory that $\{\, U(t) \,;\, t \geq 0 \,\}$ restricts to a strongly continuous analytic semigroup on X_1 and

$$\|U(t)\|_{\mathcal{L}(X_j)} + t\,\|U(t)\|_{\mathcal{L}(X_0, X_1)} \leq c, \qquad t \in \dot{J}_{\mathsf{T}}, \quad j = 0, 1. \tag{23}$$

Thus, by interpolation, given $0 \leq \xi < \eta \leq 1$,

$$\|U(t)\|_{\mathcal{L}(X_\xi)} + t^{\eta-\xi}\,\|U(t)\|_{\mathcal{L}(X_{\xi,\infty}, X_{\eta,1})} \leq c, \qquad t \in \dot{J}_{\mathsf{T}}, \tag{24}$$

where $X_\xi \in \{\, X_{\xi,q}, X^0_{\xi,\infty}, X_{[\xi]} \,;\, 1 \leq q \leq \infty \,\}$, and and U is strongly continuous on X_ξ if $X_\xi \neq X_{\xi,\infty}$ for $\xi > 0$.

In order to simplify the statements below we introduce the following convention: Let $F_{j,T}$ be Banach spaces with $F_{j,T} \hookrightarrow L_1(J_T, X_0)$ for $T \in \dot{J}_{\mathsf{T}}$ and $j = 0, 1$, and suppose that $B \in \mathcal{L}(F_{0,\mathsf{T}}, F_{1,\mathsf{T}})$ maps $F_{0,T}$ into $F_{1,T}$ for each $T \in \dot{J}_{\mathsf{T}}$. Then, given $\kappa \in \mathbb{R}$, we write

$$T^\kappa B \in \mathcal{L}(F_{0,T}, F_{1,T}) \qquad \mathsf{T}\text{-uniformly}$$

if $T^\kappa\,\|B\|_{\mathcal{L}(F_{0,T}, F_{1,T})} \leq c$ for $T \in \dot{J}_{\mathsf{T}}$.

Lemma 1 (i) *Suppose that* $0 \leq \xi < \eta \leq 1$ *and* $1 \leq p < 1/(\eta - \xi)$. *Then, setting* $\varepsilon := \xi - \eta + 1/p$,

$$T^{-\varepsilon} U \in \mathcal{L}\big(X_{\xi,\infty}, L_p(J_T, X_{\eta,1})\big) \qquad \mathsf{T}\text{-}\textit{uniformly.}$$

(ii) *If* $0 \leq \eta < \xi < 1$ *then* $U \in \mathcal{L}\big(X^0_{\xi,\infty}, C^{\xi-\eta}(J_{\mathsf{T}}, X_{\eta,1})\big)$.

PROOF. (i) follows from (24), and (ii) is a special case of [4, Theorem II.5.3.1]. ■

For $T \in \dot{J}_{\mathsf{T}}$ and $f \in L_1(J_T, X_0)$ we put

$$U \star f(t) := \int_0^t U(t-\tau) f(\tau)\, d\tau, \qquad t \in J_T,$$

and investigate mapping properties of the linear operator $U\star := (f \mapsto U \star f)$.

Lemma 2 $U\star \in \mathcal{L}\big(L_1(J_T, X_0), C(J_T, X_0)\big)$ T-*uniformly.*

PROOF. It is obvious from (23) that

$$U\star \in \mathcal{L}\big(L_1(J_T, X_0), B(J_T, X_0)\big) \qquad \mathsf{T}\text{-uniformly.}$$

Suppose that $f \in L_1(J_T, X_0)$ and $0 \leq s < t \leq T$. Then, writing $\|\cdot\|$ for the norm in X_0,

$$\|U \star f(s) - U \star f(t)\| \leq c \int_s^t \|f(\tau)\|\, d\tau + a^0(s, t),$$

where

$$a^\sigma(s,t) := \int_0^{s-\sigma} \|U(s-\tau) - U(t-\tau)\|_{\mathcal{L}(X_0)} \, \|f(\tau)\| \, d\tau, \qquad 0 \le \sigma \le s,$$

and $a^\sigma(s,t) := 0$ for $\sigma > s$. Note that

$$|a^0(s,t) - a^\sigma(s,t)| \le c \int_{(s-\sigma)_+}^{s} \|f(\tau)\| \, d\tau, \qquad 0 \le \sigma < T.$$

Thus, thanks to $f \in L_1(J_T, X_0)$, it follows that $U \star f \in C(J_T, X_0)$ if we show that $a^\sigma(s,t) \to 0$ as $t - s \to 0$ for each fixed $\sigma \in (0,T)$. But this is an immediate consequence of the uniform continuity of U on $[\sigma, \mathsf{T}]$ for $0 < \sigma < \mathsf{T}$. ■

Next we consider the case where f has values in $X_{\xi,\infty}$ for some $\xi \in (0,1)$. For later use we prove a result more general than presently needed.

Lemma 3 (i) *Suppose that* $0 \le \xi < \eta \le 1$ *and* $1 \le r \le p \le \infty$ *satisfy* $\eta - \xi < 1$. *Also suppose that*

$$\varepsilon := \xi - \eta + 1/r' + 1/p \ge 0$$

and that $r > 1$ *and* $p < \infty$ *if* $\varepsilon = 0$. *Then*

$$T^{-\varepsilon} U \star \in \mathcal{L}\big(L_r(J_T, X_{\xi,\infty}), L_p(J_T, X_{\eta,1})\big) \qquad \mathsf{T}\text{-uniformly.}$$

(ii) *Assume that* $1 < r < \infty$ *and* $0 \le \eta < 1/r'$. *Then*

$$U \star \in \mathcal{L}\big(L_r(J_T, X_0), C^\rho(J_T, X_{\eta,1})\big) \qquad \mathsf{T}\text{-uniformly}$$

for $0 \le \rho < 1/r' - \eta$.

PROOF. (i) It follows from (24) that

$$\|U \star f(t)\|_{X_{\eta,1}} \le c_0 \int_0^t (t-\tau)^{\xi-\eta} \, \|f(\tau)\|_{X_{\xi,\infty}} \, d\tau, \qquad t \in J_T, \tag{25}$$

for $T \in \dot{J}_\mathsf{T}$ and $f \in L_r(J_T, X_{\xi,\infty})$.

Set $k_T(t) := t^{\xi-\eta}\chi_T(t)$ and $g(t) := \chi_T(t)\,\|f(t)\|_{X_{\xi,\infty}}$ for $t \in \mathbb{R}$, denoting by χ_T the characteristic function of $(0,T)$. Suppose that $\eta - \xi < 1/q \le 1$ and $\varepsilon > 0$. Then

$$\|k_T\|_{L_q(\mathbb{R})} = c_1 T^{\xi-\eta+1/q}, \quad \|g\|_{L_r(\mathbb{R})} = \|f\|_{L_r(J_T, X_{\xi,\infty})}, \qquad T \in \dot{J}_\mathsf{T}.$$

Thus, by Young's inequality for convolutions,

$$\|k_T * g\|_{L_p(\mathbb{R})} \le c_1 T^{\xi-\eta+1/q} \, \|f\|_{L_r(J_T, X_{\xi,\infty})},$$

provided $1/p = 1/q + 1/r - 1$. Since $k_T * g(t)$ equals the integral on the right-hand side of (25), the assertion follows in this case.

If $1/p = \eta - \xi - 1/r' > 0$ then the assertion is a consequence of the Hardy-Littlewood inequality (cf. [27, Theorem 1.18.9.3]).

(ii) For $f \in L_r(J_T, X_0) =: L_r$ it follows that

$$U \star f(t) - U \star f(s) = \int_s^t U(t-\tau) f(\tau) \, d\tau + \int_0^s \big(U(t-\tau) - U(s-\tau)\big) f(\tau) \, d\tau, \qquad 0 \le s < t \le T. \tag{26}$$

From (24) we infer that

$$\Big\| \int_s^t U(t-\tau) f(\tau) \, d\tau \Big\|_{X_{\eta,1}} \le c \int_s^t (t-\tau)^{-\eta} \, \|f(\tau)\|_{X_0} \, d\tau \le c(t-s)^{1/r'-\eta} \, \|f\|_{L_r}, \; 0 \le s < t \le T. \tag{27}$$

Letting $s = 0$ in this estimate we see that, given $0 \le \xi < 1/r'$,

$$U\star \in \mathcal{L}\big(L_r(J_T, X_0), B(J_T, X_{\xi,1})\big) \qquad \mathsf{T}\text{-uniformly.} \tag{28}$$

Fix $\rho \in [0, 1/r' - \eta)$ and set $\xi := \rho + \eta$. Since $0 \le \xi < 1/r'$, we deduce from

$$U(t-\tau) - U(s-\tau) = \big(U(t-s) - 1\big)U(s-\tau)$$

that

$$\Big\| \int_0^s \big(U(t-\tau) - U(s-\tau)\big) f(\tau)\, d\tau \Big\|_{X_{\eta,1}} \le \|U(t-s) - 1\|_{\mathcal{L}(X_{\xi,\infty}, X_{\eta,1})} \|U \star f\|_{B(J_T, X_{\xi,\infty})}$$

for $0 \le s < t \le T$. Hence $X_{\xi,1} \hookrightarrow X_{\xi,\infty}$, Lemma 1, and (28) imply

$$\Big\| \int_0^s \big(U(t-\tau) - U(s-\tau)\big) f(\tau)\, d\tau \Big\|_{X_{\eta,1}} \le c(t-s)^\rho \|f\|_{L_r}, \qquad 0 \le s < t \le T, \tag{29}$$

T-uniformly. Now it follows from (26), (27), and (29) that

$$\|U \star f(t) - U \star f(s)\|_{X_{\eta,1}} \le c(t-s)^\rho \|f\|_{L_r}, \qquad s, t \in J_T, \quad f \in L_r,$$

T-uniformly. This and (28) imply the assertion. ■

For $1 \le p \le \infty$ we put

$$\mathbb{W}_p^1\big(J, (X_0, X_1)\big) := L_p(J, X_1) \cap W_p^1(J, X_0)$$

and

$$\mathbb{C}^1\big(J, (X_0, X_1)\big) := C(J, X_1) \cap C^1(J, X_0).$$

In the next lemma we collect some further mapping properties of U.

Lemma 4 (i) *Suppose that* $1 \le r < \infty$ *and* $0 < \xi < 1$. *Then*

$$U\star \in \mathcal{L}\Big(L_r(J_T, X_{\xi,\infty}), \mathbb{W}_r^1\big(J_T, (X_0, X_1)\big)\Big) \qquad \mathsf{T}\text{-uniformly.}$$

(ii) *If* $0 \le 1/p' < \xi \le 1$ *then, setting* $\delta := \xi - 1/p'$,

$$T^{-\delta} U \in \mathcal{L}\Big(X_{\xi,\infty}, \mathbb{W}_p^1\big(J_T, (X_0, X_1)\big)\Big) \qquad \mathsf{T}\text{-uniformly.}$$

PROOF. (i) Lemma 3(i) implies

$$U\star \in \mathcal{L}\big(L_r(J_T, X_{\xi,\infty}), L_r(J_T, X_1)\big) \qquad \mathsf{T}\text{-uniformly.}$$

Hence

$$AU\star \in \mathcal{L}\big(L_r(J_T, X_{\xi,\infty}), L_r(J_T, X_0)\big) \qquad \mathsf{T}\text{-uniformly.} \tag{30}$$

If $f \in C^1(J_T, X_0)$ then it is well-known that $u := U \star f \in \mathbb{C}^1\big(J_T, (X_0, X_1)\big)$ and $\dot u = -Au + f$. From this and (30) we deduce that

$$\|\partial_t (U \star f)\|_{L_r(J_T, X_0)} \le \|AU \star f\|_{L_r(J_T, X_0)} + \|f\|_{L_r(J_T, X_0)} \le c\, \|f\|_{L_r(J_T, X_{\xi,\infty})}$$

for $f \in C^1(J_T, X_{\xi,\infty})$ and $T \in \dot J_{\mathsf{T}}$. Now the assertion follows from the density of $C^1(J_T, X_{\xi,\infty})$ in $L_r(J_T, X_{\xi,\infty})$.

(ii) From (24) we see that $\|U(t)\|_{\mathcal{L}(X_{\xi,\infty}, X_1)} \le c t^{\xi - 1}$ for $t \in \dot J_{\mathsf{T}}$. Hence

$$\|U\|_{\mathcal{L}(X_{\xi,\infty}, L_p(J_T, X_1))} \le c T^\delta \qquad \mathsf{T}\text{-uniformly.} \tag{31}$$

Consequently,

$$\|AU\|_{\mathcal{L}(X_{\xi,\infty},L_p(J_T,X_0))} \le cT^\delta \qquad \mathsf{T}\text{-uniformly.} \tag{32}$$

Since $u := Ux \in \mathbb{C}^1\big(\dot{J}_T, (X_0, X_1)\big)$ and $\dot{u} = -AUx$ on $\dot{J}_T$ for $x \in X_0$, the assertion is implied by (31) and (32). ■

We close this section with some important embedding results.

Theorem 3 *Suppose that* $1 \le p < \infty$ *and* $0 < s < 1$. *If* $0 \le \theta < 1 - s$ *then*

$$\mathbb{W}_p^1\big(J,(X_0,X_1)\big) \hookrightarrow \begin{cases} L_{p/(1-ps)}(J, X_{\theta,1}) & \text{if} \quad 0 < s < 1/p, \\ C^{s-1/p}(J, X_{\theta,1}) & \text{if} \quad 1/p < s < 1. \end{cases} \tag{33}$$

If X_1 *is compactly injected in* X_0 *then the injections in* (33) *are compact as well.*

PROOF. Fix $\vartheta \in (\theta, 1-s)$. It follows from the more general Theorem 5.2 in [5] that

$$\mathbb{W}_p^1\big(J,(X_0,X_1)\big) \hookrightarrow W_p^s(J, X_{\vartheta,p}), \tag{34}$$

where this injection is compact if X_1 is compactly injected in X_0. It is the latter case that has been considered in [5]. But that proof obviously implies (34) if we drop the compactness assumption. Since $W_p^s(J, X_{\vartheta,p}) \hookrightarrow W_p^s(J, X_{\theta,1})$ by (22), the assertion follows from (34) and Sobolev's embedding theorem which holds in the vector-valued case also (cf. [8] and [5]). ■

Observe that the limiting case $s = 1/p$ is covered by the trace theorem which guarantees that

$$\mathbb{W}_p^1\big(J,(X_0,X_1)\big) \hookrightarrow C(J, X_{1-1/p,p}), \qquad 1 < p < \infty, \tag{35}$$

(e.g., [4, Theorem III.4.10.2]).

4. The interpolation-extrapolation setting

Now we suppose that

- E_0 is a reflexive Banach space and $A_0 \in \mathcal{H}(E_0)$.

We set $E_k := D(A_0^k)$ for $k \in \mathbb{N}$. We also put $E_0^\sharp := E_0'$ and $A_0^\sharp := A_0'$, where A_0' is the dual of A_0 in the sense of unbounded linear operators in E_0. Finally, we put $E_k^\sharp := D\big((A_0^\sharp)^k\big)$ for $k \in \mathbb{N}$. Then we define E_{-k} for $k \in \mathbb{N}\backslash\{0\}$ by $E_{-k} := (E_k^\sharp)'$, with respect to the duality pairing induced by $\langle\cdot,\cdot\rangle_{E_0}$, the ($E_0'$-$E_0$)-duality-pairing. This means that E_{-k} is a realization of the dual space of $E_k^\sharp$ and

$$\langle y, x\rangle_{E_{-k}} = \langle y, x\rangle_{E_0}, \qquad x \in E_k, \quad y \in E_0^\sharp. \tag{36}$$

Since $E_k^\sharp \overset{d}{\hookrightarrow} E_0^\sharp$ it follows that $(E_0^\sharp)' = E_0 \overset{d}{\hookrightarrow} (E_k^\sharp)'$. Thus, by density, $\langle\cdot,\cdot\rangle_{E_{-k}}$, and hence E_{-k}, are uniquely determined by (36).

For each $\theta \in (0,1)$ we fix

$$(\cdot,\cdot)_\theta \in \big\{\, (\cdot,\cdot)_{\theta,q}, (\cdot,\cdot)_{\theta,\infty}^0, [\cdot,\cdot]_\theta \ ; \ 1 \le q < \infty \,\big\}$$

and put

$$E_{k+\theta} := (E_k, E_{k+1})_\theta, \qquad k \in \mathbb{Z}.$$

It follows that

$$E_s \overset{d}{\hookrightarrow} E_t, \qquad -\infty < t < s < \infty. \tag{37}$$

If $s > 0$ then we denote by A_s the maximal restriction of A_0 to E_s whose domain equals

$$\{\, x \in E_s \cap E_1 \; ; \; A_0 x \in E_s \,\}.$$

If $s < 0$ then A_s is the well-defined closure of A_0 in E_s. The families $\big[\,(E_s, A_s) \; ; \; s \in \mathbb{R}\big]$ and $[\,E_s \; ; \; s \in \mathbb{R}]$ are called interpolation-extrapolation scale and interpolation-extrapolation space scale, respectively, generated by (E_0, A_0) and $(\cdot,\cdot)_\theta,\ 0 < \theta < 1$, and A_s is the E_s-realization of A_0.

One shows that A_r is the closure of A_s if $r < s$ so that, in particular,

$$A_r \supset A_s, \qquad r < s. \tag{38}$$

Furthermore,

$$A_s \in \mathcal{H}(E_{s+1}, E_s), \quad e^{-tA_r} \supset e^{-tA_s}, \qquad t \geq 0, \quad r < s. \tag{39}$$

We define the dual interpolation functor $(\cdot,\cdot)^\sharp_\theta$ of $(\cdot,\cdot)_\theta$ by

$$(\cdot,\cdot)^\sharp_\theta := \begin{cases} [\cdot,\cdot]_\theta & \text{if} \quad (\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta, \\ (\cdot,\cdot)_{\theta,1} & \text{if} \quad (\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty}, \\ (\cdot,\cdot)_{\theta,q'} & \text{if} \quad (\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,q}, \quad q \neq 1. \end{cases}$$

It is known that $A^\sharp \in \mathcal{H}(E^\sharp_1, E^\sharp_0)$. Thus the interpolation-extrapolation scale $\big[\,(E^\sharp_s, A^\sharp_s) \; ; \; s \in \mathbb{R}\big]$ generated by $(E^\sharp, A^\sharp)$ and $(\cdot,\cdot)^\sharp_\theta,\ 0 < \theta < 1$, the dual scale to $\big[(E_s, A_s) \; ; \; s \in \mathbb{R}\big]$, is also well-defined if $q \neq 1$.

If $(\cdot,\cdot)_\theta \neq (\cdot,\cdot)_{\theta,1}$ then

$$(E_{-s})' \doteq E^\sharp_s, \quad (A_{-s})' = A^\sharp_s, \qquad s \in \mathbb{R}, \tag{40}$$

with respect to the duality pairing $\langle\cdot,\cdot\rangle_{E_{-s}}$ induced by $\langle\cdot,\cdot\rangle_{E_0}$. Moreover,

$$\langle A^\sharp_{-s} y, x\rangle_{E_s} = \langle y, A_{s-1} x\rangle_{E_{s-1}}, \qquad (y, x) \in E^\sharp_{1-s} \times E_s, \tag{41}$$

that is, $A^\sharp_{-s} \in \mathcal{L}(E^\sharp_{1-s}, E^\sharp_{-s})$ is the dual of $A_{s-1} \in \mathcal{L}(E_s, E_{s-1})$ for $s \in \mathbb{R}$. For proofs of these facts and many more details we refer to [4, Chapter V].

Denote by $[E_{s,1} \; ; \; s \in \mathbb{R}]$ and $[E^0_{s,\infty} \; ; \; s \in \mathbb{R}]$ the interpolation-extrapolation space scales generated for $0 < \theta < 1$ by $(\cdot,\cdot)_{\theta,1}$ and $(\cdot,\cdot)^0_{\theta,\infty}$, respectively. Then [6, Lemma 1.1] guarantees that

$$(E_{s-1}, E_s)_{\theta,1} \doteq E_{s-1+\theta,1}, \quad (E_{s-1}, E_s)^0_{\theta,\infty} \doteq E^0_{s-1+\theta,\infty} \tag{42}$$

for $0 \leq s \leq 1$ and $0 < \theta < 1$ with $s + \theta \neq 1$, and

$$(E_{s-1}, E_s)_{\theta,1} \overset{d}{\hookrightarrow} E_{s-1+\theta} \overset{d}{\hookrightarrow} (E_{s-1}, E_s)^0_{\theta,\infty}, \qquad 0 \leq s \leq 1, \quad 0 < \theta < 1. \tag{43}$$

Now we restrict $(\cdot,\cdot)_\theta$ by requiring it to be *admissible* in the sense that

$$(\cdot,\cdot)_\theta \in \big\{\,(\cdot,\cdot)_{\theta,q}, [\cdot,\cdot]_\theta \; ; \; 1 < q < \infty \,\big\}, \qquad 0 < \theta < 1. \tag{44}$$

It follows that E_s is reflexive for $s \in \mathbb{R}$. We also set

$$\|\cdot\|_s := \|\cdot\|_{E_s}, \quad \|\cdot\|^\sharp_s := \|\cdot\|_{E^\sharp_s}, \qquad \langle\cdot,\cdot\rangle_s := \langle\cdot,\cdot\rangle_{E_s}, \quad \langle\cdot,\cdot\rangle^\sharp_s := \langle\cdot,\cdot\rangle_{E^\sharp_s}.$$

We fix real numbers $\alpha < \beta < \alpha + 1$ and set

$$(X_0, X_1) := (E_{\alpha-1}, E_\alpha), \quad A := A_{\alpha-1}. \tag{45}$$

Then $A \in \mathcal{H}(X_1, X_0)$ by (39), and we put

$$U(t) := e^{-tA}, \qquad t \geq 0. \tag{46}$$

We also set

$$(Y_0, Y_1) := (E^\sharp_{-\alpha}, E^\sharp_{1-\alpha}), \quad A^\top := A^\sharp_{-\alpha}. \tag{47}$$

Then $A^\top \in \mathcal{H}(Y_1, Y_0)$, and we put

$$V(t) := e^{-tA^\top}, \qquad t \geq 0. \tag{48}$$

It follows from (40) that Y_0 is the dual of X_1 and Y_1 is the dual of X_0 with respect to the duality pairing $\langle\cdot,\cdot\rangle_\alpha$ and $\langle\cdot,\cdot\rangle_{\alpha-1}$, respectively. Furthermore, (41) shows that

$$\langle A^\top y, x\rangle_\alpha = \langle y, Ax\rangle_{\alpha-1}, \qquad (y,x) \in Y_1 \times X_1,$$

that is, $A^\top \in \mathcal{L}(Y_1, Y_0)$ is the dual of $A \in \mathcal{L}(X_1, X_0)$ with respect to these duality pairings. From [4, Proposition V.2.6.5] it also follows that

$$\big\langle V(t)y, x\big\rangle_\alpha = \big\langle y, U(t)x\big\rangle_{\alpha-1}, \qquad (y,x) \in Y_1 \times X_1, \quad t > 0. \tag{49}$$

Thus $V(t) \in \mathcal{L}(Y_1, Y_0)$ is the dual of $U(t) \in \mathcal{L}(X_1, X_0)$ with respect to the duality pairings $\langle\cdot,\cdot\rangle_\alpha$ and $\langle\cdot,\cdot\rangle_{\alpha-1}$.

For $T \in \dot{J}_{\mathsf{T}}$ we define $V_T \circledast$ by

$$V_T \circledast g(t) := \int_t^T V(\tau - t) g(\tau)\, d\tau, \qquad t \in J_T, \quad g \in L_1(J_T, Y_0).$$

In the following lemma we collect some mapping properties of V and $V_T \circledast$.

Lemma 5 (i) *Suppose that $1 < p < 1/(\alpha - \beta + 1)$ and set $\varepsilon := \beta - \alpha - 1/p'$. Then*

$$T^{-\varepsilon} V_T \circledast \in \mathcal{L}\big(L_{p'}(J_T, E^\sharp_{-\alpha}), C(J_T, E^\sharp_{1-\beta})\big) \qquad \mathsf{T}\text{-uniformly}.$$

(ii) *Assume that there exists an admissible interpolation functor $\{\cdot,\cdot\}_{\beta-\alpha}$ of exponent $\beta - \alpha$ such that*

$$E_{\beta-1} \doteq \{E_{\alpha-1}, E_\alpha\}_{\beta-\alpha}. \tag{50}$$

Then

$$V \,|\, J_T \in \mathcal{L}\big(E^\sharp_{1-\beta}, C(J_T, E^\sharp_{1-\beta})\big) \qquad \mathsf{T}\text{-uniformly}.$$

PROOF. (i) From (43) we infer that $Y_{1+\alpha-\beta,1} \overset{d}{\hookrightarrow} E^\sharp_{1-\beta}$. Consequently,

$$L_{p'}(J_T, Y_{1+\alpha-\beta,1}) \hookrightarrow L_{p'}(J_T, E^\sharp_{1-\beta}) \qquad \mathsf{T}\text{-uniformly},$$

that is, the norm of this linear map in bounded, uniformly with respect to $T \in \dot{J}_{\mathsf{T}}$. Thus it follows from Lemma 3(i) (with (X_0, X_1) replaced by (Y_0, Y_1) and with $p := \infty$, $r := p'$, $\xi := 0$, and $\eta := 1 + \alpha - \beta$) by an obvious change of variables that

$$T^{-\varepsilon} V_T \circledast \in \mathcal{L}\big(L_{p'}(J_T, Y_0), B(J_T, E^\sharp_{1-\beta})\big) \qquad \mathsf{T}\text{-uniformly}. \tag{51}$$

Since $V_T \circledast g \in C(J_T, Y_1) \hookrightarrow C(J_T, E^\sharp_{1-\beta})$ for $g \in C^1(J_T, Y_1)$, the assertion is a consequence of (51) and the density of $C^1(J_T, Y_1)$ in $L_{p'}(J_T, Y_0)$.

(ii) Recall that $Vy \in C(J_{\mathsf{T}}, Y_1)$ for $y \in Y_1$. Hence $Y_1 \overset{d}{\hookrightarrow} Y_{1+\alpha-\beta,1} \overset{d}{\hookrightarrow} E^\sharp_{1-\beta}$ implies that $Vy \in C(J_{\mathsf{T}}, E^\sharp_{1-\beta})$ for $y \in Y_1$. Thanks to (50) we deduce from the duality properties of admissible interpolation functors that

$$\begin{aligned}\{Y_0, Y_1\}_{1+\alpha-\beta} &= \{E^\sharp_{-\alpha}, E^\sharp_{1-\alpha}\}^\sharp_{1+\alpha-\beta} \doteq \big\{(E_\alpha)', (E_{\alpha-1})'\big\}^\sharp_{1+\alpha-\beta}\\ &= \big(\{E_\alpha, E_{\alpha-1}\}_{1+\alpha-\beta}\big)' = \big(\{E_{\alpha-1}, E_\alpha\}_{\beta-\alpha}\big)' \doteq (E_{\beta-1})' = E^\sharp_{1-\beta}.\end{aligned}$$

Thus (24) implies $\|V(t)\|_{\mathcal{L}(E^\sharp_{1-\beta})} \le c$ for $t \in \mathsf{T}$. Now the assertion follows by the density of Y_1 in $E^\sharp_{1-\beta}$. ■

5. Convolutions of semigroups with vector measures

After the preceding preparations we can now define the convolution $U \star \mu$ of a strongly continuous semigroup with a vector measure. We also prove a generalized Green's formula which is basic for interpreting $U \star \mu$ in terms of the original evolution equation.

5.A. Definitions and basic properties For

$$\mu := f \in L_1(J_T, E_{\beta-1}) \subset \mathcal{M}(J_T, E_{\beta-1})$$

we see that

$$U \star \mu(t) := \int_0^t U(t-\tau)\,\mu(d\tau) = \int_0^t U(t-\tau) f(\tau)\,d\tau = U \star f(t), \qquad t \in J_T.$$

Now we define $U \star \mu$ for all $\mu \in \mathcal{M}(J_T, E_{\beta-1})$. Since $\tau \mapsto U(t-\tau)$ is not continuous but strongly continuous only (at $\tau = t$), the integral $\int_0^t U(t-\tau)\,\mu(d\tau)$ does not have an obvious meaning for general vector measures. We avoid this difficulty by a duality approach.

Suppose that $1 < p < 1/(\alpha - \beta + 1)$ and set

$$\Phi_T := [V_T \circledast]' \in \mathcal{L}\Big(\big[C(J_T, E^\sharp_{1-\beta})\big]', \big[L_{p'}(J_T, E^\sharp_{-\alpha})\big]'\Big), \qquad T \in \dot{J}_{\mathsf{T}}. \tag{52}$$

Thanks to Lemma 5(i), this linear map is well-defined. Since $E^\sharp_{-\alpha}$ is reflexive, it follows from (40) that $(E^\sharp_{-\alpha})' = E_\alpha$ with respect to the duality pairing $\langle\cdot,\cdot\rangle^\sharp_{-\alpha}$. Hence

$$\big[L_{p'}(J_T, E^\sharp_{-\alpha})\big]' = L_p(J_T, E_\alpha)$$

with respect to the $L_{p'}$-duality pairing induced by $\langle\cdot,\cdot\rangle^\sharp_{-\alpha}$. Furthermore, we deduce from (19) and reflexivity that

$$\big[C(J_T, E^\sharp_{1-\beta})\big]' = \mathcal{M}(J_T, E_{\beta-1})$$

with respect to the C_0-duality pairing induced by $\langle\cdot,\cdot\rangle^\sharp_{1-\alpha}$. Thus it follows from Lemma 5(i) that

$$T^{-\varepsilon}\Phi_T \in \mathcal{L}\big(\mathcal{M}(J_T, E_{\beta-1}), L_p(J_T, E_\alpha)\big) \qquad \mathsf{T}\text{-uniformly}, \tag{53}$$

where $\varepsilon := \beta - \alpha - 1/p'$.

Our next lemma identifies the restriction of Φ_T to $L_1(J_T, E_{\beta-1})$.

Lemma 6 $\Phi_T \,|\, L_1(J_T, E_{\beta-1}) = U\,\star$.

PROOF. From (24) (with $\xi := \beta - \alpha$ and $\eta := 1$), (43), and (45)–(48) we infer that

$$\left|\left\langle y, U(t)x\right\rangle_{\alpha-1}\right| \le \|y\|_{Y_0} \|U(t)x\|_{X_1} \le ct^{\beta-\alpha-1} \|y\|_{Y_0} \|x\|_{X_{\beta-\alpha,\infty}} \le ct^{\beta-\alpha-1} \|y\|^{\sharp}_{-\alpha} \|x\|_{\beta-1}$$

for $(y, x) \in E^{\sharp}_{1-\alpha} \times E_\beta$ and $t \in \dot{J}_{\mathsf{T}}$. Thus, given

$$(v, u) \in L_{p'}(J_T, E^{\sharp}_{1-\alpha}) \times L_1(J_T, E_\beta) =: X,$$

it follows that

$$\int_0^T \langle v, U \star u\rangle_{\alpha-1}\, dt = \int_0^T \int_0^t \left\langle v(t), U(t-\tau)u(\tau)\right\rangle_{\alpha-1} d\tau\, dt = \int_0^T \int_0^t \left\langle u(\tau), V(t-\tau)v(t)\right\rangle^{\sharp}_{-\alpha} d\tau\, dt$$

and

$$\begin{aligned}\int_0^T \int_0^t \left|\left\langle v(t), U(t-\tau)u(\tau)\right\rangle_{\alpha-1}\right| d\tau\, dt &\le c\int_0^T \int_0^t (t-\tau)^{\beta-\alpha-1} \|v(t)\|^{\sharp}_{-\alpha} \|u(\tau)\|_{\beta-1}\, d\tau\, dt \\ &\le c\int_0^T \|u(\tau)\|_{\beta-1} \int_\tau^T (t-\tau)^{\beta-\alpha-1} \|v(t)\|^{\sharp}_{-\alpha}\, dt\, d\tau \\ &\le cT^{\varepsilon} \|v\|_{L_{p'}(J_T, E^{\sharp}_{-\alpha})} \|u\|_{L_1(J_T, E_{\beta-1})}\end{aligned}$$

by Hölder's inequality and Tonelli's theorem. From this and Fubini's theorem we now infer that

$$\int_0^T \langle v, U \star u\rangle_{\alpha-1}\, dt = \int_0^T \langle u, V_T \circledast v\rangle^{\sharp}_{-\alpha}\, dt, \qquad (v, u) \in X. \tag{54}$$

The above estimates show that the left-hand side of (54) is continuous on X with respect to the topology induced by $Y := L_{p'}(J_T, E^{\sharp}_{-\alpha}) \times L_1(J_T, E_{\beta-1})$. We deduce from Lemma 5(i) that the right-hand side of (54) is bilinear and continuous on Y. Hence the density of X in Y implies that (54) holds for all $(v, u) \in Y$, proving the assertion. ■

This lemma justifies the following definition:

$$U \star \mu := \Phi_T \mu = [V_T \circledast]'\mu \in L_p(J_T, E_\alpha), \qquad \mu \in \mathcal{M}(J_T, E_{\beta-1}), \quad T \in \dot{J}_{\mathsf{T}}.$$

Corollary 1 *Suppose that* $1 < p < 1/(1+\alpha-\beta)$ *and set* $\varepsilon := \beta - \alpha - 1/p'$. *Then*

$$T^{-\varepsilon} U \star\; \in \mathcal{L}\big(\mathcal{M}(J_T, E_{\beta-1}), L_p(J_T, E_\alpha)\big) \qquad \mathsf{T}\text{-uniformly.}$$

PROOF. This is a restatement of (53). ■

5.B. A trace lemma From Lemma 5(ii) and the preceding arguments we know that, given assumption (50),

$$\varphi_T := \big[V(T-\cdot)\,\big|\, J_T\big]' \in \mathcal{L}\big(\mathcal{M}(J_T, E_{\beta-1}), E_{\beta-1}\big) \qquad \mathsf{T}\text{-uniformly.} \tag{55}$$

Our next lemma identifies the restriction of φ_T to $L_1(J_T, E_{\beta-1})$. Observe that Lemma 2 implies that $U \star f$ belongs to $C(J_T, E_{\beta-1})$ for $f \in L_1(J_T, E_{\beta-1})$.

Lemma 7 *Let* (50) *be satisfied. If* $\mu = f \in L_1(J_T, E_{\beta-1})$ *then* $\varphi_T \mu = U \star f(T) \in E_{\beta-1}$.

PROOF. From (49) we deduce that

$$\begin{aligned}\left\langle y, U \star f(T)\right\rangle_{\beta-1} &= \int_0^T \left\langle y, U(T-\tau)f(\tau)\right\rangle_{\beta-1} d\tau = \int_0^T \left\langle f(\tau), V(T-\tau)y\right\rangle^{\sharp}_{1-\beta} d\tau \\ &= \left\langle \mu, V(T-\cdot)y\right\rangle_{C(J_T, E^{\sharp}_{1-\beta})}\end{aligned}$$

for $y \in E^\sharp_{1-\alpha}$. Since

$$\langle \mu, V(T-\cdot)y \rangle_{C(J_T, E^\sharp_{1-\beta})} = \langle y, \varphi_T \mu \rangle_{\beta-1}, \qquad y \in E^\sharp_{1-\beta}, \tag{56}$$

the assertion follows. ■

This lemma shows that $\varphi_T \mu$ is the trace of $U \star \mu$ at $t = T$ in $E_{\beta-1}$ if μ belongs to $L_1(J_T, E_{\beta-1})$. For this reason we set

$$U \star \mu(T) := \varphi_T \mu = \big[V(T-\cdot) \,\big|\, J_T\big]' \mu \in E_{\beta-1}, \qquad \mu \in \mathcal{M}(J_T, E_{\beta-1}), \quad T \in \dot{J}_{\mathsf{T}}.$$

Corollary 2 *Let* (50) *be satisfied. Then*

$$\big(\mu \mapsto U \star \mu(T)\big) \in \mathcal{L}\big(\mathcal{M}(J_T, E_{\beta-1}), E_{\beta-1}\big) \qquad \mathsf{T}\text{-uniformly}. \tag{57}$$

PROOF. This follows from (55). ■

5.C. Green's formula After these preparations we can prove the main result of this section, a generalized Green's formula.

Proposition 1 *Let* (50) *be true and suppose that* $1 < p < 1/(1+\alpha-\beta)$. *Then*

$$\big\langle (-\partial + A^\top)v, U \star \mu \big\rangle_{L_p(J_T, E_\alpha)} = \langle \mu, v \rangle - \big\langle v(T), U \star \mu(T) \big\rangle_{\beta-1} \tag{58}$$

for all $(v, \mu) \in \mathbb{W}^1_{p'}\big(J_T, (E^\sharp_{-\alpha}, E^\sharp_{1-\alpha})\big) \times \mathcal{M}(J_T, E_{\beta-1})$ *and* $T \in \dot{J}_{\mathsf{T}}$.

PROOF. From (22) and (43) we infer that

$$Y_{1-1/p', p'} = (E^\sharp_{-\alpha}, E^\sharp_{1-\alpha})_{1/p, p'} \hookrightarrow (E^\sharp_{-\alpha}, E^\sharp_{1-\alpha})_{1+\alpha-\beta, 1} \hookrightarrow E^\sharp_{1-\beta},$$

thanks to $1/p > 1 + \alpha - \beta$. Thus (35) implies

$$\mathbb{W}^1_{p'}\big(J_T, (E^\sharp_{-\alpha}, E^\sharp_{1-\alpha})\big) = \mathbb{W}^1_{p'}\big(J_T, (Y_0, Y_1)\big) \hookrightarrow C(J_T, Y_{1-1/p', p'}) \hookrightarrow C(J_T, E^\sharp_{1-\beta}). \tag{59}$$

This shows that the right-hand side of (58) is well-defined.

Choose $v \in C^\infty(J_T, E^\sharp_{1-\alpha}) = C^\infty(J_T, Y_1)$. Then

$$w := (-\partial + A^\top)v \in C^\infty(J_T, Y_0) = C^\infty(J_T, E^\sharp_{-\alpha}).$$

Hence $v = V(T-\cdot)v(T) + V_T \circledast w$, as is seen by the substitution $t \mapsto T - t$. Thus, using the definitions of $U \star \mu$ and $U \star \mu(T)$ for $\mu \in \mathcal{M}(J_T, E_{\beta-1})$, it follows that

$$\begin{aligned} \big\langle (-\partial + A^\top)v, U \star \mu \big\rangle_{L_p(J_T, E_\alpha)} &= \langle w, U \star \mu \rangle_{L_p(J_T, E_\alpha)} = \langle \mu, V \circledast w \rangle \\ &= \big\langle \mu, v - V(T-\cdot)v(T) \big\rangle = \langle \mu, v \rangle - \big\langle v(T), U \star \mu(T) \big\rangle_{\beta-1}. \end{aligned} \tag{60}$$

The left side of (58) is continuous with respect to $v \in \mathbb{W}^1_{p'}\big(J, (E^\sharp_{-\alpha}, E^\sharp_{1-\alpha})\big)$. From (59) we see that the right-hand side is also continuous in this topology. Since $C^\infty(J, E^\sharp_{1-\alpha})$ is dense in $\mathbb{W}^1_{p'}\big(J, (E^\sharp_{-\alpha}, E^\sharp_{1-\alpha})\big)$, the assertion follows from (60). ■

6. Linear evolution equations

Throughout this section we suppose that

$$\left.\begin{array}{l}\bullet\ E_0 \text{ is a reflexive Banach space;}\\ \bullet\ A_0 \in \mathcal{H}(E_0);\\ \bullet\ [\,E_s\ ;\ s \in \mathbb{R}\,] \text{ is the interpolation-extrapolation space scale}\\ \quad \text{generated by } (E_0, A_0) \text{ and a fixed choice}\\ \quad (\cdot,\cdot)_\theta \in \{\,(\cdot,\cdot)_{\theta,q}, [\,\cdot,\cdot\,]_\theta\ ;\ 1<q<\infty\,\} \text{ for } 0<\theta<1;\\ \bullet\ \mathsf{T}>0 \text{ and } -\infty<\alpha<\beta<\alpha+1<\infty;\\ \bullet\ A := A_{\alpha-1},\ A^\top := A^\sharp_{-\alpha}.\end{array}\right\} \tag{61}$$

We also set $\widehat{p} := 1/(1+\alpha-\beta)$ and $\mathcal{M}_T := \mathcal{M}(J_T, E_{\beta-1})$ for $T \in \dot{J}_\mathsf{T}$, and

$$L_{q,T,\gamma} := L_q(J_T, E_\gamma), \qquad q \in [1,\infty], \quad T \in \dot{J}_\mathsf{T}, \quad \gamma \in \mathbb{R}.$$

6.A. Weak solutions

For $T \in \dot{J}_\mathsf{T}$ and $\mu \in \mathcal{M}_T$ we consider the linear parabolic evolution equation

$$\dot{u} + Au = \mu \qquad \text{on} \quad J_T. \tag{62}$$

Given $p \in [1, \widehat{p})$, a function u is said to be a (*weak*) $L_p(E_\alpha)$-*solution* of (62) if u belongs to $L_{p,\mathrm{loc}}\big([0,T), E_\alpha\big)$ and satisfies

$$\int_0^T \big\langle (-\partial + A^\top)v, u\big\rangle_\alpha\, dt = \langle \mu, v\rangle \tag{63}$$

for all $v \in \mathbb{C}^1_c\big([0,T), (E^\sharp_{-\alpha}, E^\sharp_{1-\alpha})\big)$, where the index c means 'compact support'.

Suppose that u is an $L_p(E_\alpha)$-solution of (62) for some $p \in (1, \widehat{p})$. Then it is an $L_q(E_\alpha)$-solution for every $q \in [1,p)$, as follows from $L_{p,\mathrm{loc}} \hookrightarrow L_{q,\mathrm{loc}}$. Thus every $L_p(E_\alpha)$-solution is a (*weak*) E_α-*solution* of (62), that is, an $L_1(E_\alpha)$-solution.

It is now almost trivial to prove the following existence, uniqueness, and regularity theorem for E_α-solutions of (62).

Theorem 4 *Suppose that $\mu \in \mathcal{M}_T$ for some $T \in \dot{J}_\mathsf{T}$. Then problem* (62) *possesses a unique E_α-solution, namely $u(\mu) := U \star \mu$. It belongs to*

$$\bigcap_{1 \le p < \widehat{p}} L_{p,T,\alpha}. \tag{64}$$

PROOF. Suppose that $p \in (1, \widehat{p})$. Then Proposition 1 shows that $u(\mu)$ is an $L_p(E_\alpha)$-solution, hence an E_α-solution. Indeed, if v vanishes near T then the last term in (58) does not appear so that assumption (50) is not needed.

For $T' \in (0,T)$ and $g \in \mathcal{D}\big((0,T'), E^\sharp_{1-\alpha}\big)$, it follows that the unique solution in $\mathbb{C}^1\big(J_{T'}, (E^\sharp_{-\alpha}, E^\sharp_{1-\alpha})\big)$ of the final value problem

$$(-\partial + A^\sharp_{-\alpha})w = g \quad \text{in} \quad [0,T), \qquad w(T') = 0,$$

is given by $v := V_{T'} \circledast g$. Hence (the proof of) [4, Proposition V.2.6.3] guarantees that (62) has at most one solution. This proves everything. ■

The next corollary shows that $u(\mu)$ depends linearly and continuously on $\mu \in \mathcal{M}_T$ in the natural projective limit topology of (64).

Corollary 3 *Suppose that* $1 < p < \widehat{p}$ *and set* $\varepsilon := 1/p - 1/\widehat{p}$. *Then*

$$\big(\mu \mapsto T^{-\varepsilon} u(\mu)\big) \in \mathcal{L}(\mathcal{M}_T, L_{p,T,\alpha}) \qquad \mathsf{T}\text{-uniformly.}$$

PROOF. This follows from Corollary 1. ■

6.B. Strong solutions Now we consider cases where μ has better regularity properties. Suppose that (x, f) belongs to $E_{\beta-1} \times L_{1,T,\beta-1}$. Then Examples 1(a) and (b) imply $\mu := f + x \otimes \delta_0 \in \mathcal{M}_T$ and

$$\langle \mu, v\rangle = \langle v, f\rangle_{L_{1,T,\beta-1}} + \big\langle v(0), x\big\rangle_{\beta-1}, \qquad v \in C(J, E^\sharp_{1-\beta}).$$

Hence it follows from (58) that (63) is a weak formulation of the initial value problem

$$\dot{u} + Au = f(t), \quad t \in \dot{J}_T, \qquad u(0) = x. \tag{65}$$

Suppose that $1 \le p < \infty$ and $\alpha \le \gamma < \beta$. A *strong* $L_p(E_\gamma)$*-solution* of (65) is a function $u \in \mathbb{W}^1_p\big(J_T, (E_{\gamma-1}, E_\gamma)\big)$ satisfying

$$(\partial + A)u = f, \quad u(0) = x. \tag{66}$$

Note that $(\partial + A)u \in L_{p,T,\gamma-1} \hookrightarrow L_{1,T,\gamma-1}$ and $E_{\beta-1} \hookrightarrow E_{\gamma-1}$ imply that the first equation makes sense in $L_{1,T,\gamma-1}$. Similarly, the second equation in (66) is meaningful in $E_{\gamma-1}$ since $u(0) \in E_{\gamma-1}$ by Theorem 3.

Lastly, u is a *classical* E_γ-solution of (65) if

$$u \in \mathbb{C}^1\big(\dot{J}_T, (E_{\gamma-1}, E_\gamma)\big) \cap C(J_T, E_{\gamma-1})$$

and u satisfies (65) point-wise. It is said to be *strict* if $u \in \mathbb{C}^1\big(J_T, (E_{\gamma-1}, E_\gamma)\big)$.

Henceforth, we set $\mathbb{W}^1_{p,T,\gamma} := \mathbb{W}^1_p\big(J_T, (E_{\gamma-1}, E_\gamma)\big)$.

Theorem 5 *Suppose that* $1 \le r < \infty$ *and* $(x, f) \in E_{\beta-1/r} \times L_{r,T,\beta-1}$, *and set* $u(x, f) := u(f + x \otimes \delta_0)$. *Then* $u(x, f) \in \mathbb{W}^1_{r,T,\gamma}$ *for* $\alpha \le \gamma < \beta$, *and it is represented by*

$$u(x, f) = Ux + U \star f,$$

that is, by the variation of constants formula. *Furthermore,* $u(x, f)$ *is the unique strong* $L_r(E_\gamma)$*-solution of* (65).

PROOF. We can assume that $\gamma > \beta - 1/r$. Set $(X_0, X_1) := (E_{\gamma-1}, E_\gamma)$. Then (43) implies

$$X_{\xi,1} \hookrightarrow E_{\xi+\gamma-1} \hookrightarrow X^0_{\xi,\infty}, \qquad 0 \le \xi \le 1. \tag{67}$$

Consequently, setting $\xi := \beta - \gamma$, we find that $E_{\beta-1} \hookrightarrow X_{\beta-\gamma,\infty}$ and infer from Lemma 4(i) that

$$U\star \in \mathcal{L}(L_{r,T,\beta-1}, \mathbb{W}^1_{r,T,\gamma}) \qquad \mathsf{T}\text{-uniformly.} \tag{68}$$

From (67) it follows that $E_{\beta-1/r} \hookrightarrow X_{\beta-\gamma+1-1/r,\infty}$. Setting $\xi := \beta - \gamma + 1/r'$, we deduce from Lemma 4(ii) that

$$T^{\gamma-\beta} U \in \mathcal{L}(E_{\beta-1/r}, \mathbb{W}^1_{r,T,\gamma}) \qquad \mathsf{T}\text{-uniformly.} \tag{69}$$

From this we infer that

$$w := Ux + U \star f \in \mathbb{W}^1_{r,T,\gamma}.$$

If $(x, f) \in E_\gamma \times C^1(J_T, E_{\beta-1})$ then w is a strict classical solution of (65) (see [4, Theorem II.1.2.1]). Thus (68), (69), and the density of $E_\gamma \times C^1(J_T, E_{\beta-1})$ in $E_{\beta-1/r} \times L_{r,T,\beta-1}$ imply that w is a strong $L_r(E_\gamma)$-solution in the general case also. It follows from [4, Proposition V.2.6.2] that every strong $L_r(E_\gamma)$-solution is a weak $L_r(E_\gamma)$-solution, hence an E_α-solution. Thus $w = u(f + x \otimes \delta_0)$ follows from the uniqueness part of Theorem 4. ■

Recall that $Ux + U \star f$ is said to be the *mild* solution of (65). Thus Theorem 5 shows that, given the assumptions of that theorem, the mild solution is in fact a strong solution.

In the next proposition we collect continuity properties of $u(x, f)$.

Proposition 2 (i) *If* $1 \le r < \infty$ *and* $\alpha \le \gamma < \beta$ *then*

$$\big((x,f) \mapsto u(x,f)\big) \in \mathcal{L}(E_{\beta-1/r} \times L_{r,T,\beta-1}, \mathbb{W}^1_{r,T,\gamma}) \qquad \mathsf{T}\text{-}uniformly.$$

(ii) *Suppose that* $1 \le r < 1/(\beta-\alpha)$ *and* $r \le p < 1\big/(\alpha-\beta+1/r)$.
Then, setting $\varepsilon := \beta-\alpha-1/r+1/p$,

$$\big((x,f) \mapsto T^{-\varepsilon}u(x,f)\big) \in \mathcal{L}(E_{\beta-1/r} \times L_{r,T,\beta-1}, L_{p,T,\alpha}) \qquad \mathsf{T}\text{-}uniformly.$$

(iii) *Assume that* $1/(\beta-\alpha) < r < \infty$ *and* $0 \le \rho < \beta-\alpha-1/r$. *Then*

$$\big((x,f) \mapsto u(x,f)\big) \in \mathcal{L}\big(E_{\beta-1/r} \times L_{r,T,\beta-1}, C^\rho(J_T,E_\alpha)\big) \qquad \mathsf{T}\text{-}uniformly.$$

(iv) *Suppose that* $1/(\beta-\alpha) < r < \infty$. *Then*

$$\big((x,f) \mapsto u(x,f)\big) \in \mathcal{L}\big(E_\alpha \times L_{r,T,\beta-1}, C(J_T,E_\alpha)\big) \qquad \mathsf{T}\text{-}uniformly.$$

PROOF. (i) This is a consequence of (68) and (69).

(ii) Set $(X_0,X_1) := (E_{\alpha-1},E_\alpha)$. Then the assertion follows from (67) (with γ replaced by α), from Lemma 1(i) (with $\xi := \beta-\alpha+1/r'$ and $\eta := 1$), and from Lemma 3(i) (with $\xi := \beta-\alpha$ and $\eta := 1$).

(iii) We put $(X_0,X_1) := (E_{\beta-1},E_\beta)$ and recall from (38) that $A_{\beta-1} \subset A$, so that $A \in \mathcal{H}(X_1,X_0)$ by (39).

Let $\xi := 1/r'$ and $\eta := \alpha-\beta+1$ so that $\xi-\eta > \rho$. Then (67) (with γ replaced by β) and Lemma 1(ii) show that

$$U \in \mathcal{L}\big(E_{\beta-1/r}, C^\rho(J_{\mathsf{T}},E_\alpha)\big). \tag{70}$$

From Lemma 3(ii) we similarly infer that

$$U\star \in \mathcal{L}\big(L_{r,T,\beta-1}, C^\rho(J_T,E_\alpha)\big) \qquad \mathsf{T}\text{-uniformly}. \tag{71}$$

Now (70), (71), and the variation-of-constants formula imply the assertion.

(iv) The strong continuity of U on E_α implies $U \in \mathcal{L}\big(E_\alpha, C(J_{\mathsf{T}},E_\alpha)\big)$. Hence the assertion follows from (71). ■

6.C. Positivity Finally, we derive a positivity result which can be viewed as an abstract form of a parabolic maximum principle. For this we introduce the following additional assumption:

$$E_0 \text{ is an ordered Banach space (OBS) with positive cone } P_0. \tag{72}$$

Then $E_0^\sharp$ is given the natural dual preorder induced by the dual positive cone $P_0^\sharp := P_0'$. Moreover, E_γ and $E^\sharp_{1-\gamma}$ are naturally ordered for $\gamma \in \mathbb{R}$ by the positive cones P_γ and $P^\sharp_{1-\gamma}$, respectively, where $P_\gamma := P_0 \cap E_\gamma$ if $\gamma > 0$, and P_γ is the closure of P_0 in E_γ if $\gamma < 0$, with a similar definition for $P^\sharp_{1-\gamma}$ (see [4, Section V.2.7]).

For $q \in [1,\infty]$ and $T > 0$ we set $L_q^+(J_T,E_\gamma) := L_q(J_T,P_\gamma)$. It is a closed convex cone in $L_q(J_T,E_\gamma)$ inducing the natural preorder determined by the order of E_0. Similarly, $L_q^+(J_T,E^\sharp_{1-\gamma}) := L_q(J_T,P^\sharp_{1-\gamma})$ induces the natural preorder in $L_q(J_T,E^\sharp_{1-\gamma})$ and in each of its vector subspaces, thus on $C(J_T,E^\sharp_{1-\gamma})$, for example. We always refer to these preorders if (72) is presupposed.

Proposition 3 *Let assumption* (72) *be satisfied and suppose that* A_0 *is resolvent positive. If* $\mu \in \mathcal{M}_{\beta-1}$ *is positive then the* E_α*-solution* $u(\mu)$ *of* (62) *is also positive, that is,* $\mu \ge 0$ *implies* $u(\mu) \ge 0$.

PROOF. Suppose that A_0 is resolvent positive. Then A and $A^\top$ are also resolvent positive thanks to [4, Theorem V.2.7.2]. Thus it follows from [4, Theorem II.6.4.1] that V_T is positive. From this and Lemma 5(i) we deduce that $V_T \circledast v$ belongs to $C^+(J_T, E^\sharp_{1-\beta})$ for $v \in L^+_{p'}(J_T, E^\sharp_{-\alpha})$ and $1 < p < \widehat{p}$. Thus $\mu \geq 0$ and the definition of $u(\mu) = U \star \mu$ imply

$$\langle v, U \star \mu\rangle_{L_p(J_T,E_\alpha)} = \langle \mu, V_T \circledast v\rangle_{C(J_T,E^\sharp_{1-\beta})} \geq 0, \qquad v \in L^+_{p'}(J_T, E^\sharp_{-\alpha}),$$

for $1 < p < \widehat{p}$. This shows that $U \star \mu$ belongs to the dual cone of $L^+_{p'}(J_T, E^\sharp_{-\alpha})$. Using $P^\sharp_{-\alpha} = (P_\alpha)'$ (see [4, Theorem V.2.7.2]) and reflexivity it is not difficult to see that $L^+_{p}(J_T, E^\sharp_{-\alpha})' = L^+_p(J_T, E_\alpha)$. Hence $u(\mu) \in L^+_p(J_T, E_\alpha)$, which proves the assertion. ■

7. Second order parabolic equations

In this section we illustrate the abstract results of Section 6 by applying them to one of the most important concrete situations, namely to second order parabolic boundary value problems. At the end we indicate generalizations and other applications as well.

7.A. Elliptic boundary value problems We put

$$\mathcal{A}u := -\nabla\cdot(\boldsymbol{a}\nabla u) + \vec{a}\cdot\nabla u + a_0 u,$$

where ∇ denotes the gradient on Ω and $\nabla\cdot$ divergence, and assume that

$$(\boldsymbol{a}, \vec{a}, a_0) \in BUC^\infty(\Omega, \mathbb{R}^{n\times n}\times\mathbb{R}^n\times\mathbb{R}),$$

with $\boldsymbol{a}$ being symmetric and uniformly positive definit. Thus $\mathcal{A}$ is strongly uniformly elliptic.

We denote by $\vec{\boldsymbol{\nu}} := (\gamma\boldsymbol{a})\vec{\nu}$ the outer conormal on Γ with respect to $\boldsymbol{a}$ and by $\partial_{\boldsymbol{\nu}} u$ the corresponding directional derivative. Then we set

$$\mathcal{B}u := \begin{cases} \gamma u & \text{on} \quad \Gamma_0, \\ \partial_{\boldsymbol{\nu}} u + b\gamma u & \text{on} \quad \Gamma_1, \end{cases} \tag{73}$$

where we assume that $b \in C^\infty(\Gamma)$. (Clearly, we can always set $b\,|\,\Gamma_0 = 0$.) Thus $(\mathcal{A}, \mathcal{B})$ is a normally elliptic second order boundary value problem with smooth coefficients (using the terminology of [3]). We also set

$$\mathcal{A}^\sharp v := -\nabla\cdot(\boldsymbol{a}^\sharp\nabla v) + \vec{a}^\sharp\cdot\nabla v + a_0^\sharp v$$

with $\boldsymbol{a}^\sharp := \boldsymbol{a}$, $\vec{a}^\sharp := -\vec{a}$, and $a_0^\sharp := a_0 - \nabla\cdot\vec{a}$, and

$$\mathcal{B}^\sharp v := \begin{cases} \gamma v & \text{on} \quad \Gamma_0, \\ \partial^\sharp_{\boldsymbol{\nu}} v + b^\sharp\gamma v & \text{on} \quad \Gamma_1, \end{cases}$$

where $b^\sharp := b + (\gamma\vec{a})\cdot\vec{\nu}$ and $\partial^\sharp_{\boldsymbol{\nu}} := \partial_{\boldsymbol{\nu}}$. Then $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$ is a normally elliptic boundary value problem formally adjoint to $(\mathcal{A}, \mathcal{B})$. (We continue writing $\partial^\sharp_{\boldsymbol{\nu}}$ instead of $\partial_{\boldsymbol{\nu}}$ in view of the generalizations to systems described in Subsection 7.H.)

The Dirichlet form $\mathfrak{a}$ induced by $(\mathcal{A}, \mathcal{B})$ is defined by

$$\mathfrak{a}(v, u) := \langle\nabla v, \boldsymbol{a}\nabla u\rangle + \langle v, \vec{a}\cdot\nabla u + a_0 u\rangle + \langle\gamma v, b\gamma u\rangle_\Gamma \tag{74}$$

for $u, v \in \mathcal{D}(\overline{\Omega})$, where

$$\langle u, v\rangle_\Gamma := \int_\Gamma uv\, d\sigma, \qquad u, v \in C^\infty(\Gamma),$$

with $d\sigma$ denoting the volume measure of Γ. Recall that the last term in (74) is omitted if $\Gamma = \emptyset$. Note that

$$\mathfrak{a}(v,u) = \langle \boldsymbol{a}^\sharp \nabla v, \nabla u\rangle + \langle \vec{a}^\sharp \cdot \nabla v + a_0^\sharp v, u\rangle + \langle b^\sharp \gamma v, \gamma u\rangle_\Gamma \tag{75}$$

for $u, v \in \mathcal{D}(\overline{\Omega})$. Also note that Green's formulas

$$\mathfrak{a}(v,u) = \langle v, \mathcal{A}u\rangle + \langle \gamma v, \partial_{\boldsymbol{\nu}} u + b\gamma u\rangle_\Gamma = \langle \mathcal{A}^\sharp v, u\rangle + \langle \partial_{\boldsymbol{\nu}}^\sharp v + b^\sharp \gamma v, \gamma u\rangle_\Gamma \tag{76}$$

are valid for $u, v \in \mathcal{D}(\overline{\Omega})$.

7.B. The Sobolev-Slobodeckii interpolation-extrapolation scale Suppose that $1 < q < \infty$. If $\Gamma = \emptyset$ then we set

$$W_{q,\mathcal{B}}^s := W_q^s, \qquad s \in \mathbb{R}.$$

Otherwise,

$$W_{q,\mathcal{B}}^s := \begin{cases} \{\, u \in W_q^s \ ;\ \mathcal{B}u = 0 \,\}, & 1 + 1/q < s \le 2 + 1/q, \\ \{\, u \in W_q^s \ ;\ \gamma u = 0 \text{ on } \Gamma_0 \,\}, & 1/q < s < 1 + 1/q, \\ W_q^s, & 0 \le s < 1/q, \\ (W_{q',\mathcal{B}^\sharp}^{-s})', & -3 + 1/q \le s < 0, \quad s \notin \mathbb{Z} + 1/q, \end{cases}$$

where $(W_{q',\mathcal{B}^\sharp}^{-s})'$ is determined by the duality pairing $\langle\cdot,\cdot\rangle$ being induced by the $L_{q'}$-duality pairing. Furthermore, the values $s = 1 + 1/q$ and $s = -2 + 1/q$ are admitted if $\Gamma = \Gamma_0$, and $s = 1/q$ and $s = -1 + 1/q$ are included if $\Gamma = \Gamma_1$. Observe that

$$W_{q,\mathcal{B}}^s = W_q^s \qquad \text{for} \quad \begin{cases} -2 + 1/q < s < 1/q & \text{if} \quad \Gamma = \Gamma_0 \ne \emptyset, \\ -1 + 1/q < s < 1/q & \text{if} \quad \Gamma_1 \ne \emptyset, \end{cases} \tag{77}$$

where $s \ne -1 + 1/q$ if $\Gamma = \Gamma_0 \ne \emptyset$. Moreover, $W_{q,\mathcal{B}}^s$ is a closed linear subspace of W_q^s for

$$s \in I_q := \begin{cases} \mathbb{R} & \text{if} \quad \Gamma = \emptyset, \\ I \backslash \{1/q, -1 + 1/q\} & \text{if} \quad \Gamma = \Gamma_0 \ne \emptyset, \\ I \backslash \{1 + 1/q, -2 + 1/q\} & \text{if} \quad \Gamma = \Gamma_1 \ne \emptyset, \\ I \setminus (\mathbb{Z} + 1/q) & \text{otherwise,} \end{cases}$$

where $I := (-3 + 1/q, 2 + 1/q)$.

Now we define the L_q-realization $A_0 := A_{0,q}$ of $(\mathcal{A}, \mathcal{B})$ by

$$\operatorname{dom}(A_0) := W_{q,\mathcal{B}}^2, \quad A_0 u := \mathcal{A}u.$$

Similarly, $A_0^\sharp := A_{0,q'}^\sharp$, the $L_{q'}$-realization of $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$, is given by

$$\operatorname{dom}(A_0^\sharp) := W_{q',\mathcal{B}^\sharp}^2, \quad A_0^\sharp v := \mathcal{A}^\sharp v.$$

Then A_0 and $A_0^\sharp$ are densely defined in $E_0 := L_q$ and $E_0^\sharp := L_{q'}$, respectively. However, much more is true as the next theorem shows.

Theorem 6 (i) $A_0 \in \mathcal{H}(W_{q,\mathcal{B}}^2, L_q)$ *and* $A_0^\sharp \in \mathcal{H}(W_{q',\mathcal{B}^\sharp}^2, L_{q'})$.

(ii) $A_0' = A_0^\sharp$ *in the sense of unbounded linear operators.*

(iii) *Suppose that* $[\,E_s\ ;\ s \in \mathbb{R}\,]$ *is the interpolation-extrapolation space scale generated by* (E_0, A_0) *and the interpolation functors*

$$(\cdot,\cdot)_\theta := \begin{cases} (\cdot,\cdot)_{\theta,q}, & \theta \in (0,1)\backslash\{1/2\}, \\ [\cdot,\cdot]_\theta, & \theta = 1/2, \end{cases} \tag{78}$$

and let $[\,E_s^\sharp\ ;\ s \in \mathbb{R}\,]$ *be its dual scale. Then*

$$E_s \doteq W_{q,\mathcal{B}}^{2s}, \quad 2s \in I_q, \qquad \text{and} \qquad E_s^\sharp \doteq W_{q',\mathcal{B}^\sharp}^{2s}, \quad 2s \in I_{q'}.$$

PROOF. For this we refer to [3] and [8]. (It is easily seen that the results of [3] can be extended to the full range I_q.) ■

7.C. The existence theorem Let E be a Banach space and $J := \mathbb{R}^+$ or $J := \dot{\mathbb{R}}^+ = (0,\infty)$. Then

$$\mathcal{M}_{\mathrm{loc}}(J,E) := \bigcap_{T>0} \mathcal{M}\big(J \cap [0,T], E\big),$$

and this space is given its natural Fréchet space topology.

Now we suppose that

$$\left.\begin{array}{l} \bullet\ s,\sigma \in I_q,\ s-2<\sigma<s; \\ \bullet\ s,\sigma > -1+1/q \text{ if } \Gamma \neq \emptyset; \\ \bullet\ \mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, W_{q,\mathcal{B}}^{s-2}). \end{array}\right\} \tag{79}$$

Then we study the linear parabolic problem

$$\dot{u} + Au = \mu \qquad \text{on} \qquad \mathbb{R}^+, \tag{80}$$

where $A := A_{\sigma/2-1}$. We also investigate the case where μ is more regular, that is, we consider the additional assumption

$$\left.\begin{array}{l} \bullet\ 1 \le r < \infty; \\ \bullet\ (u^0, f) \in W_{q,\mathcal{B}}^{s-2/r} \times L_{r,\mathrm{loc}}(\mathbb{R}^+, W_{q,\mathcal{B}}^{s-2}); \\ \bullet\ \mu = f + u^0 \otimes \delta_0. \end{array}\right\} \tag{81}$$

This case is related to the initial value problem

$$\dot{u} + Au = f(t) \quad \text{in } (0,\infty), \qquad u(0) = u^0\ . \tag{82}$$

Clearly, weak and strong $L_p(W_{q,\mathcal{B}}^\sigma)$-solutions, respectively, are defined as in Subsections 6.A and 6.B, respectively, by replacing there J_T by $\mathbb{R}^+$.

Now we can prove the following fundamental existence, uniqueness, continuity, regularity, and positivity theorem. Here and below, all concrete spaces are ordered by the natural order induced by standard point-wise positivity and canonical product orders.

Theorem 7 *Let* (79) *be satisfied.*

(i) *Problem* (80) *possesses a unique* $W_{q,\mathcal{B}}^\sigma$*-solution* u, *and* $u \in L_{p,\mathrm{loc}}(\mathbb{R}^+, W_{q,\mathcal{B}}^\sigma)$ *for* $1 \le p < 2/(\sigma - s + 2)$.

(ii) *Let also* (81) *be true. Then* $u \in \mathbb{W}_{r,\mathrm{loc}}^1\big(\mathbb{R}^+, (W_{q,\mathcal{B}}^{\sigma-2}, W_{q,\mathcal{B}}^\sigma)\big)$, *and it is the unique strong* $L_r(W_{q,\mathcal{B}}^\sigma)$*-solution of* (82). *Furthermore, if* $\sigma > s - 2/r$ *then*

$$u \in L_{p,\mathrm{loc}}(\mathbb{R}^+, W_{q,\mathcal{B}}^\sigma), \qquad 1 \le p < 2/(\sigma - s + 2/r),$$

and if $\sigma < s - 2/r$ *then*

$$u \in C^\rho(\mathbb{R}^+, W_{q,\mathcal{B}}^\sigma), \qquad 0 \le 2\rho < s - \sigma - 2/r.$$

In the latter case $u \in C(\mathbb{R}^+, W_{q,\mathcal{B}}^\sigma)$ *if* $u^0 \in W_{q,\mathcal{B}}^\sigma$.

(iii) *In each case the map* $\mu \mapsto u$ *is linear and continuous in the respective topologies.*

(iv) *If* $\mu \geq 0$ *then* $u \geq 0$.

PROOF. (a) In case (i) fix $p \in \big[1, 2/(\sigma - s + 2)\big)$. In case (ii) choose $p \in \big[1, 2/(\sigma - s + 2/r)\big)$ if $\sigma > s - 2/r$, and fix 2ρ in $[0, s - \sigma - 2/r)$ otherwise. Set $2\alpha := \sigma$ and $2\beta := s$. Given any $\mathsf{T} > 0$, it follows from Theorem 6 that assumption (61) is satisfied. Hence Theorems 4 and 5 and Proposition 2 apply to problems (80) and (82), respectively, with $\mathbb{R}^+$ replaced by J_T. Since this is true for every $\mathsf{T} > 0$ and the corresponding solutions are unique, assertions (i)–(iii) follow.

(b) It is a consequence of the maximum principle that A_0 is resolvent positive (see [1], where the case without sign restriction for b has been treated, also see [3, Theorem 8.7]). ■

It remains to give interpretations of this theorem in more classical terms. This is done in the following subsections.

7.D. Weak solutions First we identify the extrapolated operators A_α for $\alpha < 0$ with appropriate realizations of $(\mathcal{A}, \mathcal{B})$. This is the content of the following assertions.

Theorem 8 (i) *If* $2s \in I_q$ *with* $2s > 1 + 1/q$ *then* $A_{s-1} = \mathcal{A} | W_{q,\mathcal{B}}^{2s-2}$.

(ii) *Assume that* $1/q < 2s < 1 + 1/q$. *Then* A_{s-1} *is determined by*

$$\langle v, A_{s-1} u \rangle_{s-1} = \mathfrak{a}(v, u), \qquad (v, u) \in W_{q',\mathcal{B}^\sharp}^{2-2s} \times W_{q,\mathcal{B}}^{2s}.$$

(iii) *If* $2s < 1/q$ *and either* $\Gamma = \emptyset$ *or* $2s > -1 + 1/q$ *then* $A_{s-1} = (\mathcal{A}^\sharp | W_{q',\mathcal{B}^\sharp}^{2-2s})'$, *that is,*

$$\langle v, A_{s-1} u \rangle_{s-1} = \langle \mathcal{A}^\sharp v, u \rangle, \qquad (v, u) \in W_{q',\mathcal{B}^\sharp}^{2-2s} \times W_q^{2s}.$$

PROOF. If $\Gamma \neq \emptyset$ then this is a special case of [3, Theorem 8.3], thanks to (77). The assertions are obvious if $\Gamma = \emptyset$. ■

We also need the following approximation result.

Lemma 8 (i) $\mathcal{D}_\mathcal{B} := \big\{ u \in \mathcal{D}(\overline{\Omega}) \ ; \ \mathcal{B}u = 0 \big\}$ *is dense in* $W_{q,\mathcal{B}}^s$ *for* $s \in I_q$.

(ii) *If either* $\Gamma = \emptyset$ *or* $s < 1/q$ *then* $\mathcal{D}$ *is dense in* W_q^s.

PROOF. (i) Theorem 6(iii) and (37) imply that $W_{q,\mathcal{B}}^s$ is dense in $W_{q,\mathcal{B}}^t$ for $s, t \in I_q$ with $t < s$. Thus it remains to show that $\mathcal{D}_\mathcal{B}$ is dense in $W_{q,\mathcal{B}}^s$ for $2 < s < 2 + 1/q$, since the case where $\Gamma = \emptyset$ is clear.
Fix $2\alpha := s \in (2, 2 + 1/q)$ and $\lambda > 0$ such that $\lambda + A_{\alpha-1}$ is an isomorphism from $W_{q,\mathcal{B}}^s$ onto W_q^{s-2}. Then

$$(\lambda + A_{\alpha-1})^{-1}\big(\mathcal{D}(\Omega)\big) \subset C^\infty \cap W_{q,\mathcal{B}}^s$$

by elliptic regularity theory. Since $\mathcal{D}(\Omega)$ is dense in W_q^{s-2}, we see that $C^\infty \cap W_{q,\mathcal{B}}^s$ is dense in $W_{q,\mathcal{B}}^s$. Now a standard argument based on multiplication with smooth cutoff functions shows that $\mathcal{D}_\mathcal{B}$ is dense in $C^\infty \cap W_{q,\mathcal{B}}^s$, hence in $W_{q,\mathcal{B}}^s$.

(ii) Since $\mathcal{D}$ is dense in L_q and $L_q \hookrightarrow W_q^s$ for $s \leq 0$, the assertion is clear if $s \leq 0$. It is well-known if $\Gamma = \emptyset$ or $0 < s < 1/q$ (e.g., [27]). ■

Now we are ready for the first step in clarifying the concept of weak solutions.

Proposition 4 *Let assumption* (79) *be satisfied. Then* $u \in L_{1,\mathrm{loc}}(\mathbb{R}^+, W_{q,\mathcal{B}}^\sigma)$ *is the* $W_{q,\mathcal{B}}^\sigma$*-solution of* (80) *iff*

$$\int_0^\infty \langle -\partial_t v + \mathcal{A}^\sharp v, u \rangle \, dt = \int_0^\infty v \, d\mu, \qquad v \in \mathcal{D}(\mathbb{R}^+, \mathcal{D}_{\mathcal{B}^\sharp}). \tag{83}$$

If $\sigma > 1/q$ *then this is equivalent to*

$$\int_0^\infty \big\{-\langle \partial_t v, u\rangle + \mathfrak{a}(v,u)\big\}\,dt = \int_0^\infty v\,d\mu, \qquad v \in \mathcal{D}\big(\mathbb{R}^+, \mathcal{D}(\Omega\cup\Gamma_1)\big),$$

and, if $\sigma > 1 + 1/q$, *to*

$$\int_0^\infty \big\{-\langle \partial_t v, u\rangle + \langle v, \mathcal{A}u\rangle\big\}\,dt = \int_0^\infty v\,d\mu, \qquad v \in \mathcal{D}(\mathbb{R}^+, \mathcal{D}).$$

PROOF. Set $2\alpha := \sigma$ and $2\beta := s$. Then

$$\langle A^\top v, u\rangle_\alpha = \langle v, Au\rangle_{\alpha-1}, \qquad (v,u) \in E^\sharp_{1-\alpha} \times E_\alpha,$$

and Theorem 8 imply

$$\langle A^\top v, u\rangle_\alpha = \begin{cases} \langle v, \mathcal{A}u\rangle, & 1+1/q < \sigma < 2+1/q, \\ \mathfrak{a}(v,u), & 1/q < \sigma < 1+1/q, \\ \langle \mathcal{A}^\sharp v, u\rangle, & -1+1/q < \sigma < 1/q, \end{cases}$$

for $(u,v) \in W^{2-\sigma}_{q',\mathcal{B}^\sharp} \times W^\sigma_{q,\mathcal{B}}$, where the restriction $-1+1/q < \sigma < 2+1/q$ can be dropped if $\Gamma = \emptyset$. Since $\sigma > -1 + 1/q$ if $\Gamma \neq \emptyset$ it follows from (77) that

$$\langle \partial_t v, u\rangle_\alpha = \langle \partial_t v, u\rangle, \qquad (v,u) \in \mathcal{D}_{\mathcal{B}^\sharp} \times W^\sigma_{q,\mathcal{B}}.$$

It is an easy consequence of Lemma 8(i) that $\mathcal{D}(\mathbb{R}^+, \mathcal{D}_{\mathcal{B}^\sharp})$ is dense in

$$\mathbb{C}^1_c\big(\mathbb{R}^+, (W^{-\sigma}_{q',\mathcal{B}^\sharp}, W^{2-\sigma}_{q',\mathcal{B}^\sharp})\big). \tag{84}$$

If $1/q < \sigma < 1 + 1/q$ then $\mathcal{D}_{\mathcal{B}^\sharp} \subset \mathcal{D}(\Omega\cup\Gamma_1) \subset W^{2-\sigma}_{q',\mathcal{B}^\sharp}$ so that $\mathcal{D}\big(\mathbb{R}^+, \mathcal{D}(\Omega\cup\Gamma_1)\big)$ is dense in (84). Finally, in the remaining case $\mathcal{D}(\mathbb{R}^+, \mathcal{D})$ is dense in (84), thanks to $W^{2-\sigma}_{q',\mathcal{B}^\sharp} = W^{2-\sigma}_{q'}$ and to Lemma 8(ii). Now the assertions are obvious since $W^\rho_{q,\mathcal{B}} \hookrightarrow W^\tau_{q,\mathcal{B}}$ for $\rho, \tau \in I_q$ with $\rho > \tau$. ■

Suppose that $s - 2 < \sigma_0 < \sigma_1 < s$ with $\sigma_j \in I_q$ and denote by u_j the $W^{\sigma_j}_{q,\mathcal{B}}$-solution of (80) for $j = 0, 1$. Thanks to $W^{\sigma_1}_{q,\mathcal{B}} \hookrightarrow W^{\sigma_0}_{q,\mathcal{B}}$ it follows that u_1 is a $W^{\sigma_0}_{q,\mathcal{B}}$-solution also. Hence $u_1 = u_0$ by uniqueness. This shows that the $W^\sigma_{q,\mathcal{B}}$-solution u is independent of $\sigma \in (s-2, s)$. Since u belongs to $L_{1,\mathrm{loc}}(\mathbb{R}^+, W^{s-2}_{q,\mathcal{B}})$ we say that u is a $W^{s-2}_{q,\mathcal{B}}$-*solution* of (80) to express its independence of σ.

Now suppose that $q_j \in (1,\infty)$ and $s_j \in I_{q_j}$ for $j = 0, 1$ with $s_j > -1 + 1/q_j$ if $\Gamma \neq \emptyset$. Also assume that

$$\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, W^{s_0-2}_{q_0,\mathcal{B}} \cap W^{s_1-2}_{q_1,\mathcal{B}}) \tag{85}$$

and let u_j be the $W^{s_j-2}_{q_j,\mathcal{B}}$-solution of (80). The following theorem shows that $u_0 = u_1$ in this case also .

Theorem 9 *The* $W^{s-2}_{q,\mathcal{B}}$-*solution is independent of* s *and* q *in the following sense: if* (85) *is satisfied and* u_1 *belongs to* $L_{1,\mathrm{loc}}(\mathbb{R}^+, W^\sigma_{q_0,\mathcal{B}})$ *for some* $\sigma \in I_{q_0}$ *with* $\sigma \le (s_0 \wedge s_1) - 2$, *where* $\sigma > -1 + 1/q_0$ *if* $\Gamma \neq \emptyset$, *then* $u_0 = u_1$.

PROOF. Our assumptions imply that $w := u_1 - u_0$ belongs to $L_{1,\mathrm{loc}}(\mathbb{R}^+, W^\sigma_{q_0,\mathcal{B}})$. Fix s in I_{q_0} satisfying $s - 2 < \sigma < s$, where $s > -1 + 1/q_0$ if $\Gamma \neq \emptyset$. Example 1(c) and (85) imply that $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, W^{s-2}_{q_0,\mathcal{B}})$. Since u_0 and u_1 satisfy (83) it follows that

$$\int_0^\infty \langle -\partial_t v + \mathcal{A}^\sharp v, w\rangle\,dt = 0, \qquad v \in \mathcal{D}(\mathbb{R}^+, \mathcal{D}_{\mathcal{B}^\sharp}).$$

Thus Proposition 4 guarantees that function w is a weak $W^\sigma_{q_0,\mathcal{B}}$-solution of $\dot u + \mathcal{A}u = 0$. Hence $w = 0$ by uniqueness. ■

Corollary 4 *Let* (85) *be true and one of the conditions below be satisfied:*

(i) $\Omega = \mathbb{R}^n$*;*

(ii) Ω *is bounded.*

Then $u_0 = u_1$.

PROOF. (i) We can assume that $q_0 \geq q_1$. Set $\rho := s_1 - n(1/q_1 - 1/q_0)$. Then $W_{q_1}^{s_1-2} \hookrightarrow W_{q_0}^{\rho-2}$ by Sobolev's embedding theorem. Consequently, it follows that $\sigma := (\rho \wedge s_0) - 2 \leq (s_0 \wedge s_1) - 2$ and

$$u_1 \in L_{1,\mathrm{loc}}(\mathbb{R}^+, W_{q_1}^{s_1-2}) \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^+, W_{q_0}^{\sigma}).$$

Hence the hypotheses of Theorem 9 are satisfied and the assertion follows.

(ii) We can assume that $q_1 \geq q_0$. Since $s_0 \wedge s_1 > 1 + 1/q_0 > 0$ it follows from Theorem 7(i) that u_j belongs to $L_{1,\mathrm{loc}}(\mathbb{R}^+, L_{q_j})$. By the boundedness of Ω,

$$u_1 \in L_{1,\mathrm{loc}}(\mathbb{R}^+, L_{q_1}) \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^+, L_{q_0}) = L_{1,\mathrm{loc}}(\mathbb{R}^+, W_{q_0,\mathcal{B}}^{0}).$$

Thus the assertion is also a consequence of Theorem 9. ■

7.E. The structure of extrapolation spaces In order to appreciate completely the significance of a $W_{q,\mathcal{B}}^{\sigma}$-solution of (80) we have to understand the meaning of $\int v\, d\mu$ for $\mu \in \mathcal{M}(\mathbb{R}^+, W_{q,\mathcal{B}}^{s-2})$. By (77) this is clear if either $\Gamma = \emptyset$ or $s > 1 + 1/q$. However, if $\Gamma \neq \emptyset$ and $s < 1 + 1/q$ then $W_{q,\mathcal{B}}^{s}$ is not a space of distributions on Ω, but also contains distributions supported on Γ. This is made precise in the following theorem, and it is the key observation for our treatment of nonhomogeneous boundary value problems. In the particular case where $\Gamma = \Gamma_0$ it has been proven in [7, Theorem 1.1], where we have used the Bessel potential instead of the Sobolev-Slobodeckii scale.

For $s \in \mathbb{R}$ and $1 \leq p < \infty$ we define a vector subspace ∂W_p^s of $W_p^{s-1/p}(\Gamma_0) \times W_p^{s-1-1/p}(\Gamma_1)$ by

$$\partial W_p^s := \begin{cases} \{0\} \times W_p^{s-1-1/p}(\Gamma_1), & 1/p < s < 1 + 1/p, \quad \Gamma_1 \neq \emptyset, \\ W_p^{s-1/p}(\Gamma_0) \times W_p^{s-1-1/p}(\Gamma_1), & -1 + 1/p < s < 1/p, \quad \Gamma \neq \emptyset, \end{cases}$$

and by putting $\partial W_p^s := \{0\}$ in all other cases. As usual, we often omit any reference to a trivial space $\{0\}$, that is, we identify ∂W_p^s with $W_p^{s-1-1/p}(\Gamma_1)$ if $1/p < s < 1 + 1/p$ and $\Gamma_1 \neq \emptyset$, for example.

Theorem 10 *Suppose that* $\Gamma \neq \emptyset$ *and* $s \in I_q$ *satisfies* $s < 1/q$. *Then* $W_{q,\mathcal{B}}^{s} \cong W_q^s \times \partial W_q^{s+2}$.

PROOF. (a) If $-1 + 1/q < s < 1/q$ then $W_{q,\mathcal{B}}^{s} = W_q^s$ by (77).

(b) Suppose that $-2 + 1/q < s < -1 + 1/q$. Then $W_{q',\mathcal{B}^\sharp}^{-s} = \{\, v \in W_{q'}^{-s} \,;\, \gamma_0 v = 0 \,\}$, where γ_j denotes the restriction of γ to Γ_j. Thus

$$\ker(\gamma_1 \,|\, W_{q',\mathcal{B}^\sharp}^{-s}) = \{\, v \in W_{q'}^{-s} \,;\, \gamma v = 0 \,\} = \mathring{W}_{q'}^{-s}.$$

Recall that $\gamma_1 \in \mathcal{L}\big(W_{q',\mathcal{B}^\sharp}^{-s}, W_{q'}^{-s-1/q'}(\Gamma_1)\big)$ is a retraction, that is, there exists a right inverse

$$\gamma_1^c \in \mathcal{L}\big(W_{q'}^{-s-1/q'}(\Gamma_1), W_{q',\mathcal{B}^\sharp}^{-s}\big),$$

a coretraction for γ_1. Hence (see [4, Lemma I.2.3.1]) $W_{q',\mathcal{B}^\sharp}^{-s} = \mathrm{im}(\gamma_1^c) \oplus \mathring{W}_{q'}^{-s}$, and

$$p_\Omega := 1 - \gamma_1^c \gamma_1 \in \mathcal{L}(W_{q',\mathcal{B}^\sharp}^{-s})$$

is the projection onto $\mathring{W}_{q'}^{-s}$ parallel to $\mathrm{im}(\gamma_1^c)$. Thus

$$\mathring{W}_{q'}^{-s} = \mathrm{im}(1 - \gamma_1^c \gamma_1). \tag{86}$$

It follows that

$$r_1 := (\gamma_1^c)' \in \mathcal{L}\big(W_{q,\mathcal{B}}^s, W_q^{s+1-1/q}(\Gamma_1)\big)$$

and

$$r_1^c := (\gamma_1 \,|\, W_{q',\mathcal{B}^\sharp}^{-s})' \in \mathcal{L}\big(W_q^{s+1-1/q}(\Gamma_1), W_{q,\mathcal{B}}^s\big)$$

are well-defined. Furthermore,

$$r_1 r_1^c = \big((\gamma_1 \,|\, W_{q',\mathcal{B}^\sharp}^{-s})\gamma_1^c\big)' = (\gamma_1\gamma_1^c)' = 1_{W_q^{s+1-1/q}(\Gamma_1)}.$$

Consequently,

$$W_{q,\mathcal{B}}^s = \ker(r_1) \oplus \operatorname{im}(r_1^c) \cong \ker(r_1) \times W_q^{s+1-1/q}(\Gamma_1) \tag{87}$$

because r_1^c is an isomorphism from $W_q^{s+1-1/q}(\Gamma_1)$ onto its image. But $u \in W_{q,\mathcal{B}}^s$ belongs to $\ker(r_1)$ iff $\langle u, \gamma_1^c g\rangle = 0$ for all $g \in W_{q'}^{-s-1/q'}(\Gamma_1)$. Since $v = (1-\gamma_1^c\gamma_1)v + \gamma_1^c\gamma_1 v$, we see that $u \in \ker(r_1)$ iff $\langle u, v\rangle = \big\langle u, (1-\gamma_1^c\gamma_1)v\big\rangle$ for all $v \in W_{q',\mathcal{B}^\sharp}^{-s}$. Hence we infer from (86) that $u \in \ker(r_1)$ iff $u = r_\Omega u$, where

$$r_\Omega \in \mathcal{L}(W_{q,\mathcal{B}}^s, W_q^s)$$

is the restriction map $u \mapsto u \,|\, \mathring{W}_{q'}^{-s}$. This shows that $\ker(r_1) = W_q^s$ so that (87) proves the assertion.

(c) Now suppose that $-3 + 1/q < s < -2 + 1/q$. Set

$$\partial^\sharp W_{q'}^{-s-2} := W_{q'}^{-s-2+1/q}(\Gamma_0) \times W_{q'}^{-s-1+1/q}(\Gamma_1).$$

Then $\partial W_q^{s+2} = (\partial^\sharp W_{q'}^{-s-2})'$ with respect to the duality pairing $\langle\cdot,\cdot\rangle_{\Gamma_0} + \langle\cdot,\cdot\rangle_{\Gamma_1}$. We also put

$$\partial_{\boldsymbol{\nu},j}^\sharp u := \partial_{\boldsymbol{\nu}}^\sharp u \,|\, \Gamma_j, \quad \mathcal{B}_1^\sharp u := \mathcal{B}^\sharp u \,|\, \Gamma_1, \qquad u \in W_{q'}^{-s}, \quad j = 0, 1.$$

Then

$$\partial_{\boldsymbol{\nu},j}^\sharp \in \mathcal{L}\big(W_{q'}^{-s}, W_{q'}^{-s-2+1/q}(\Gamma_j)\big), \quad \mathcal{B}_1^\sharp \in \mathcal{L}\big(W_{q'}^{-s}, W_{q'}^{-s-2+1/q}(\Gamma_1)\big).$$

It follows from (the proof of) [2, Theorem B.3] that there exist

$$\mathcal{R}_j^\sharp \in \mathcal{L}\big(W_{q'}^{-s-2+1/q}(\Gamma_j), W_{q'}^{-s}\big)$$

satisfying

$$\partial_{\boldsymbol{\nu},0}^\sharp \mathcal{R}_0^\sharp = 1, \quad \partial_{\boldsymbol{\nu},0}^\sharp \gamma_1 = 0, \quad \partial_{\boldsymbol{\nu},1}^\sharp \mathcal{R}_1^\sharp = 1$$

and

$$\gamma_j \mathcal{R}_j^\sharp = 0, \quad \gamma_{\boldsymbol{\nu},j}^\sharp \mathcal{R}_k^\sharp = \gamma_j \mathcal{R}_k^\sharp = 0, \qquad j \neq k,$$

as well as

$$\gamma_1^c \in \mathcal{L}\big(W_{q'}^{-s-1+1/q}(\Gamma_1), W_{q'}^{-s}\big)$$

such that $\gamma_1\gamma_1^c = 1$, $\gamma_0\gamma_1^c = 0$ and $\partial_{\boldsymbol{\nu},1}^\sharp \gamma_j^c = 0$, where 1 denotes the identity in the appropriate spaces. Set

$$\delta v := (-\partial_{\boldsymbol{\nu},0}^\sharp v, \gamma_1 v), \qquad v \in W_{q',\mathcal{B}^\sharp}^{-s},$$

so that $\delta \in \mathcal{L}(W_{q',\mathcal{B}^\sharp}^{-s}, \partial^\sharp W_{q'}^{-s-2})$. Also put

$$\delta^c g := -\mathcal{R}_0^\sharp g_0 + (\gamma_1^c - \mathcal{R}_1^\sharp b^\sharp) g_1, \qquad g = (g_0, g_1) \in \partial^\sharp W_{q'}^{-s-2}.$$

Then $\mathcal{B}^\sharp \delta^c = 0$ so that

$$\delta^c \in \mathcal{L}(\partial^\sharp W_{q'}^{-s-2}, W_{q',\mathcal{B}^\sharp}^{-s}), \qquad \delta\delta^c = 1_{\partial^\sharp W_{q'}^{-s-2}}.$$

Hence δ is a retraction from $W_{q',\mathcal{B}^\sharp}^{-s}$ onto $\partial^\sharp W_{q'}^{-s-2}$, and δ^c is a corresponding coretraction. Note that

$$\ker(\delta) = \{\, v \in W_{q'}^{-s} \; ; \; \gamma v = 0,\ \partial_{\boldsymbol{\nu}}^\sharp v = 0 \,\} = \mathring{W}_{q'}^{-s}.$$

Consequently, as above, $W_{q',\mathcal{B}^\sharp}^{-s} = \operatorname{im}(\delta^c) \oplus \mathring{W}_{q'}^{-s}$, and

$$p_\Omega := 1 - \delta^c\delta \in \mathcal{L}(W_{q',\mathcal{B}^\sharp}^{-s})$$

is the projection onto $\mathring{W}_{q'}^{-s}$ parallel to $\operatorname{im}(\delta^c)$. Thus

$$\mathring{W}_{q'}^{-s} = \operatorname{im}(1 - \delta^c\delta). \tag{88}$$

It follows that $r := (\delta^c)' \in \mathcal{L}(W_{q,\mathcal{B}}^s, \partial W_q^{s+2})$ is a retraction and $r^c := \delta' \in \mathcal{L}(\partial W_q^{s+2}, W_{q,\mathcal{B}}^s)$ is a corresponding coretraction. Hence

$$W_{q,\mathcal{B}}^s = \ker(r) \oplus \operatorname{im}(r^c) \cong \ker(r) \times \partial W_q^{s+2}. \tag{89}$$

Note that $u \in W_{q,\mathcal{B}}^s$ belongs to $\ker(r)$ iff $\langle u, \delta^c g\rangle = 0$ for all $g \in \partial^\sharp W_{q'}^{-s-2}$, that is, iff

$$\langle u, v\rangle = \big\langle u, (1 - \delta^c\delta)v\big\rangle, \qquad v \in W_{q',\mathcal{B}^\sharp}^{-s}.$$

Thus (88) implies $u \in \ker(r)$ iff $u = r_\Omega u$. So, $\ker(r) = W_q^s$ and the assertion follows from (89). ■

If E and F are Banach spaces then we denote by $\mathcal{L}\mathrm{is}(E,F)$ the set of all isomorphisms in $\mathcal{L}(E,F)$.

Corollary 5 *Suppose that $\Gamma \neq \emptyset$ and $s \in I_q$ satisfies $s < 1/q$. Put*

$$R(w,g) := p_\Omega' w - (\partial_{\boldsymbol{\nu},0}^\sharp)' g_0 + \gamma_1' g_1, \qquad w \in W_q^s, \quad g := (g_0, g_1) \in \partial W_q^{s+2},$$

where p_Ω is the identity on $W_{q'}^{-s}$ if $s > -1 + 1/q$. Then

$$R \in \mathcal{L}\mathrm{is}(W_q^s \times \partial W_q^{s+2}, W_{q,\mathcal{B}}^s),$$

and its inverse is given by $u \mapsto (r_\Omega u, r_1 u)$ if $s > -2 + 1/q$ and by $u \mapsto (r_\Omega u, ru)$ otherwise.

Of course, p_Ω and, consequently, the isomorphism R depend on the choice of the coretractions γ_1^c and δ^c, respectively.

Let X be an open subset of $\overline{\Omega}$. For $s \in \mathbb{R}^+$ we denote by $C_0^s(X)$ the closure of $\mathcal{D}(X)$ in $BUC^s(X)$, and $C_0^s := C_0^s(\Omega)$. Then

$$(C_0^s)' \doteq W_1^{-s}, \qquad s \in \mathbb{R}^+ \setminus \mathbb{N}, \tag{90}$$

with respect to the duality pairing $\langle \cdot, \cdot \rangle$ (e.g., [8]). Since $C_0' = \mathcal{M} := \mathcal{M}(\Omega)$, we set

$$\mathcal{M}^k(X) := C_0^k(X)', \qquad k \in \mathbb{N},$$

and $\mathcal{M}^k := \mathcal{M}^k(\Omega)$. We also set

$$C_{\mathcal{B}}^s := C_0^s, \qquad s \in \mathbb{R}^+, \quad \Gamma = \emptyset,$$

and

$$C_{\mathcal{B}}^s := \begin{cases} \{\, v \in C_0^s(\overline{\Omega}) \; ; \quad \gamma_0 v = 0 \,\}, & 0 \le s < 1, \\ \{\, v \in C_0^s(\overline{\Omega}) \; ; \quad \mathcal{B}v = 0 \,\}, & 1 \le s < 2, \end{cases}$$

if $\Gamma \neq \emptyset$. Note that $C_{\mathcal{B}}^s = C_0(\Omega \cup \Gamma_1)$ for $0 \le s < 1$.

Lemma 9 *The following embeddings are true:*

(i) *If* $q_j \in [1, \infty)$ *and* $s_j \in \mathbb{R}$ *satisfy*

$$\frac{1}{q_1} \geq \frac{1}{q_0} \geq \frac{1}{q_1} - \frac{s_1 - s_0}{n} \tag{91}$$

then $W_{q_1}^{s_1} \overset{d}{\hookrightarrow} W_{q_0}^{s_0}$.

(ii) *If* $q_j \in (1, \infty)$ *and* $s_j \in I_{q_j}$ *satisfy* (91) *then* $W_{q_1,\mathcal{B}}^{s_1} \overset{d}{\hookrightarrow} W_{q_0,\mathcal{B}}^{s_0}$.

(iii) *If* $s \in I_q$ *satisfies* $s > n/q$ *then* $W_{q,\mathcal{B}}^{s} \overset{d}{\hookrightarrow} C_{\mathcal{B}}^{\rho}$ *for* $0 \leq \rho \leq s - n/q$, *with* $\rho \neq s - n/q$ *if* $s - n/q \in \mathbb{N}$.

(iv) *If* $0 \leq \rho \leq -s - n/q'$, *with* $\rho \neq -s - n/q'$ *for* $-s - n/q' \in \mathbb{N}$, *then* $(C_{\mathcal{B}^\sharp}^{\rho})' \hookrightarrow W_{q,\mathcal{B}}^{s}$.

PROOF. (i) is a well-known Sobolev type embedding theorem.

(ii) If $s_0 > -1 + 1/q_0$ then the assertion follows from (i) and (77), thanks to Lemma 8(i). The case $s_1 < 1/q_1$ is now obtained by duality and again by Lemma 8(i). If $s_1 > 1/q_1$ and $s_0 < -1 + 1/q_0$ then the assertion is implied by $W_{q_1,\mathcal{B}}^{s_1} \overset{d}{\hookrightarrow} L_{q_0} \overset{d}{\hookrightarrow} W_{q_0,\mathcal{B}}^{s_0}$ and what has just been shown.

(iii) is also a consequence of well-known Sobolev type embedding theorems and Lemma 8(i); and (iv) follows from (iii) by duality. ■

For $k \in \mathbb{N}$ we define the vector subspace $\partial\mathcal{M}^k$ of $\mathcal{M}^k(\Gamma)$ by

$$\partial\mathcal{M}^k := \begin{cases} \{0\} \times \mathcal{M}(\Gamma_1), & k = 0, \quad \Gamma_1 \neq \emptyset, \\ \mathcal{M}(\Gamma_0) \times \mathcal{M}(\Gamma_1), & k = 1, \quad \Gamma \neq \emptyset, \end{cases}$$

and by $\partial\mathcal{M}^k := \{0\}$ in all other cases. Of course, $\partial\mathcal{M} := \partial\mathcal{M}^0$.

Using these notations we can prove the following analogue to Theorem 10.

Theorem 11 *Suppose that* $0 < s < 2$ *with* $s \neq 1$. *Then*

$$(C_{\mathcal{B}^\sharp}^{s})' \cong W_1^{-s} \times \partial W_1^{-s+2}.$$

If $k \in \{0, 1\}$ *then*

$$(C_{\mathcal{B}^\sharp}^{k})' \cong \mathcal{M}^k \times \partial\mathcal{M}^k.$$

For each $t \in [0, 2]$ *an isomorphism onto* $(C_{\mathcal{B}^\sharp}^{t})'$ *is given by the map* R *defined in Corollary 5.*

PROOF. The assertion follows by literally the same arguments as those used in the proof of Theorem 10, since [2, Theorem B.3] guarantees that the operators γ, γ^c, δ, and δ^c have the required properties in this case also. (In [2] the case $s \geq 1$ has been treated only. But those facts are obvious if $0 \leq s < 1$.) ■

Corollary 6 *Suppose that* $1 \leq r < \infty$ *and* $s < 2/r - n/q'$ *with* $s - 2/r \in I_q$. *Also suppose that*

$$0 \leq t \leq 2/r - n/q' - s \quad \textit{and} \quad t < 2$$

with $t < 1$ *if* $s = 2/r - n/q' - 1$. *Then*

$$R \in \mathcal{L}(W_1^{-t} \times \partial W_1^{-t+2}, W_{q,\mathcal{B}}^{s-2/r}),$$

and, if $t \in \{0, 1\}$,

$$R \in \mathcal{L}(\mathcal{M}^t \times \partial\mathcal{M}^t, W_{q,\mathcal{B}}^{s-2/r}).$$

PROOF. This follows from Lemma 9 and Theorem 11. ■

7.F. Parabolic boundary value problems Let assumption (79) be satisfied. If $\Gamma \neq \emptyset$ then choose an isomorphism R of the form specified in Corollary 5 and set $(\mu_\Omega, \mu_\Gamma) := R^{-1}\mu$. It follows from Example 1(c) that

$$(\mu_\Omega, \mu_\Gamma) \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, W_q^{s-2} \times \partial W_q^s), \qquad (\mu_0, \mu_1) := \mu_\Gamma. \tag{92}$$

The definition of R implies that u is the $W^\sigma_{q,\mathcal{B}}$-solution of (80) iff $u \in L_{1,\mathrm{loc}}(\mathbb{R}^+, W^\sigma_{q,\mathcal{B}})$ and

$$\int_0^\infty \langle -\partial_t v + \mathcal{A}^\sharp v, u\rangle\, dt = \int_0^\infty p_\Omega v\, d\mu_\Omega + \int_0^\infty v\, d\mu_1 - \int_0^\infty \partial^\sharp_{\nu,0} v\, d\mu_0, \qquad v \in \mathcal{D}(\mathbb{R}^+, \mathcal{D}_{\mathcal{B}^\sharp}). \tag{93}$$

This is being expressed by saying that u is the (unique *weak*) W_q^σ*-solution of the boundary value problem* (BVP)

$$\left.\begin{aligned} \partial_t u + \mathcal{A}u &= \mu_\Omega \quad &&\text{in} \quad \Omega \times \mathbb{R}^+, \\ \mathcal{B}u &= \mu_\Gamma \quad &&\text{on} \quad \Gamma \times \mathbb{R}^+. \end{aligned}\right\} \tag{94}$$

Let also assumption (81) be satisfied and set

$$(f_\Omega, g) := R^{-1} f \in L_{r,\mathrm{loc}}(\mathbb{R}^+, W^{s-2}_{q,\mathcal{B}}), \quad (g_0, g_1) := g.$$

Then (93) takes the form

$$\int_0^\infty \langle -\partial_t v + \mathcal{A}^\sharp v, u\rangle\, dt = \int_0^\infty \big\{\langle p_\Omega v, f_\Omega\rangle + \langle \gamma_1 v, g_1\rangle_{\Gamma_1} - \langle \partial^\sharp_{\nu,0} v, g_0\rangle_{\Gamma_0}\big\}\, dt + \big\langle v(0), u^0\big\rangle, \tag{95}$$

and u is said to be the (unique *weak*) W_q^σ*-solution of the initial boundary value problem* (IBVP)

$$\left.\begin{aligned} \partial_t u + \mathcal{A}u &= f_\Omega(t) \quad &&\text{in} \quad \Omega \times (0,\infty), \\ \mathcal{B}u &= g(t) \quad &&\text{on} \quad \Gamma \times (0,\infty), \\ u(\cdot, 0) &= u^0 \quad &&\text{on} \quad \Omega. \end{aligned}\right\} \tag{96}$$

Theorem 12 *Let assumption* (79) *be satisfied. Then the* W_q^σ*-solution of* (94) *is a distributional solution of* $\partial_t u + \mathcal{A}u = \mu_\Omega$*, that is,*

$$\int_0^\infty \langle -\partial_t \varphi + \mathcal{A}^\sharp \varphi, u\rangle\, dt = \int_0^\infty \varphi\, d\mu_\Omega, \qquad \varphi \in \mathcal{D}(Q), \tag{97}$$

where $Q := \Omega \times (0,\infty)$.

PROOF. It follows from (93) and $p_\Omega \,|\, \mathcal{D} = 1_{\mathcal{D}}$ that (97) is true for all φ belonging to $\mathcal{D}\big((0,\infty), \mathcal{D}\big)$. It is known that $\mathcal{D}\big((0,\infty), \mathcal{D}\big)$ can be naturally identified with $\mathcal{D}(Q)$ (e.g., Corollary 1 to Theorem 40.1 in [26]). Since $s > -1 + 1/q$ if $\Gamma \neq \emptyset$ and since we can choose σ close to s it follows that

$$u \in L_{1,\mathrm{loc}}(\mathbb{R}^+, W_q^\sigma) \hookrightarrow \mathcal{D}'(Q),$$

so that u is a distribution on Q. ■

Suppose that μ_Ω has the property that

$$\int_0^\infty p_\Omega v\, d\mu_\Omega = \int_0^\infty v\, d\mu_\Omega, \qquad v \in \mathcal{D}(\mathbb{R}^+, \mathcal{D}_{\mathcal{B}^\sharp}). \tag{98}$$

Then p_Ω, hence the isomorphism R, does not appear explicitly in (93) and (95). In this case (93), resp. (95), is *formally* obtained by 'multiplying' (94), resp. (96), by $v \in \mathcal{D}(\mathbb{R}^+, \mathcal{D}_{\mathcal{B}^\sharp})$, integrating over $\mathbb{R}^+$, integrating by parts, and using Green's formulas (76) and the boundary, resp. initial and boundary, conditions.

In the following lemma we collect some important cases for which (98) is true. Of course, it is trivially true if $\Gamma = \emptyset$.

Lemma 10 *Suppose that* (79) *is true.*

(i) *Let one of the following assumptions be satisfied:*

(α) $\mu_\Omega \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, W_q^t)$ *for some* $t \geq s-2$ *with* $t > -2+1/q$*, and* $t > -1+1/q$ *if* $\Gamma_1 \neq \emptyset$*;*

(β) $L_1 \hookrightarrow W_{q,\mathcal{B}}^{s-2}$ *and* $\mu_\Omega \in L_{1,\mathrm{loc}}(\mathbb{R}^+, L_1)$.

Then (98) *holds for every choice of* p_Ω.

(ii) *For each* $\varepsilon > 0$ *there exists* p_Ω *such that* (98) *is true whenever* $\mu_\Omega \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, W_q^{s-2})$ *satisfies*

$$\operatorname{dist}\bigl(\operatorname{supp}(\mu_\Omega(B)), \Gamma\bigr) \geq \varepsilon$$

for every Borel subset B *of* $\mathbb{R}^+$.

(iii) *Suppose that* $\mu_\Omega \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{M})$ *and let* $\mu_\Omega = \mu_a + \mu_s$ *be the Lebesgue decomposition of* $\mu_\Omega \in \mathcal{M}(\Omega \times \mathbb{R}^+)$ *with respect to Lebesgue's measure* $dx\,dt$*, where* μ_a *is the absolutely continuous and* $\mu_s \perp \mu_a$ *the corresponding singular part. If* $\operatorname{dist}\bigl(\operatorname{supp}(\mu_s), \Gamma \times \mathbb{R}^+\bigr) > 0$ *then there exists* p_Ω *such that* (98) *is true.*

PROOF. (i) Suppose that $w \in W_q^t$ or $w \in L_1$, where we can assume that $t < 1/q$. Hence $(W_q^t)' = \mathring{W}_{q'}^{-t} = W_{q',\mathcal{B}^\sharp}^{-t}$ so that $\mathcal{D} \overset{d}{\hookrightarrow} (W_{q,\mathcal{B}^\sharp}^{-t})'$. Of course, $\mathcal{D} \overset{d}{\hookrightarrow} L_1$. Since $p_\Omega \varphi = \varphi$ for $\varphi \in \mathcal{D}$ we see that

$$\langle \varphi, w\rangle = \langle p_\Omega \varphi, w\rangle = \langle \varphi, p_\Omega' w\rangle, \qquad \varphi \in \mathcal{D}.$$

Thus $p_\Omega' w \in W_{q,\mathcal{B}}^{s-2}$ has the unique continuous extension $w \in W_q^t$, resp. $w \in L_1$. This implies the assertion.

(ii) Let $\psi \in \mathcal{D}(\overline{\Omega})$ be equal to 1 in a neighborhood of Γ such that $\psi(x) = 0$ for $x \in \Omega$ with $\operatorname{dist}(x, \Gamma) \geq \varepsilon/2$. Let γ_1^c, resp. δ^c, be a coretraction for γ_1, resp. δ. Then $\psi\gamma_1^c$, resp. $\psi\delta^c$, is a coretraction for γ_1, resp. δ, as well. Hence we can assume that

$$\varphi p_\Omega v = \varphi v, \qquad v \in \mathcal{D}_{\mathcal{B}^\sharp},$$

for each $\varphi \in BUC^\infty$ with $\operatorname{dist}\bigl(\operatorname{supp}(\varphi), \Gamma\bigr) > \varepsilon/2$. Fix any such φ satisfying $\varphi(x) = 1$ if $\operatorname{dist}(x, \Gamma) \geq \varepsilon$. Then, given $v \in \mathcal{D}_{\mathcal{B}^\sharp}$, it follows that $\varphi v \in \mathcal{D}$. Now suppose that $w \in W_q^{s-2}$ satisfies $\operatorname{dist}\bigl(\operatorname{supp}(w), \Gamma\bigr) \geq \varepsilon$. Then

$$\langle \varphi v, w\rangle = \langle \varphi p_\Omega v, w\rangle, \qquad v \in \mathcal{D}_{\mathcal{B}^\sharp}. \tag{99}$$

Let $\widetilde{\varphi}$ be another function having the same properties as φ. Then

$$\operatorname{supp}\bigl((\varphi - \widetilde{\varphi})v\bigr) \cap \operatorname{supp}(w) = \emptyset, \quad (\varphi - \widetilde{\varphi})v \in \mathcal{D}, \qquad v \in \mathcal{D}_{\mathcal{B}^\sharp}.$$

Thus $\bigl\langle (\varphi - \widetilde{\varphi})v, w\bigr\rangle = 0$, that is, $\langle \varphi v, w\rangle = \langle \widetilde{\varphi} v, w\rangle$ for $v \in \mathcal{D}_{\mathcal{B}^\sharp}$. Consequently, setting $\widetilde{w}(v) := \langle \varphi v, w\rangle$ for $v \in \mathcal{D}_{\mathcal{B}^\sharp}$, it follows that $\widetilde{w} \in W_{q,\mathcal{B}}^{s-2}$ with $\widetilde{w}\,|\,\mathring{W}_{q'}^{2-s} = w$, and $\widetilde{w}$ is uniquely determined by w. Hence, writing again w for $\widetilde{w}$, we see from (99) that $\langle v, w\rangle = \langle p_\Omega v, w\rangle$ for $v \in \mathcal{D}_{\mathcal{B}^\sharp}$. Now the assertion is obvious.

(iii) Since $\mu_a = f\,dx\,dt$, where $f \in L_{1,\mathrm{loc}}(\mathbb{R}^+, L_1)$, the assertion follows from (i) and (ii). ■

Lastly, we study positivity properties.

Proposition 5 *Let* (79) *be satisfied and suppose that* (98) *is also true. If* $\mu_\Omega \geq 0$ *and* $\mu_\Gamma \geq 0$ *then the* W_q^σ*-solution of BVP* (94) *is also positive.*

PROOF. Suppose that $v \in \mathcal{D}_{\mathcal{B}^\sharp}$ is positive. Since the trace operator is positive, $\langle \gamma_1 v, \mu_0\rangle_{\Gamma_0} \geq 0$. Moreover, $\gamma_0 v = 0$ implies $\partial_{\nu,0}^\sharp v \leq 0$. Hence $\langle \partial_{\nu,0}^\sharp v, \mu_1\rangle_{\Gamma_1} \leq 0$. Consequently, $\mu_\Omega \geq 0$ and $\mu_\Gamma = (\mu_0, \mu_1) \geq 0$ imply $\mu = R(\mu_\Omega, \mu_\Gamma) \geq 0$. Now the assertion follows from Theorem 7(iv). ■

7.G. Examples We begin by considering the case where μ_Ω and μ_Γ are Radon measures on $\Omega \times \mathbb{R}^+$ and $\Gamma \times \mathbb{R}^+$, respectively.

Theorem 13 *Let* (3) *and* (4) *be satisfied and set* $\delta := 0$ *if* $\mu_0 = 0$*, and* $\delta := 1$ *otherwise. Also suppose that* $\sigma \in I_q$ *satisfies* $0 \le \sigma < 2 - n/q' - \delta$*. Then BVP* (94) *has a unique* W_q^σ*-solution* u*, and* u *belongs to* $L_{p,\mathrm{loc}}(\mathbb{R}^+, W_q^\sigma)$ *for* $p \ge 1$ *with* $2/p + n/q > n + \delta + \sigma$*. Furthermore, the map* $(\mu_\Omega, \mu_\Gamma) \mapsto u$ *is linear and continuous in the respective topologies. If* $(\mu_\Omega, \mu_\Gamma) \ge 0$ *then* $u \ge 0$.

PROOF. Fix p_Ω such that (98) is true. Thanks to (4) and Lemma 10(iii) this is possible. Also fix $s \in I_q$ satisfying $\sigma < s < 2 - n/q' - \delta$. Then Corollary 6 and $\mathcal{M} \times \big(\{0\} \times \mathcal{M}(\Gamma_1)\big) \hookrightarrow \mathcal{M}^k \times \partial\mathcal{M}^k$ imply

$$\big((\mu_\Omega, \mu_\Gamma) \mapsto R(\mu_\Omega, \mu_\Gamma)\big) \in \mathcal{L}\big(\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{M} \times \mathcal{M}(\Gamma)), \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^+, W_{q,\mathcal{B}}^{s-2})\big).$$

Now it follows from the considerations of Subsection 7.F. and Theorem 7(i) that (94) has a unique W_q^σ-solution u, and

$$u \in L_{p,\mathrm{loc}}(\mathbb{R}^+, W_q^\sigma), \qquad 1 \le p < 2/(\sigma - s + 2).$$

Theorem 9 guarantees that u is independent of s. Thanks to Proposition 5, the assertions are now clear. ∎

Remarks 2 Let (3) and (4) be true and let u be the solution of (94).
(a) Suppose that $\mu_0 = 0$. Then u is characterized by $u \in L_{1,\mathrm{loc}}(\mathbb{R}^+, W_{q,\mathcal{B}}^1)$ and

$$\int_0^\infty \big\{-\langle \partial_t v, u\rangle + \mathfrak{a}(v,u)\big\}\,dt = \int_0^\infty v\,d\mu_\Omega + \int_0^\infty v\,d\mu_1, \qquad v \in \mathcal{D}(\mathbb{R}^+, \mathcal{D}_\mathcal{B}).$$

Furthermore, $\gamma_1 u \in L_{p,\mathrm{loc}}\big(\mathbb{R}^+, L_r(\Gamma_1)\big)$ for $2/p + (n-1)/r > n$.
PROOF. The first assertion follows from Proposition 4. Since

$$\gamma_1 \in \mathcal{L}\big(W_q^\sigma, W_q^{\sigma - 1/q}(\Gamma_1)\big)$$

and $W_q^{\sigma-1/q}(\Gamma_1) \hookrightarrow L_r(\Gamma_1)$ for $1/r \ge 1/q - (\sigma - 1/q)/(n-1)$, we obtain the second one. ∎
(b) For simplicity, we have imposed C^∞-smoothness for the coefficients of $(\mathcal{A}, \mathcal{B})$ and Ω. However, these requirements can be considerably relaxed. For example, it suffices to assume that $(a_0, b_0) \in L_\infty \times L_\infty(\Gamma_1)$, and these restrictions can be relaxed even further (see [3] and [9, Appendix]). ∎

Now we briefly consider examples involving distributions which are not necessarily measures.

Theorem 14 *Let* (3)*,* (4)*, and* (13) *be satisfied and consider*

$$\left.\begin{aligned} \partial_t u + \mathcal{A}u &= \mu_\Omega + \textstyle\sum_j \partial_j \rho_j \quad &&\text{in} \quad \Omega \times \mathbb{R}^+, \\ \mathcal{B}u &= \mu_\Gamma &&\text{on} \quad \Gamma \times \mathbb{R}^+. \end{aligned}\right\} \tag{100}$$

Given $\sigma \in [0, 1 - n/q')$*, problem* (100) *has a unique* W_q^σ*-solution* u*, and*

$$u \in L_{p,\mathrm{loc}}(\mathbb{R}^+, W_q^\sigma) \qquad \text{for} \quad 2/p + n/q > n + 1 + \sigma, \quad p \ge 1. \tag{101}$$

PROOF. This follows from Theorem 7 and the arguments of the proof of Theorem 13. ∎

Remarks 3 **(a)** Solution (101) of (100) is smooth on $(\Omega \times \mathbb{R}^+)\backslash\mathrm{supp}\big(\mu_\Omega + \sum_j \partial_j \rho_j\big)$, as follows from standard regularity theory.
(b) Suppose that $\Omega = \mathbb{R}^n$ and $m \in \mathbb{N}$. Also assume that $\rho_\alpha \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^+)$ for $\alpha \in \mathbb{N}^n$ with $|\alpha| \le m$. Given any $\sigma \in [-m - n/q', 2 - m - n/q')$, the BVP

$$\partial_t u + \mathcal{A}u = \sum_{|\alpha| \le m} \partial^\alpha \rho_\alpha \qquad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+$$

has a unique W_q^σ-solution u, and $u \in L_{p,\mathrm{loc}}(\mathbb{R}^+, W_q^\sigma)$ for $2/p + n/q > \sigma + m + n$ with $p \ge 1$. Moreover, u is independent of σ and q.

PROOF. This is a consequence of Theorem 7 and Corollary 4. ■

7.H. Generalizations For the purpose of illustration we have treated second order parabolic initial boundary value problems for a single equation in detail. However, our abstract results apply to many other problems as well, in particular, to fully coupled systems.

More precisely, we now suppose that $N \in \mathbb{N}$ with $N > 1$, that

$$\boldsymbol{\alpha} := [\alpha_{jk}] \in BUC^{\infty}(\Omega, \mathbb{R}^{n\times n})$$

is symmetric and uniformly positive definit, and that

$$\mathsf{A}, a_j, a_0 \in BUC^{\infty}(\Omega, \mathbb{R}^{N\times N}), \quad b \in C^{\infty}(\Gamma, \mathbb{R}^{N\times N}), \qquad 1 \le j \le n.$$

We also assume that there exists $\varepsilon > 0$ such that

$$\sigma\big(\mathsf{A}(x)\big) \subset \{\, z \in \mathbb{C}\,;\ \mathrm{Re}(z) \ge \varepsilon \,\}, \qquad x \in \Omega,$$

where $\sigma(\cdot)$ denotes the spectrum. Then we set

$$a_{jk} := \mathsf{A}\alpha_{jk} \in BUC^{\infty}(\Omega, \mathbb{R}^{N\times N}), \quad \boldsymbol{a} := [a_{jk}], \quad \vec{a} := [a_1, \ldots, a_n],$$

and, using the summation convention with j and k running from 1 to n,

$$\nabla \cdot (\boldsymbol{a}\nabla u) := \partial_j(a_{jk}\partial_k u), \quad \vec{a}\cdot\nabla u := a_j\partial_j u, \qquad u \in \mathcal{D}(\overline{\Omega}, \mathbb{R}^N).$$

We also set

$$\partial_{\boldsymbol{\nu}} u := \nu^j \gamma(a_{jk}\partial_k u), \qquad u \in \mathcal{D}(\overline{\Omega}, \mathbb{R}^N).$$

Using these notations we define $(\mathcal{A}, \mathcal{B})$ as before. Then $(\mathcal{A}, \mathcal{B})$ is a normally elliptic boundary value problem of separated divergence form, using the terminology of [3].

Now we put

$$\boldsymbol{a}^{\sharp} := \boldsymbol{a}^{\top}, \quad \vec{a}^{\sharp} := -[a_1^{\top}, \ldots, a_n^{\top}], \quad a_0^{\sharp} := a_0^{\top}$$

and

$$b^{\sharp} := b^{\top} - (\gamma a_j^{\top})\nu^j, \quad \partial_{\boldsymbol{\nu}}^{\sharp} u := \nu^j(\gamma a_{jk}^{\top}\partial_k u), \qquad u \in \mathcal{D}(\overline{\Omega}, \mathbb{R}^N).$$

With these conventions we define $(\mathcal{A}^{\sharp}, \mathcal{B}^{\sharp})$ as before. Finally, we set

$$\langle \nabla v, \boldsymbol{a}\nabla u\rangle := \langle \partial_j v, a_{jk}\partial_k u\rangle, \qquad u, v \in \mathcal{D}(\overline{\Omega}, \mathbb{R}^N).$$

Then, by obvious modifications, everything above, except the assertions concerning positivity, remains valid for the parabolic system corresponding to (94).

PROOF. This follows from the results in [3] (see, in particular, Example 4.3(d) therein). ■

Of course, it is also possible to consider more general systems not being of separated divergence form. For this we again refer to [3].

The general theory applies equally well to higher order systems or to triangular systems of possibly different orders. Moreover, it can be suitably modified to cover parabolic problems with dynamic boundary conditions.

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